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Doubly nonlinear evolution equation and its applications

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1 Introduction

Let V and H be a real reflexive Banach space and a real Hilbert space respectively, and let V^* and H^* be dual spaces of V and H respectively. Moreover let H be identified with its dual space H^* and suppose that

$$(1) \qquad V \subset H \equiv H^* \subset V^*$$

with densely defined continuous natural injections.

This paper is concerned with doubly nonlinear evolution equations such as

$$\text{(DE)} \quad \frac{dv}{dt}(t) + \partial_V \varphi(u(t)) \ni f(t) \ \text{ in } V^*, \quad v(t) \in \partial_V \psi(u(t)),$$

where $\varphi, \psi : V \to (-\infty, +\infty]$ are proper lower semi-continuous convex (p-l.s.c. for short) functions and their subdifferentials $\partial_V \varphi, \partial_V \psi$ are defined as follows:

DEFINITION 1.1 *Let X be a linear topological space and let $\phi \in \Phi(X) := \{\phi : X \to (-\infty, +\infty]; \phi$ is p-l.s.c.$\}$. Then the effective domain $D(\phi)$ and the subdifferential $\partial_X \phi$ of ϕ are given by*

$$\begin{aligned} D(\phi) &:= \{u \in X; \phi(u) < +\infty\}, \\ \partial_X \phi(u) &:= \{\xi \in X^*; \phi(v) - \phi(u) \geq \langle \xi, v - u \rangle, \forall v \in D(\phi)\}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes a duality pairing between X and X^*.

In particular, for every $\phi \in \Phi(H)$, its subdifferential $\partial_H\phi$ is given as follows

$$\partial_H\phi(u) \quad = \quad \{\xi \in H; \phi(v) - \phi(u) \geq (\xi, v-u)_H, \forall v \in D(\phi)\},$$

where $(\cdot,\cdot)_H$ denotes an inner product in H.

In the next section, we prove the existence of a strong solution to Cauchy problem for (DE) without supposing that $\partial_V\psi$ is Lipschitz continuous.

Moreover as an application of (DE) to PDEs, we introduce the following doubly nonlinear parabolic equation (DP).

$$\text{(DP)} \quad \frac{\partial}{\partial t}|u|^{m-2}u - \Delta_p u = f \text{ in } \Omega \times (0,T), \quad u = 0 \text{ on } \partial\Omega \times (0,T),$$

where Δ_p is the so-called p-Laplacian defined by $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$, and Ω denotes a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. We then discuss the existence of a weak solution to the initial-boundary value problem for (DP), and its periodic problem as well.

2 Abstract Evolution Equation

Let us consider the following Cauchy problem (CP) for (DE).

$$\text{(CP)} \quad \begin{cases} \dfrac{dv}{dt}(t) + g(t) = f(t) \text{ in } V^*, \quad 0 < t < T, \\ v(t) \in \partial_V\psi(u(t)), \ g(t) \in \partial_V\varphi(u(t)), \\ v(0) = v_0. \end{cases}$$

Sufficient conditions for the existence of strong solutions to (CP) were studied by Kenmochi [13] and Kenmochi-Pawlow [14] in the Hilbert space framework (i.e., the case where $V = H$). However since they assume that $\partial_H\psi$ is Lipschitz continuous in H in [13] and [14], their results can not be directly applied to (DP); so we make an attempt to construct a strong solution of (CP) without any Lipschitz continuity of $\partial_V\psi$.

We first give a definition of strong solutions for (CP) as follows.

Definition 2.1 *A pair of functions $(u, v) : [0,T] \to V \times V^*$ is said to be a strong solution of* (CP) *on* $[0,T]$ *if the following* (i)-(iv) *hold true.*

(i) *v is a V^*-valued absolutely continuous function on* $[0,T]$.

(ii) *$u(t) \in D(\partial_V\psi) \cap D(\partial_V\varphi)$ for a.e. $t \in (0,T)$.*

(iii) *There exists $g(t) \in \partial_V\varphi(u(t))$ such that*

$$\frac{dv}{dt}(t) + g(t) = f(t), \quad v(t) \in \partial_V\psi(u(t)) \text{ in } V^*, \text{ for a.e. } t \in (0,T). \tag{2}$$

(iv) *$v(t) \to v_0$ strongly in V^* as $t \to +0$.*

The following result is concerned with the existence of strong solutions for (CP).

THEOREM 2.2 *Suppose that* (A1)-(A4) *are all satisfied.*

(A1) *There exist numbers* C_1, C_2 *such that* $|u|_V^p \le C_1\varphi(u) + C_2$ *for all* $u \in D(\varphi)$.

(A2) *There exists a non-decreasing function* $l : \mathbf{R} \to [0, +\infty)$ *such that* $|\xi|_{V^*} \le l(\varphi(u))$ *for all* $[u, \xi] \in \partial_V\varphi$.

(A3) *There exists* $\tilde{\psi} \in \Phi(H)$ *such that* $\tilde{\psi}(u) = \psi(u)$ *for all* $u \in V$, *and* $\varphi(J_\lambda u) \le \varphi(u)$ *for all* $u \in D(\varphi)$ *and* $\lambda > 0$, *where* $J_\lambda := (I + \lambda\partial_H\tilde{\psi})^{-1}$.

(A4) *For any* $r > 0$, *the set* $\{v \in R(\partial_V\psi); \psi^*(v) + |v|_H \le r\}$ *is precompact in* V^*, *where* $\psi^*(u) := \sup_{w\in V}\{\langle u, w\rangle - \psi(w)\}$.

Then for any $f \in W^{1,p'}(0, T; V^*) \cap L^2(0, T; H)$ *and* $v_0 \in (\partial_H\tilde{\psi})^\circ(D(\varphi) \cap D(\partial_H\tilde{\psi}))$, (CP) *has at least one strong solution* (u, v) *satisfying*:

$$\begin{aligned}
&u \in L^\infty(0, T; V), \quad u(t) \in D(\partial_H\tilde{\psi}) \ \text{ for a.e. } t \in (0, T),\\
&v \in C_w([0, T]; H) \cap W^{1,\infty}(0, T; V^*), \quad v(t) \in \partial_H\tilde{\psi}(u(t)) \ \text{ for a.e. } t \in (0, T),\\
&\text{the function } t \mapsto \tilde{\psi}^*(v(t)) \in W^{1,\infty}(0, T), \quad g \in L^\infty(0, T; V^*),
\end{aligned}$$

where $g(t)$ *denotes the sections of* $\partial_V\varphi(u(t))$ *in* (2). *Moreover* $C_w([0, T]; H)$ *denotes the set of all weakly continuous functions from* $[0, T]$ *into* H.

Before describing the proof of Theorem 2.2, we provide a remark on (A3).

REMARK 2.3 Since $\tilde{\psi}|_V = \psi$, we can derive

$$(3) \qquad D(\partial_H\tilde{\psi}) \cap V \subset D(\partial_V\psi) \ \text{ and } \ \partial_H\tilde{\psi}(u) \subset \partial_V\psi(u) \quad \forall u \in D(\partial_H\tilde{\psi}) \cap V.$$

Indeed, let $[u, f] \in \partial_H\tilde{\psi}$ be such that $u \in V$. Then we have

$$\begin{aligned}
\psi(v) - \psi(u) &= \tilde{\psi}(v) - \tilde{\psi}(u)\\
&\ge (f, v - u)_H = \langle f, v - u\rangle \quad \forall v \in D(\psi),
\end{aligned}$$

which implies $u \in D(\partial_V\psi)$ and $f \in \partial_V\psi(u)$.

In the rest of this section, for simplicity, we suppose that

$$V \text{ is separable, } 0 \in D(\varphi), \ \varphi \ge 0 \text{ and } \psi \ge 0.$$

However the above assumptions are not essential and can be easily removed by slight modifications on the following arguments.

We now proceed to the proof of Theorem 2.2. Here and henceforth, we denote by C non-negative constants, which do not depend on the elements of the corresponding space or set.

