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混合単調性をもつ写像に対する不動点定理と混合単調性をもたない写像に対する不動点定理

TOSHIKAZU WATANABE*

1. Introduction

Bhaskar and Lakshmikantham [2] obtained some coupled fixed point results for mixed monotone operators $F: X \times X \to X$ which satisfy a certain contractive type condition, where X is a partially ordered metric space.

Definition 1. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F: X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

If (X, d) is a metric space and $F: X \times X \to X$ is an operator, then, by definition, a coupled fixed point for F is a pair $(x, y) \in X \times X$ satisfying the system ;

$$\begin{cases} x = F(x, y) \\ y = F(y, x). \end{cases} \tag{1.1}$$

In order to consider this in the ordered set, for the mapping F we need the following mixed monotone property.

Definition 2. We say that a mapping F of X^n into X has mixed monotone property, if it satisfies the following, see [1,4]: for any $t_1, t_2, \ldots, t_n, \in X$,

$$\begin{cases} x_1, x_1' \in X, x_1 \succeq x_1', \Rightarrow F(x_1, t_2, t_3, \ldots, t_n) \succeq F(x_1', t_2 \ldots, t_n), \\ x_2, x_2' \in X, x_2 \succeq x_2', \Rightarrow F(t_1, x_2, t_3, \ldots, t_n) \succeq F(t_1, x_2', \ldots, t_n), \\ \cdots \\ x_n, x_n' \in X, x_n \succeq x_n', \Rightarrow F(t_1, t_2, \ldots, x_n) \succeq F(t_1, t_2, \ldots, x_n'), \end{cases}$$

Using this, we have several results [1].

Theorem 3. *Let $(X, d, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \to X$ be a mixed monotone mapping for which there exists a constant $k \in [0, 1)$ such that for each $x \leq u$, $y \geq v$,*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq k[d(x, u) + d(y, v)].$$

If there exist x_0, $y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, or $x_0 \geq F(x_0, y_0)$ and $y_0 \leq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

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*Presenting author.

However for the mapping $F : X \times X \to X$ there are some abstract concept without mixed monotone.

It is easy to see that the above coupled fixed point problem can be represented as a fixed point problem for the operator $T_F : Z \to Z$ defined by

$$T_F(x, y) = (F(x, y), F(y, x)),$$

where $Z := X \times X$. On the other hand, any solution (x, y) of the coupled fixed point problem with $x = y$ gives a fixed point for F, i.e., a solution of the equation $x = F(x, x)$.

Moreover if we consider two operators $F_1 : X \times X \to X$ and $F_2 : X \times X \to X$ and define $T : Z \to Z$ by

$$T(x, y) = (F_1(x, y), F_2(x, y))$$

where $Z := X \times X$. Then if $F_1(x, y) = x$ and $F_2(x, y) = y$, then this result represent ordinary fixed point theorem.

In this talk, according to the [4, 5, 6], we introduce several notions for the mapping $F : X \times X \to X$ without mixed monotone property and consider the coupled fixed point theorem. Moreover, we introduce these notion for the mapping $f : X \to X$ and consider the fixed point theorem. And as a our result, we give some application of the fixed point theorem.

2. Coupled fixed point theorem and fixed point theorem

Definition 4. (Samet and Vetro [6]) Let (X, d) be a metric space and $F : X \times X \to X$ be a given mapping. Let M be a nonempty subset of $X \times X$. We say that M is an F-invariant subset of $X \times X$ if, for all $x