PROOF OF THEOREM 2.2 Let $u_0 \in D(\varphi) \cap D(\partial_H\tilde{\psi})$ be such that $(\partial_H\tilde{\psi})^\circ(u_0) = v_0$. To construct a strong solution of (CP), we introduce the following approximate problem:

$$(\mathrm{CP})_\lambda \quad \begin{cases} \lambda\dfrac{du_\lambda}{dt}(t) + \dfrac{d}{dt}\partial_H\tilde{\psi}_\lambda(u_\lambda(t)) + g_\lambda(t) = f(t) \ \text{ in } H, \ 0 < t < T,\\ g_\lambda(t) \in \partial_H\varphi_H(u_\lambda(t)), \quad u_\lambda(0) = u_0, \end{cases}$$

where $\tilde{\psi}_\lambda$ denotes the Moreau-Yosida regularization of $\tilde{\psi}$ and φ_H denotes an extension of φ on H given by

$$(4) \qquad \varphi_H(u) := \begin{cases} \varphi(u) & \text{if } u \in V, \\ +\infty & \text{if } u \in H \setminus V. \end{cases}$$

We then remark that (A1) ensures that $\varphi_H \in \Phi(H)$, $D(\varphi_H) = D(\varphi)$, $D(\partial_H \varphi_H) \subset D(\partial_V \varphi)$ and $\partial_H \varphi_H(u) \subset \partial_V \varphi(u)$ for all $u \in D(\partial_H \varphi_H)$ (see [2] for more details). Moreover $\lambda I + \partial_H \tilde{\psi}_\lambda$ becomes bi-Lipschitz continuous in H; hence we can assure the existence of a strong solution u_λ for (CP) on $[0,T]$ in much the same way as in Kenmochi [13] or [14].

We next establish a priori estimates in the following Lemmas 2.4-2.7. To this end, we employ fundamental properties of resolvents and Yosida approximations of maximal monotone operators, which can be found, e.g., in [4], [5] and [7].

LEMMA 2.4 *There exists a constant C such that*

$$(5) \qquad \sup_{t \in [0,T]} \varphi(u_\lambda(t)) \leq C,$$

$$(6) \qquad \lambda \int_0^T \left| \frac{du_\lambda}{dt}(t) \right|_H^2 dt \leq C.$$

PROOF OF LEMMA 2.4 Multiplying the first equation in $(\mathrm{CP})_\lambda$ by $du_\lambda(t)/dt$, we have

$$(7) \qquad \begin{aligned} &\lambda \left| \frac{du_\lambda}{dt}(t) \right|_H^2 + \left(\frac{d}{dt} \partial_H \tilde{\psi}_\lambda(u_\lambda(t)), \frac{du_\lambda}{dt}(t) \right)_H + \frac{d}{dt} \varphi_H(u_\lambda(t)) \\ &\leq \left(f(t), \frac{du_\lambda}{dt}(t) \right)_H \\ &= \frac{d}{dt} (f(t), u_\lambda(t))_H - \left\langle \frac{df}{dt}(t), u_\lambda(t) \right\rangle \quad \text{for a.e. } t \in (0,T). \end{aligned}$$

From the monotonicity of $\partial_H \tilde{\psi}_\lambda$, it is easily seen that

$$0 \leq \left(\frac{d}{dt} \partial_H \tilde{\psi}_\lambda(u_\lambda(t)), \frac{du_\lambda}{dt}(t) \right)_H .$$

Hence integrating both sides of (7) over $(0,t)$, we have

$$(8) \qquad \begin{aligned} &\lambda \int_0^t \left| \frac{du_\lambda}{d\tau}(\tau) \right|_H^2 d\tau + \varphi_H(u_\lambda(t)) \\ &\leq \varphi_H(u_0) + (f(t), u_\lambda(t))_H - (f(0), u_0)_H \\ &\quad - \int_0^t \left\langle \frac{df}{d\tau}(\tau), u_\lambda(\tau) \right\rangle d\tau \quad \forall t \in [0,T]. \end{aligned}$$

Moreover by Young's inequality, we get by (A1)

$$(f(t), u_\lambda(t))_H \leq C(|f(t)|_{V^*}^{p'} + 1) + \frac{1}{2} \varphi_H(u_\lambda(t))$$

and

$$\left|\int_0^t \left\langle \frac{df}{d\tau}(\tau), u_\lambda(\tau) \right\rangle d\tau\right| \le C\left(\int_0^T \left|\frac{df}{d\tau}(\tau)\right|_{V^*}^{p'} d\tau + 1\right) + \int_0^t \varphi(u_\lambda(\tau))d\tau.$$

Thus Gronwall's inequality implies (5). Moreover (6) follows from (5) and (8). ∎

LEMMA 2.5 *There exists a constant C such that*

$$\sup_{t\in[0,T]} |\partial_H \tilde{\psi}_\lambda(u_\lambda(t))|_H \le C. \tag{9}$$

PROOF OF LEMMA 2.5 Multiplying the first equation in $(\mathrm{CP})_\lambda$ by $\partial_H \tilde{\psi}_\lambda(u_\lambda(t))$, we obtain

$$\begin{aligned} &\lambda \frac{d}{dt}\tilde{\psi}_\lambda(u_\lambda(t)) + \frac{1}{2}\frac{d}{dt}|\partial_H\tilde{\psi}_\lambda(u_\lambda(t))|_H^2 + \left(g_\lambda(t), \partial_H\tilde{\psi}_\lambda(u_\lambda(t))\right)_H \\ &= \left(f(t), \partial_H\tilde{\psi}_\lambda(u_\lambda(t))\right)_H \le |f(t)|_H |\partial_H\tilde{\psi}_\lambda(u_\lambda(t))|_H \quad \text{for a.e. } t\in(0,T). \end{aligned} \tag{10}$$

From the fact that $g_\lambda(t) \in \partial_H \varphi_H(u_\lambda(t))$, Theorem 4.4 of [6] and (A3) imply

$$0 \le \left(g_\lambda(t), \partial_H\tilde{\psi}_\lambda(u_\lambda(t))\right)_H.$$

Hence integrating (10) over $(0,t)$, we get

$$\begin{aligned} &\lambda\tilde{\psi}_\lambda(u_\lambda(t)) + \frac{1}{2}|\partial_H\tilde{\psi}_\lambda(u_\lambda(t))|_H^2 \\ &\le \lambda\tilde{\psi}_\lambda(u_0) + \frac{1}{2}|\partial_H\tilde{\psi}_\lambda(u_0)|_H^2 + \frac{1}{2}\int_0^T |f(\tau)|_H^2 d\tau + \frac{1}{2}\int_0^t |\partial_H\tilde{\psi}_\lambda(u_\lambda(\tau))|_H^2 d\tau \end{aligned}$$

for all $t \in [0,T]$. Therefore since

$$\tilde{\psi}_\lambda(u_0) \le \tilde{\psi}(u_0) \quad \text{and} \quad |\partial_H\tilde{\psi}_\lambda(u_0)|_H \le |v_0|_H,$$

Gronwall's inequality yields (9). ∎

LEMMA 2.6 *There exists a constant C such that*

$$\sup_{t\in[0,T]} \tilde{\psi}^*(\partial_H\tilde{\psi}_\lambda(u_\lambda(t))) \le C, \tag{11}$$

$$\sup_{t\in[0,T]} \psi^*(\partial_H\tilde{\psi}_\lambda(u_\lambda(t))) \le C, \tag{12}$$

where $\tilde{\psi}^*$ denotes the conjugate function of $\tilde{\psi} \in \Phi(H)$ given by $\tilde{\psi}^*(u) := \sup_{w\in H}\{(u,w)_H - \tilde{\psi}(w)\}$.

PROOF OF LEMMA 2.6 Multiplying the first equation in $(\mathrm{CP})_\lambda$ by $u_\lambda(t)$ and noting that

$$\left(\frac{d}{dt}\partial_H\tilde{\psi}_\lambda(u_\lambda(t)), J_\lambda u_\lambda(t)\right)_H = \frac{d}{dt}\tilde{\psi}^*(\partial_H\tilde{\psi}_\lambda(u_\lambda(t))),$$

we get by (A1)

$$\begin{aligned}&\frac{\lambda}{2}\frac{d}{dt}|u_\lambda(t)|_H^2 + \frac{d}{dt}\tilde{\psi}^*(\partial_H\tilde{\psi}_\lambda(u_\lambda(t))) + \frac{\lambda}{2}\frac{d}{dt}|\partial_H\tilde{\psi}_\lambda(u_\lambda(t))|_H^2 + \frac{1}{2}\varphi_H(u_\lambda(t))\\ &\leq \varphi_H(0) + C(|f(t)|_{V^*}^{p'} + 1) \quad \text{for a.e. } t \in (0,T).\end{aligned}$$

Hence integrating this over $(0,t)$, we have

$$\begin{aligned}(13)\quad &\frac{\lambda}{2}|u_\lambda(t)|_H^2 + \tilde{\psi}^*(\partial_H\tilde{\psi}_\lambda(u_\lambda(t))) + \frac{\lambda}{2}|\partial_H\tilde{\psi}_\lambda(u_\lambda(t))|_H^2 + \frac{1}{2}\int_0^t \varphi_H(u_\lambda(\tau))d\tau\\ &\leq \frac{\lambda}{2}|u_0|_H^2 + \tilde{\psi}^*(\partial_H\tilde{\psi}_\lambda(u_0)) + \frac{\lambda}{2}|v_0|_H^2 + T\varphi_H(0) + C\left(\int_0^T |f(\tau)|_{V^*}^{p'}d\tau + 1\right).\end{aligned}$$

We here note that

$$\begin{aligned}\tilde{\psi}^*(\partial_H\tilde{\psi}_\lambda(u_0)) &= \left(\partial_H\tilde{\psi}_\lambda(u_0), J_\lambda u_0\right)_H - \tilde{\psi}(J_\lambda u_0)\\ &\leq |v_0|_H|u_0|_H.\end{aligned}$$

Thus we can derive (11) from (13).

Moreover from the definition of ψ^*, (11) implies

$$\begin{aligned}\psi^*(\partial_H\tilde{\psi}_\lambda(u_\lambda(t))) &= \sup_{v\in V}\left\{\langle\partial_H\tilde{\psi}_\lambda(u_\lambda(t)), v\rangle - \psi(v)\right\}\\ &\leq \sup_{v\in H}\left\{(\partial_H\tilde{\psi}_\lambda(u_\lambda(t)), v)_H - \tilde{\psi}(v)\right\}\\ &= \tilde{\psi}^*(\partial_H\tilde{\psi}_\lambda(u_\lambda(t))) \leq C \quad \forall t \in [0,T],\end{aligned}$$

which completes the proof. ∎

LEMMA 2.7 *There exists a constant C such that*

$$(14)\qquad \sup_{t\in[0,T]}|u_\lambda(t)|_V \leq C,$$

$$(15)\qquad \sup_{t\in[0,T]}|J_\lambda u_\lambda(t)|_V \leq C,$$

$$(16)\qquad \sup_{t\in[0,T]}|g_\lambda(t)|_{V^*} \leq C,$$

$$(17)\qquad \int_0^T\left|\frac{d}{dt}\partial_H\tilde{\psi}_\lambda(u_\lambda(t))\right|_{V^*}^2 dt \leq C.$$

PROOF OF LEMMA 2.7 First (14) follows immediately from (A1) and (5). Moreover by (A1), (A3) and (5), we can verify (15). Furthermore since $g_\lambda(t) \in \partial_H\varphi_H(u_\lambda(t)) \subset \partial_V\varphi(u_\lambda(t))$, (A2) and (5) imply (16). Finally we can derive (17) from (6), (16) and $(\mathrm{CP})_\lambda$. ∎

On account of a priori estimates stated above, we can take a sequence λ_n such that $\lambda_n \to +0$ as $n \to +\infty$ and the following lemmas hold true.

LEMMA 2.8 *There exists $u \in L^\infty(0,T;V)$ such that*

$$
\begin{aligned}
&(18) \qquad & u_{\lambda_n} &\to u & &\textit{weakly star in } L^\infty(0,T;V),\\
&(19) \qquad & J_{\lambda_n} u_{\lambda_n} &\to u & &\textit{weakly star in } L^\infty(0,T;V),\\
&(20) \qquad & \lambda_n \frac{du_{\lambda_n}}{dt} &\to 0 & &\textit{strongly in } L^2(0,T;H).
\end{aligned}
$$

PROOF OF LEMMA 2.8 By (14) and (15), we can derive (18) and the following

$$
(21) \qquad J_{\lambda_n} u_{\lambda_n} \to v \qquad \text{weakly star in } L^\infty(0,T;V)
$$

respectively for some $v \in L^\infty(0,T;V)$. Moreover it follows from (9) that

$$
|u_{\lambda_n}(t) - J_{\lambda_n} u_{\lambda_n}(t)|_H \;\leq\; \lambda_n |\partial_H \tilde{\psi}_{\lambda_n}(u_{\lambda_n}(t))|_H \leq \lambda_n C \to 0
$$

as $\lambda_n \to 0$. Hence by (18) and (21), we have $v = u$. Finally (6) implies (20). ∎

LEMMA 2.9 *There exist $g \in L^\infty(0,T;V^*)$ and $v \in W^{1,\infty}(0,T;V^*) \cap C_w([0,T];H)$ such that*

$$
\begin{aligned}
&(22) \qquad & g_{\lambda_n} &\to g & &\textit{weakly star in } L^\infty(0,T;V^*),\\
&(23) \qquad & \partial_H \tilde{\psi}_{\lambda_n}(u_{\lambda_n}(\cdot)) &\to v & &\textit{weakly star in } L^\infty(0,T;H),\\
&(24) \qquad & \partial_H \tilde{\psi}_{\lambda_n}(u_{\lambda_n}(\cdot)) &\to v & &\textit{weakly in } W^{1,2}(0,T;V^*).
\end{aligned}
$$

Moreover we have

$$
\frac{dv}{dt}(t) + g(t) = f(t) \quad \textit{in } V^*, \quad \textit{for a.e. } t \in (0,T).
$$

PROOF OF LEMMA 2.9 (9), (16) and (17) imply (22)-(24) immediately. Hence it follows from (20) that $dv/dt = f - g \in L^\infty(0,T;V^*)$. ∎

LEMMA 2.10 *We have*

$$
\begin{aligned}
&(25) \qquad & \partial_H \tilde{\psi}_{\lambda_n}(u_{\lambda_n}(\cdot)) &\to v & &\textit{strongly in } C([0,T];V^*),\\
&(26) \qquad & \partial_H \tilde{\psi}_{\lambda_n}(u_{\lambda_n}(t)) &\to v(t) & &\textit{weakly in } H \textit{ for all } t \in [0,T].
\end{aligned}
$$

PROOF OF LEMMA 2.10 Since (3) and (15) imply $\partial_H \tilde{\psi}_\lambda(u_\lambda(t)) \subset R(\partial_V \psi)$ for all $t \in [0,T]$, it follows from (A4), (9) and (12) that

$$
(27) \qquad \{\partial_H \tilde{\psi}_\lambda(u_\lambda(t))\}_{\lambda \in (0,1]} \text{ is precompact in } V^* \text{ for each } t \in [0,T].
$$

Moreover (17) implies that the function

$$
t \mapsto \partial_H \tilde{\psi}_\lambda(u_\lambda(t)) \text{ is equi-continuous in } C([0,T];V^*) \text{ for each } \lambda \in (0,1].
$$

Thus Ascoli-Arzela's lemma yields (25). Moreover (26) follows from (9) and (25). ∎

LEMMA 2.11 *We have*

$$v(t) \in \partial_H \tilde{\psi}(u(t)) \subset \partial_V \psi(u(t)) \quad \textit{for a.e. } t \in (0,T), \tag{28}$$

$$g(t) \in \partial_V \varphi(u(t)) \quad \textit{for a.e. } t \in (0,T). \tag{29}$$

PROOF OF LEMMA 2.11 For simplicity of notation, we drop n. It follows from (19) and (25) that

$$\lim_{\lambda \to 0} \int_0^T (\partial_H \tilde{\psi}_\lambda(u_\lambda(t)), J_\lambda u_\lambda(t))_H dt = \int_0^T (v(t), u(t))_H dt.$$

Hence by Lemma 1.2 of [4, Chap.II] and Proposition 1.1 of [12], it follows from (19) and (23) that $u(t) \in D(\partial_H \tilde{\psi})$ and $v(t) \in \partial_H \tilde{\psi}(u(t))$ for a.e. $t \in (0,T)$. Moreover by (3) and the fact that $u(t) \in D(\partial_H \tilde{\psi}) \cap V$ for a.e. $t \in (0,T)$, we get $\partial_H \tilde{\psi}(u(t)) \subset \partial_V \psi(u(t))$.

Now integrating $\langle g_\lambda(t), u_\lambda(t) \rangle$ over $(0,T)$, we have

$$\begin{aligned}
\int_0^T \langle g_\lambda(t), u_\lambda(t) \rangle dt &= \int_0^T \left\langle f(t) - \lambda \frac{du_\lambda}{dt}(t) - \frac{d}{dt} \partial_H \tilde{\psi}_\lambda(u_\lambda(t)), u_\lambda(t) \right\rangle dt \\
&= \int_0^T \left\langle f(t) - \lambda \frac{du_\lambda}{dt}(t), u_\lambda(t) \right\rangle dt \\
&\quad - \tilde{\psi}^*(\partial_H \tilde{\psi}_\lambda(u_\lambda(T))) + \tilde{\psi}^*(\partial_H \tilde{\psi}_\lambda(u_0)) \\
&\quad - \frac{\lambda}{2} |\partial_H \tilde{\psi}_\lambda(u_\lambda(T))|_H^2 + \frac{\lambda}{2} |\partial_H \tilde{\psi}_\lambda(u_0)|_H^2.
\end{aligned}$$

Now it follows from (18) and (20) that

$$\int_0^T \left\langle f(t) - \lambda \frac{du_\lambda}{dt}(t), u_\lambda(t) \right\rangle dt \to \int_0^T \langle f(t), u(t) \rangle \, dt.$$

On the other hand, since $\tilde{\psi}^* \in \Phi(H)$, (26) yields

$$\liminf_{\lambda \to 0} \tilde{\psi}^*(\partial_H \tilde{\psi}_\lambda(u_\lambda(T))) \geq \tilde{\psi}^*(v(T)) \geq \psi^*(v(T)).$$

Moreover we see

$$\begin{aligned}
\lim_{\lambda \to 0} \tilde{\psi}^*(\partial_H \tilde{\psi}_\lambda(u_0)) &= \lim_{\lambda \to 0} (\partial_H \tilde{\psi}_\lambda(u_0), J_\lambda u_0)_H - \lim_{\lambda \to 0} \tilde{\psi}(J_\lambda u_0) \\
&= (v_0, u_0)_H - \tilde{\psi}(u_0) \\
&= \langle v_0, u_0 \rangle - \psi(u_0) = \psi^*(v_0).
\end{aligned}$$

Therefore combining these inequalities, we have

$$\begin{aligned}
\limsup_{\lambda \to 0} \int_0^T \langle g_\lambda(t), u_\lambda(t) \rangle dt &\leq \int_0^T \langle f(t), u(t) \rangle \, dt - \psi^*(v(T)) + \psi^*(v_0) \\
&= \int_0^T \left\langle f(t) - \frac{dv}{dt}(t), u(t) \right\rangle dt = \int_0^T \langle g(t), u(t) \rangle \, dt.
\end{aligned}$$

Consequently by (18) and (22), we can deduce that $g(t) \in \partial_V \varphi(u(t))$ for a.e. $t \in (0,T)$.

∎

Finally we claim that $v(+0) = v_0$ in V^*. Indeed, we get by (17) and (25),

$$\begin{aligned}|v(t) - v_0|_{V^*} &= \lim_{\lambda_n \to 0} |\partial_H \tilde{\psi}_{\lambda_n}(u_{\lambda_n}(t)) - \partial_H \tilde{\psi}_{\lambda_n}(u_0)|_{V^*} \\ &\leq \lim_{\lambda_n \to 0} \left(\int_0^t \left| \frac{d}{d\tau} \partial_H \tilde{\psi}_{\lambda_n}(u_{\lambda_n}(\tau)) \right|_{V^*}^{p'} d\tau \right)^{1/p'} t^{1/p} \\ &\leq C^{1/p'} t^{1/p},\end{aligned}$$

which implies $v(t) \to v_0$ strongly in V^* as $t \to +0$. Hence (u, v) becomes a strong solution of (CP) on $[0, T]$, which completes the proof. ∎

In order to discuss the smoothing effect of (CP), we establish the following theorem.

THEOREM 2.12 *Suppose that* (A1), (A3) *and the following* (A2)′ *and* (A4)′ *are all satisfied.*

(A2)′ *There exists a constant C_3 such that $|\xi|_{V^*}^{p'} \leq C_3\{\varphi(u) + 1\}$ for all $[u, \xi] \in \partial_V \varphi$.*
(A4)′ *For any $r > 0$, the set $\{v \in R(\partial_V \psi); \psi^*(v) \leq r\}$ is precompact in V^*.*

Then for all $f \in L^{p'}(0, T; V^*)$, if $v_0 \in V^*$ satisfies the following:

$$(30) \qquad \begin{cases} \exists v_{0,n} \in (\partial_H \tilde{\psi})^\circ(D(\varphi) \cap D(\partial_H \tilde{\psi})); \\ v_{0,n} \to v_0 \text{ strongly in } V^*, \quad \psi^*(v_{0,n}) \to \psi^*(v_0) \text{ as } n \to +\infty, \end{cases}$$

then (CP) *has a strong solution (u, v) on $[0, T]$ such that*

$$\begin{aligned}&u \in L^p(0, T; V), \quad v \in W^{1,p'}(0, T; V^*), \\ &\text{the function } t \mapsto \psi^*(v(t)) \in W^{1,1}(0, T), \quad g \in L^{p'}(0, T; V^*),\end{aligned}$$

where $g(t)$ denotes a section of $\partial_V \varphi(u(t))$ in (2).

REMARK 2.13 It is obvious that (A2)′ and (A4)′ imply (A2) and (A4) respectively.

PROOF OF THEOREM 2.12 Let (f_n) be a sequence in $C^1([0, T]; H)$ such that $f_n \to f$ strongly in $L^{p'}(0, T; V^*)$ as $n \to +\infty$, and consider

$$(\mathrm{CP})_n \qquad \begin{cases} \dfrac{dv_n}{dt}(t) + g_n(t) = f_n(t) \ \text{ in } V^*, \ 0 < t < T, \\ v_n(t) \in \partial_V \psi(u_n(t)), \ g_n(t) \in \partial_V \varphi(u_n(t)), \ v_n(0) = v_{0,n}. \end{cases}$$

Then the existence of a strong solution (u_n, v_n) of $(\mathrm{CP})_n$ on $[0, T]$ is assured by Theorem 2.2. Hence multiplying the first equation in $(\mathrm{CP})_n$ by $u_n(t)$, just as in the proof of Lemma 2.6, we have

$$(31) \qquad \frac{d}{dt}\psi^*(v_n(t)) + \frac{1}{2}\varphi(u_n(t)) \leq \varphi(0) + C(|f_n(t)|_{V^*}^{p'} + 1) \ \text{ for a.e. } t \in (0, T).$$

Thus we can derive the following estimates.

LEMMA 2.14 *There exists a constant C such that*

$$\sup_{t\in[0,T]} \psi^*(v_n(t)) \leq C, \tag{32}$$

$$\int_0^T \varphi(u_n(t))dt \leq C. \tag{33}$$

Moreover by (A1) and (A2)$'$, we have

LEMMA 2.15 *There exists a constant C such that*

$$\int_0^T |u_n(t)|_V^p dt \leq C, \tag{34}$$

$$\int_0^T |g_n(t)|_{V^*}^{p'} dt \leq C. \tag{35}$$

Consequently by $(\mathrm{CP})_n$, we have

LEMMA 2.16 *There exists a constant C such that*

$$\int_0^T \left|\frac{dv_n}{dt}(t)\right|_{V^*}^{p'} dt \leq C. \tag{36}$$

From a priori estimates described above, just as in the proof of Lemmas 2.8-2.10, we can take a subsequence (n_k) of (n) and derive the following convergences.

LEMMA 2.17 *There exist $u \in L^p(0,T;V)$, $v \in W^{1,p'}(0,T;V^*)$ and $g \in L^{p'}(0,T;V^*)$ such that*

$$u_{n_k} \to u \quad \text{weakly in } L^p(0,T;V), \tag{37}$$

$$v_{n_k} \to v \quad \text{weakly in } W^{1,p'}(0,T;V^*), \tag{38}$$

$$g_{n_k} \to g \quad \text{weakly in } L^{p'}(0,T;V^*). \tag{39}$$

Hence we find that $dv/dt + g = f$ in $L^{p'}(0,T;V^*)$. Moreover by (A4)$'$, it follows from (32) and (36) that

$$v_{n_k} \to v \quad \text{strongly in } C([0,T];V^*). \tag{40}$$

Therefore we also have $v(t) \in \partial_V\psi(u(t))$ for a.e. $t \in (0,T)$.

In the rest of this proof, to simplify the notations, we drop k. Now multiplying $g_n(t)$ by $u_n(t)$ and integrating this over $(0,T)$, we get

$$\int_0^T \langle g_n(t), u_n(t)\rangle dt = \int_0^T \langle f_n(t), u_n(t)\rangle dt - \psi^*(v_n(T)) + \psi^*(v_{0,n}). \tag{41}$$

Now from the fact that $\psi^* \in \Phi(V^*)$, (40) yields

$$\liminf_{n\to+\infty} \psi^*(v_n(T)) \geq \psi^*(v(T)).$$

Hence since $\psi^*(v_{0,n}) \to \psi^*(v_0)$, we get by (37),

$$\begin{aligned}\limsup_{n\to+\infty}\int_0^T \langle g_n(t), u_n(t)\rangle &\le \int_0^T \langle f(t), u(t)\rangle dt - \psi^*(v(T)) + \psi^*(v_0)\\ &= \int_0^T \left\langle f(t) - \frac{dv}{dt}(t), u(t)\right\rangle dt.\end{aligned}$$

Thus Lemma 1.3 of [4, Chap.II] and Proposition 1.1 of [12] yield that $g(t) \in \partial_V\varphi(u(t))$ for a.e. $t \in (0,T)$. In much the same way as in the proof of Theorem 2.2, we can also verify that $v(+0) = v_0$ in V^*, which completes the proof. ∎

3 Initial-Boundary Value Problem for (DP)

To exemplify the applicability of the preceding abstract theory to PDEs, let us introduce the initial-boundary value problem (IBVP) for the doubly nonlinear parabolic equation (DP).

$$\text{(IBVP)}\begin{cases} \dfrac{\partial}{\partial t}|u|^{m-2}u(x,t) - \Delta_p u(x,t) = f(x,t) & (x,t) \in \Omega\times(0,T),\\ u(x,t) = 0 & (x,t) \in \partial\Omega\times(0,T),\\ |u|^{m-2}u(x,0) = v_0(x) & x \in \Omega,\end{cases}$$

where Ω denotes a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$.

In this section, we provide a couple of results on the existence of weak solutions to (IBVP). Before them, we give a definition of weak solutions as follows.

DEFINITION 3.1 *A pair of functions* $(u,v) : \Omega\times(0,T) \to \mathbf{R}^2$ *is said to be a weak solution of* (IBVP) *on* $[0,T]$ *if the following* (i)-(iv) *are all satisfied.*

(i) *The function* $t \mapsto v(\cdot,t)$ *is* $W^{-1,p'}(\Omega)$*-valued absolutely continuous on* $[0,T]$.

(ii) $u(\cdot,t) \in W_0^{1,p}(\Omega) \cap L^m(\Omega)$ *and* $v(\cdot,t) = |u|^{m-2}u(\cdot,t)$ *for a.e.* $t \in (0,T)$.

(iii) *The following identity holds true:*

$$\left\langle \frac{\partial v}{\partial t}(\cdot,t), \phi\right\rangle_{W_0^{1,p}(\Omega)} + \int_\Omega |\nabla u|^{p-2}\nabla u(x,t)\cdot\nabla\phi(x)dx = \langle f(\cdot,t), \phi\rangle_{W_0^{1,p}(\Omega)}$$

for a.e. $t \in (0,T)$ *and all* $\phi \in W_0^{1,p}(\Omega)$.

(iv) $v(\cdot,t) \to v_0$ *strongly in* $W^{-1,p'}(\Omega)$ *as* $t \to +0$.

The existence of weak solutions for (IBVP) was already studied by several authors. Raviart [17] proved the existence under some restriction on m by semi-descritization method. We can also find some results only for 1-dimensional case in [16], where Faedo-Galerkin's method is employed. Moreover Tsutsumi [20] and Ishige [11] employed the theory of quasi-linear parabolic equations developed in [15] to construct a weak solution of (IBVP) for the case where $f \equiv 0$.

In the rest of this paper, we put

$$H = L^2(\Omega), \quad V = W_0^{1,p}(\Omega)$$

with the norms $|\cdot|_V = |\nabla \cdot|_{L^p(\Omega)}$ and $|\cdot|_H = |\cdot|_{L^2(\Omega)}$ respectively. Then (1) holds true under the assumption that $p \geq 2N/(N+2)$. Moreover define

$$\begin{aligned}\psi_m(u) &:= \begin{cases} \dfrac{1}{m}\displaystyle\int_\Omega |u(x)|^m dx & \text{if } u \in V \cap L^m(\Omega), \\ +\infty & \text{if } u \in V \cap L^m(\Omega)^c, \end{cases} \\ \varphi_p(u) &:= \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx \quad \forall u \in V.\end{aligned}$$

Then it is easily seen that $\psi_m, \varphi_p \in \Phi(V)$ and $\partial_V \varphi_p(u)$ coincides with $-\Delta_p u$ with homogeneous Dirichlet boundary condition $u|_{\partial\Omega} = 0$ in the sense of distribution. Now just as in (4), we define an extension $\tilde{\psi}_m$ of ψ_m on H as follows.

$$\tilde{\psi}_m(u) := \begin{cases} \psi_m(u) & \text{if } u \in V, \\ +\infty & \text{if } u \in H \setminus V. \end{cases}$$

Then we can verify that $\tilde{\psi}_m \in \Phi(H)$ and $\tilde{\psi}_m|_V = \psi_m$ (see [2]); and it is well known that $\partial_H \tilde{\psi}_m(u)$ coincides with $|u|^{m-2}u$ in H for every $m \in (1, +\infty)$ (see e.g. [7]). On the other hand, for the case where $m \leq p^*$, V is continuously embedded in $L^m(\Omega)$; hence ψ_m is Fréchet differentiable in V and its Fréchet derivative $\partial_V \psi_m(u)$ coincides with $|u|^{m-2}u$ in $L^{m'}(\Omega)$ for every $u \in D(\partial_V \psi_m) = V$. Therefore we observe that every strong solution (u, v) of the following $(\mathrm{CP})^{p,m}$ becomes a weak solution of (IBVP) if $v(t) \in \partial_H \tilde{\psi}(u(t))$ for a.e. $t \in (0, T)$ or $m \leq p^*$.

$$(\mathrm{CP})^{p,m} \quad \begin{cases} \dfrac{dv}{dt}(t) + g(t) = f(t) \text{ in } V^*, \quad 0 < t < T, \\ v(t) = \partial_V \psi_m(u(t)), \ g(t) = \partial_V \varphi_p(u(t)), \\ v(0) = v_0. \end{cases}$$

Now employing Theorem 2.2, we can derive the following theorem.

THEOREM 3.2 *Suppose that* $p \in [2N/(N+2), +\infty)$ *and*

$$m \in \begin{cases} (1, +\infty) & \text{if } p > 2N/(N+2), \\ (1, p^*) & \text{if } p = 2N/(N+2), \end{cases}$$

where p^* *denotes the so-called Sobolev's critical exponent.*
Then for any $f \in W^{1,p'}(0,T; W^{-1,p'}(\Omega)) \cap L^2(0,T; L^2(\Omega))$ *and* $v_0 \in L^2(\Omega)$ *with* $u_0 := |v_0|^{m'-2}v_0 \in W_0^{1,p}(\Omega) \cap L^{2(m-1)}(\Omega)$, (IBVP) *has at least one weak solution* (u, v) *on* $[0, T]$ *satisfying*:

$u \in L^\infty(0,T; W_0^{1,p}(\Omega)) \cap C([0,T]; L^m(\Omega))$,
$v \in C_w([0,T]; L^2(\Omega)) \cap C([0,T]; L^{m'}(\Omega)) \cap W^{1,\infty}(0,T; W^{-1,p'}(\Omega))$,
the function $t \mapsto |v(\cdot,t)|^{m'}_{L^{m'}(\Omega)} \in W^{1,\infty}(0,T)$, $\quad \Delta_p u(\cdot,\cdot) \in L^\infty(0,T; W^{-1,p'}(\Omega))$.

PROOF OF THEOREM 3.2 For the case where $2N/(N+2) < p$, H is compactly embedded in V^*, which implies (A4) immediately. For the case where $m < p^*$, $L^{m'}(\Omega)$ is compactly embedded in V^*; hence observing

$$\psi_m^*(v) \;=\; \frac{1}{m'}\int_\Omega |v(x)|^{m'}dx \quad \forall v \in R(\partial_V\psi_m) \subset L^{m'}(\Omega),$$

we deduce that (A4)$'$ holds true.

From the definition of φ_p, it is obvious that (A1) is satisfied. Moreover we have

$$\begin{aligned}\langle \partial_V\varphi_p(u), v\rangle \;&=\; \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)dx \\ &\le\; |u|_V^{p-1}|v|_V \quad \forall u, v \in V,\end{aligned}$$

which implies (A2)$'$.

Moreover (A3) is derived from the following lemma, whose proof can be found in [7] or [3].

LEMMA 3.3 *Let $j \in \Phi(\mathbf{R})$ and define $\psi : H \to (-\infty, +\infty]$ as follows:*

$$\psi(u) \;:=\; \begin{cases} \displaystyle\int_\Omega j(u(x))dx & \text{if } u \in H \text{ and } j(u(\cdot)) \in L^1(\Omega), \\ +\infty & \text{otherwise.}\end{cases}$$

Then $\psi \in \Phi(H)$ and

$$f \in \partial_H\psi(u) \quad \text{if and only if} \quad f(x) \in \partial_{\mathbf{R}} j(u(x)) \text{ for a.e. } x \in \Omega.$$

Moreover the following inequality holds true.

$$\varphi_p(J_\lambda u) \;\le\; \varphi_p(u) \quad \forall u \in V,\ \forall \lambda > 0,$$

where J_λ denotes the resolvent of $\partial_H\psi$.

Therefore by Theorem 2.2, we conclude that $(\mathrm{CP})^{p,m}$ admits at least one strong solution on $[0, T]$. ∎

Moreover as for the case where $v_0 \in L^{m'}(\Omega)$, Theorem 2.12 implies the following result, where we can also observe the smoothing effect of (IBVP).

THEOREM 3.4 *Suppose that $p \in [2N/(N+2), +\infty)$ and $m \in (1, p^*)$. Then for all $f \in L^{p'}(0,T; W^{-1,p'}(\Omega))$ and $v_0 \in L^{m'}(\Omega)$, there exists at least one weak solution (u, v) of* (IBVP) *on $[0, T]$ satisfying:*

$$(42) \qquad \begin{cases} u \in L^p(0,T; W_0^{1,p}(\Omega)) \cap C([0,T]; L^m(\Omega)), \\ v \in C([0,T]; L^{m'}(\Omega)) \cap W^{1,p'}(0,T; W^{-1,p'}(\Omega)), \\ \text{the function } t \mapsto |v(\cdot,t)|_{L^{m'}(\Omega)}^{m'} \in W^{1,1}(0,T), \\ \Delta_p u(\cdot,\cdot) \in L^{p'}(0,T; W^{-1,p'}(\Omega)).\end{cases}$$

PROOF OF THEOREM 2.12 Let $v_0 \in L^{m'}(\Omega)$ and put $u_0 := |v_0|^{m'-2}v_0$. Then since $u_0 \in L^m(\Omega)$, we can take a sequence $(u_{0,n})$ in $C_0^\infty(\Omega)$ such that $u_{0,n} \to u_0$ strongly in $L^m(\Omega)$ as $n \to +\infty$. Moreover put $v_{0,n} := |u_{0,n}|^{m-2}u_{0,n} \in C_0(\Omega)$. Then $v_{0,n} \to v_0$ strongly in $L^{m'}(\Omega)$. The rest of proof can be derived as in the proof of Theorem 3.2. ∎

In general, it is difficult to derive the uniqueness of weak solutions for (IBVP) with a non-smooth initial data, e.g., $v_0 \in L^{m'}(\Omega)$. Now let $\mathcal{S}_{f,v_0}$ be the set of all strong solutions for $(\mathrm{CP})^{p,m}$ on $[0,T]$ with an initial data v_0 and a forcing term f; we are then going to construct a class of unique solutions to $(\mathrm{CP})^{p,m}$ as a subclass of $\mathcal{S}_{f,v_0}$.

For the case where $f \in \mathcal{X} := W^{1,p'}(0,T;V^*) \cap L^2(0,T;H)$, $v_0 \in D := \{v \in H; |v|^{m'-2}v \in V \cap L^{2(m-1)}(\Omega)\}$, define

$$\begin{aligned}\mathcal{S}^1_{f,v_0} := \Big\{&(u,v) \in \mathcal{S}_{f,v_0};\ \text{there exists a sequence } (u_\lambda) \text{ such that}\\ &u_\lambda \text{ is a strong solution of } (\mathrm{CP})_\lambda \text{ on } [0,T] \text{ with } u_0, \varphi \text{ and } \psi \text{ replaced}\\ &\text{by } |v_0|^{m'-2}v_0, \varphi_p \text{ and } \psi_m \text{ respectively, } u_\lambda \to u \text{ weakly star in}\\ &L^\infty(0,T;V) \text{ and } \partial_H\tilde{\psi}_{m,\lambda}(u_\lambda(\cdot)) \to v \text{ strongly in } C([0,T];V^*)\Big\};\end{aligned}$$

for the case where $f \in L^{p'}(0,T;V^*)$ and $v_0 \in L^{m'}(\Omega)$, define

$$\begin{aligned}\mathcal{S}^1_{f,v_0} := \Big\{&(u,v) \in \mathcal{S}_{f,v_0};\ \text{there exist } \{f_n\} \subset \mathcal{X} \text{ and } \{v_{0,n}\} \subset D \text{ such that } f_n \to f\\ &\text{strongly in } L^{p'}(0,T;V^*) \text{ and } v_{0,n} \to v_0 \text{ strongly in } L^{m'}(\Omega). \text{ Moreover}\\ &\text{there exists } (u_n,v_n) \in \mathcal{S}^1_{f_n,v_{0,n}} \text{ such that } u_n \to u \text{ weakly in } L^{p'}(0,T;V)\\ &\text{and } v_n \to v \text{ strongly in } C([0,T];V^*)\Big\}.\end{aligned}$$

Then we have

THEOREM 3.5 *Suppose that* $2N/(N+2) \le p$ *and* $m < p^*$. *Then for all* $f \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, *it follows that*

$$\begin{aligned}&\left|v^1(t) - v^2(t)\right|_{L^1(\Omega)} \le \left|v_0^1 - v_0^2\right|_{L^1(\Omega)} \quad \forall t \in [0,T],\\ &\quad \forall (u^1,v^1) \in \mathcal{S}^1_{f,v_0^1},\ \forall (u^2,v^2) \in \mathcal{S}^1_{f,v_0^2},\ \forall v_0^1, v_0^2 \in L^{m'}(\Omega).\end{aligned}$$

Hence $\mathcal{S}^1_{f,v_0}$ *has a unique element for every* $f \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ *and* $v_0 \in L^{m'}(\Omega)$.

PROOF OF THEOREM 3.5 We first suppose that $f \in \mathcal{X}$ and $v_0^i \in D$ $(i = 1,2)$. Now let $u_0^i := |v_0^i|^{m'-2}v_0^i \in V \cap L^{2(m-1)}(\Omega)$ and let $(u^i,v^i) \in \mathcal{S}^1_{f,v_0^i}$ for each $i = 1,2$. Then there exists a strong solution u_λ^i of the following $(\mathrm{CP})^i_\lambda$ on $[0,T]$:

$$(\mathrm{CP})^i_\lambda \quad \begin{cases} \lambda \dfrac{du_\lambda^i}{dt}(t) + \dfrac{d}{dt}\partial_H\tilde{\psi}_{m,\lambda}(u_\lambda^i(t)) + g_\lambda^i(t) = f \ \text{ in } H,\ 0 < t < T,\\ g_\lambda^i(t) = \partial_H\varphi_{p,H}(u_\lambda^i(t)),\quad u_\lambda^i(0) = u_0^i,\end{cases}$$

where $\varphi_{p,H}$ denotes an extension of φ_p on H given as in (4), such that

$$\begin{aligned} u_\lambda \to u \qquad & \text{weakly star in } L^\infty(0,T;V),\\ \partial_H\tilde{\psi}_{m,\lambda}(u_\lambda(\cdot)) \to v \qquad & \text{strongly in } C([0,T];V^*).\end{aligned}$$

For simplicity of notation, we write φ and ψ simply for φ_p and ψ_m respectively in the rest of this proof.

Now let $\eta_n \in C^1(\mathbf{R}) \cap W^{1,\infty}(\mathbf{R})$ be such that

$$\eta_n(s) \quad = \quad \begin{cases} 1 & \text{if } s \geq \dfrac{1}{n}, \\ 0 & \text{if } s = 0, \\ -1 & \text{if } s \leq -\dfrac{1}{n} \end{cases}$$

and

$$0 \leq \eta_n'(s) \leq 2n, \quad -1 \leq \eta_n(s) \leq 1 \quad \forall s \in \mathbf{R}.$$

Then we can easily verify that for any measurable function u,

$$\eta_n(u(\cdot)) \to \eta(u(\cdot)) \text{ strongly in } L^q(\Omega), \quad 1 \leq q < +\infty,$$

where $\eta(\cdot)$ is given by

$$\eta(s) \quad = \quad \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ -1 & \text{if } s < 0. \end{cases}$$

Now we see

$$0 \quad \leq \quad \left\langle g_\lambda^1(t) - g_\lambda^2(t), \eta_n(u_\lambda^1(\cdot,t) - u_\lambda^2(\cdot,t)) \right\rangle.$$

Hence multiplying $(\mathrm{CP})_\lambda^1 - (\mathrm{CP})_\lambda^2$ by $\eta_n(u_\lambda^1(\cdot,t) - u_\lambda^2(\cdot,t))$ and letting $n \to +\infty$, we find

$$\begin{aligned} &\lambda \frac{d}{dt} \int_\Omega |u_\lambda^1(x,t) - u_\lambda^2(x,t)| dx \\ &\qquad + \left(\frac{d}{dt} \left\{ \partial_H \tilde{\psi}_\lambda(u_\lambda^1(t)) - \partial_H \tilde{\psi}_\lambda(u_\lambda^2(t)) \right\}, \eta(u_\lambda^1(\cdot,t) - u_\lambda^2(\cdot,t)) \right)_H \leq 0, \end{aligned}$$

where we note that $\eta(s) \in \partial_{\mathbf{R}}|s|$ for all $s \in \mathbf{R}$. Moreover we observe that

$$\begin{aligned} \eta\left(u_\lambda^1(x,t) - u_\lambda^2(x,t)\right) &= \eta\left(J_\lambda(u_\lambda^1(t))(x) - J_\lambda(u_\lambda^2(t))(x)\right) \\ &= \eta\left(\partial_H \tilde{\psi}_\lambda(u_\lambda^1(t))(x) - \partial_H \tilde{\psi}_\lambda(u_\lambda^2(t))(x)\right). \end{aligned}$$

Then it follows that

$$\begin{aligned} &\lambda \frac{d}{dt} |u_\lambda^1(t) - u_\lambda^2(t)|_{L^1(\Omega)} \\ &\qquad + \frac{d}{dt} \left| \partial_H \tilde{\psi}_\lambda(u_\lambda^1(t)) - \partial_H \tilde{\psi}_\lambda(u_\lambda^2(t)) \right|_{L^1(\Omega)} \leq 0. \end{aligned}$$

Therefore integrating this over $(0,t)$, we get

$$\begin{aligned} &\lambda |u_\lambda^1(t) - u_\lambda^2(t)|_{L^1(\Omega)} + \left| \partial_H \tilde{\psi}_\lambda(u_\lambda^1(t)) - \partial_H \tilde{\psi}_\lambda(u_\lambda^2(t)) \right|_{L^1(\Omega)} \\ &\leq \lambda |u_0^1 - u_0^2|_{L^1(\Omega)} + \left| \partial_H \tilde{\psi}_\lambda(u_0^1) - \partial_H \tilde{\psi}_\lambda(u_0^2) \right|_{L^1(\Omega)} \quad \forall t \in [0,T]. \end{aligned}$$

Just as in the proof of Theorem 2.2, letting $\lambda \to +0$, we can also derive the following for $i = 1, 2$:

$$\begin{aligned} \lambda u_\lambda^i(t) &\to 0 && \text{strongly in } V \text{ for all } t \in [0,T], \\ \partial_H \tilde{\psi}_\lambda(u_\lambda^i(t)) &\to v^i(t) && \text{weakly in } H \text{ for all } t \in [0,T], \\ \partial_H \tilde{\psi}_\lambda(u_0^i) &\to v_0^i && \text{strongly in } H. \end{aligned}$$

Hence v^1 and v^2 satisfy

$$\left| v^1(t) - v^2(t) \right|_{L^1(\Omega)} \leq \left| v_0^1 - v_0^2 \right|_{L^1(\Omega)} \quad \forall t \in [0,T].$$

As for the case where $f \in L^{p'}(0,T;V^*)$ and $v_0^i \in L^{m'}(\Omega)$, let $(u^i, v^i) \in \mathcal{S}^1_{f,v_0^i}$ for $i = 1, 2$. Then there exist $f_n \in \mathcal{X}$ and $v_{0,n}^i \in D$ such that

$$\begin{aligned} f_n &\to f && \text{strongly in } L^{p'}(0,T;V^*), \\ v_{0,n}^i &\to v_0^i && \text{strongly in } L^{m'}(\Omega); \end{aligned}$$

moreover there exists $(u_n^i, v_n^i) \in \mathcal{S}^1_{f_n, v_{0,n}^i}$ such that

$$\begin{aligned} u_n^i &\to u^i && \text{weakly in } L^p(0,T;V), \\ v_n^i &\to v^i && \text{strongly in } C([0,T];V^*). \end{aligned}$$

Hence (u_n^i, v_n^i) is a strong solution of the following $(\mathrm{CP})_n^i$ on $[0,T]$ for $i = 1, 2$.

$$(\mathrm{CP})_n^i \quad \begin{cases} \dfrac{dv_n^i}{dt}(t) + g_n^i(t) = f_n(t) \ \text{ in } V^*, \quad 0 < t < T, \\ v_n^i(t) = \partial_V \psi(u_n^i(t)), \ g_n^i(t) = \partial_V \varphi(u_n^i(t)), \\ v_n^i(0) = v_{0,n}^i. \end{cases}$$

Moreover according to the last case, v_n^1 and v_n^2 satisfy

$$\left| v_n^1(t) - v_n^2(t) \right|_{L^1(\Omega)} \leq \left| v_{0,n}^1 - v_{0,n}^2 \right|_{L^1(\Omega)} \quad \forall t \in [0,T]. \tag{43}$$

Now as in the proof of Lemma 2.14, we get

$$\sup_{t \in [0,T]} |v_n^i(t)|_{L^{m'}(\Omega)} \leq C, \quad i = 1, 2,$$

which implies

$$v_n^i(t) \to v^i(t) \quad \text{weakly in } L^{m'}(\Omega) \ \ \forall t \in [0,T], \quad i = 1, 2. \tag{44}$$

Therefore combining (43) and (44), we conclude that

$$\left| v^1(t) - v^2(t) \right|_{L^1(\Omega)} \leq \left| v_0^1 - v_0^2 \right|_{L^1(\Omega)} \quad \forall t \in [0,T]. \ \blacksquare$$

4 Periodic Problem for (DP)

We next proceed to discuss the following periodic problem (PP) for the doubly nonlinear parabolic equation (DP):

$$
\text{(PP)}\begin{cases} \dfrac{\partial}{\partial t}|u|^{m-2}u(x,t) - \Delta_p u(x,t) = f(x,t) & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T), \\ |u|^{m-2}u(x,0) = |u|^{m-2}u(x,T) & x \in \Omega. \end{cases}
$$

As mentioned in the last section, several studies on the existence of solutions for (IBVP) are already done; however as for the periodic problem (PP), any studies have not appeared yet.

For the case where $m = 2$, one can construct a periodic solution by finding a fixed point of the Poincaré map $\mathcal{P}_f : u_0 \mapsto u(T)$ for the corresponding initial-boundary value problem: $u_t - \Delta_p u = f$, $u|_{\partial\Omega} = 0$, $u(0) = u_0$. Actually if u_0 is a fixed point of $\mathcal{P}_f$, then it follows that $u(0) = u_0 = u(T)$, which implies u becomes a periodic solution. To this end, we observe that the Poincaré map $\mathcal{P}_f$ is non-expansive in $L^2(\Omega)$; hence since $L^2(\Omega)$ is uniformly convex, Browder-Petryshyn's fixed point lemma ensures the existence of a unique fixed point of $\mathcal{P}_f$ (see [2] and [9]).

Moreover for the case where $p = 2$, the Poincare map $\mathcal{P}_f$ corresponding to (IBVP) with $p = 2$ is non-expansive in $H^{-1}(\Omega)$; hence we can also find a periodic solution in much the same way as in the case where $m = 2$.

However for the case where $m \neq 2$ and $p \neq 2$, it becomes more difficult to verify that the Poincaré map $\mathcal{P}_f : v_0 \mapsto v(T) = |u|^{m-2}u(T)$ is non-expansive in some Hilbert space. Moreover for non-smooth initial data, e.g., $v_0 \in L^{m'}(\Omega)$, it is difficult even to construct a unique weak solution for (IBVP).

In the last section, we have already constructed a class of unique weak solutions for (IBVP). So we define

$$\mathcal{P}_f : v_0 \mapsto v(T),$$

where v denotes a second component of a unique element of $\mathcal{S}^1_{f,v_0}$. Then $\mathcal{P}_f$ maps from $L^{m'}(\Omega)$ into itself; moreover it follows that

$$\left|\mathcal{P}_f v_0^1 - \mathcal{P}_f v_0^2\right|_{L^1(\Omega)} \leq \left|v_0^1 - v_0^2\right|_{L^1(\Omega)} \quad \forall v_0^1, v_0^2 \in L^{m'}(\Omega).$$

However since $L^1(\Omega)$ is no longer uniformly convex, Browder-Petryshyn's fixed point lemma does not work well in our case. To avoid this difficulty, we find a sequence $(v_{0,n})$ of quasi-fixed points of $\mathcal{P}_f$ and construct a periodic solution as a limit of the solutions (u_n, v_n) for (IBVP) with the initial data $v_{0,n}$.

THEOREM 4.1 *Suppose that $p \in [2N/(N+2), +\infty)$ and $m \in (1, p^*)$. Then for all $f \in L^\infty(0,T; W^{-1,p'}(\Omega))$,* (PP) *has at least one weak solution (u, v) on $[0,T]$ satisfying* (42).

PROOF OF THEOREM 4.1 In order to find quasi-fixed points of the Poincaré map $\mathcal{P}_f$, we employ the following lemma.

LEMMA 4.2 *Let X be a Banach space and let B be a closed convex subset of X. Let $T : B \to B$ be a non-expansive mapping in X, i.e., $T(B) \subset B$ and $|Tu - Tv|_X \leq |u - v|_X$ for all $u, v \in X$. If $T(B)$ is bounded in X, then there exists $u_n \in B$ such that $|Tu_n - u_n|_X \leq 1/n$ for each $n \in \mathbf{N}$.*

PROOF OF LEMMA 4.2 Let $M := \sup_{u \in B} |T(u)|_X < +\infty$. For each $n \in \mathbf{N}$, take $r_n \in (0, 1)$ such that

$$(1 - r_n)M \leq 1/n.$$

Then we see

$$|r_n T(u) - r_n T(v)|_X \leq r_n |u - v|_X \quad \forall u, v \in B.$$

Hence since $r_n T : B \to B$ becomes a strictly contractive mapping in X, there exists a fixed point $u_n \in B$ of $r_n T$, i.e., $r_n T(u_n) = u_n$. Therefore it follows that

$$\begin{aligned} |T(u_n) - u_n|_X &= |T(u_n) - r_n T(u_n)|_X \\ &= (1 - r_n)|T(u_n)|_X \\ &\leq (1 - r_n)M \leq 1/n. \blacksquare \end{aligned}$$

In Theorem 3.5, we have already seen that $\mathcal{P}_f$ is non-expansive in $L^1(\Omega)$; hence we next show that $\mathcal{P}_f$ maps from a bounded closed convex set into itself.

LEMMA 4.3 *Let $f \in L^\infty(0, T; V^*)$ and let $v_0 \in L^{m'}(\Omega)$. Then there exists a constant $R = R(T, p, m, N, |\Omega|, \|f\|_{L^\infty(0,T;V^*)})$ independent of $|v_0|_{L^{m'}(\Omega)}$ such that any strong solution (u, v) of $(\mathrm{CP})^{p,m}$ on $[0, T]$ satisfies the following estimate:*

$$|v(T)|_{L^{m'}(\Omega)} = |u(T)|^{m-1}_{L^m(\Omega)} \leq R.$$

PROOF OF LEMMA 4.3 Multiplying the first equation of $(\mathrm{CP})^{p,m}$ by $u(t)$, just as in (31), we find

$$\frac{1}{m'} \frac{d}{dt} |u(t)|^m_{L^m(\Omega)} + \frac{1}{2} |u(t)|^p_V \leq C |f(t)|^{p'}_{V^*} \text{ for a.e. } t \in (0, T).$$

Hence since $m < p^*$, Sobolev's inequality implies

$$\frac{d}{dt} |u(t)|^m_{L^m(\Omega)} + C |u(t)|^p_{L^m(\Omega)} \leq C_0 \quad \text{for a.e. } t \in (0, T), \tag{45}$$

where $C_0 := m' C \|f\|^{p'}_{L^\infty(0,T;V^*)}$. Then by improving the Ghidaglia-type differential inequality (see e.g. [19], [20]), we obtain the desired result. $\blacksquare$

Now set

$$B_R := \{v \in L^{m'}(\Omega); |v|_{L^{m'}(\Omega)} \leq R\}.$$

Then B_R is bounded, closed and convex in $L^1(\Omega)$. Moreover by Theorem 3.5 and Lemma 4.3, $\mathcal{P}_f$ maps from B_R into B_R. Therefore by Lemma 4.2, we can take a sequence $(v_{0,n})$ in $L^{m'}(\Omega)$ such that

$$|\mathcal{P}_f v_{0,n} - v_{0,n}|_{L^1(\Omega)} \le \frac{1}{n} \quad \forall n \in \mathbf{N}. \tag{46}$$

Hence to complete the proof, it suffices to show that $v_{0,n}$ converges to some element v_0, which becomes a fixed point of $\mathcal{P}_f$, i.e., $\mathcal{P}_f v_0 = v_0$. To this end, we remark that $L^{m'}(\Omega)$ is compactly embedded in V^*; then since $v_{0,n}$ and $v_n(T) := \mathcal{S}_f v_{0,n}$ belong to B_R, we can take a subsequence, which is denoted by the same letter n, and functions $v_0, w \in L^{m'}(\Omega)$ such that

$$\begin{aligned} v_{0,n} \to v_0 \quad & \text{strongly in } V^* \text{ and weakly in } L^{m'}(\Omega), \\ v_n(T) \to w \quad & \text{strongly in } V^* \text{ and weakly in } L^{m'}(\Omega). \end{aligned}$$

Now let $(u_n, v_n) \in \mathcal{S}^1_{f,v_{0,n}}$. Then repeating the same procedure as in the proof of Theorem 2.12, we can obtain the following convergences:

$$u_n \to u \quad \text{weakly in } L^p(0,T;V), \tag{47}$$

$$v_n \to v \quad \text{weakly in } W^{1,p'}(0,T;V^*), \tag{48}$$

$$v_n \to v \quad \text{strongly in } C([0,T];V^*), \tag{49}$$

$$v_n(t) \to v(t) \quad \text{weakly in } L^{m'}(\Omega) \ \text{ for all } t \in [0,T], \tag{50}$$

$$g_n \to g \quad \text{weakly in } L^{p'}(0,T;V^*), \tag{51}$$

where $g_n := f - dv_n/dt$. Hence we have $w = v(T)$ and $v(t) \in \partial_V \psi(u(t))$ for a.e $t \in (0,T)$. Moreover it follows from (47) and (49) that

$$\begin{aligned} & \int_0^T \int_\Omega |u_n(x,t) - u(x,t)|^m dxdt \\ \le\ & C \int_0^T \langle v_n(t) - v(t), u_n(t) - u(t) \rangle \, dt \to 0 \quad \text{as } n \to +\infty, \end{aligned}$$

which implies

$$u_n \to u \quad \text{strongly in } L^m(0,T;L^m(\Omega)). \tag{52}$$

Now set $I := \{t \in [0,T]; u_n(t) \to u(t) \ \text{ strongly in } L^m(\Omega)\}$ and let $\delta \in I$ be fixed. We then find

$$\begin{aligned} & \limsup_{n\to+\infty} \int_\delta^T \langle g_n(t), u_n(t) \rangle dt \\ =\ & \lim_{n\to+\infty} \int_\delta^T \langle f(t), u_n(t) \rangle dt - \liminf_{n\to+\infty} \frac{1}{m'} |u_n(T)|^m_{L^m(\Omega)} + \lim_{n\to+\infty} \frac{1}{m'} |u_n(\delta)|^m_{L^m(\Omega)} \\ \le\ & \int_\delta^T \langle f(t), u(t) \rangle dt - \frac{1}{m'} |u(T)|^m_{L^m(\Omega)} + \frac{1}{m'} |u(\delta)|^m_{L^m(\Omega)} \\ =\ & \int_\delta^T \left\langle f(t) - \frac{dv}{dt}(t), u(t) \right\rangle dt, \end{aligned}$$

which yields $g(t) = f(t) - dv(t)/dt = \partial_V \varphi_p(u(t))$ for a.e. $t \in (\delta, T)$. Hence since $|[0,T] \setminus I| = 0$, the arbitrariness of δ implies $g(t) = \partial_V \varphi_p(u(t))$ for a.e. $t \in (0,T)$. Moreover just as in the proof of Theorem 2.2, we can also derive that $v(+0) = v_0$ in V^* from (48) and (49).

Therefore (u, v) becomes a strong solution of $(\mathrm{CP})^{p,m}$ with an initial data v_0. Furthermore since $v_n(T) \to w = v(T)$ weakly in $L^{m'}(\Omega)$, we get by (46),

$$|v(T) - v_0|_{L^1(\Omega)} \leq \liminf_{n \to +\infty} |v_n(T) - v_{0,n}|_{L^1(\Omega)} \leq \lim_{n \to +\infty} \frac{1}{n} = 0,$$

which implies $v(T) = v_0$. Hence (u, v) is a weak solution of (PP) on $[0, T]$. ∎

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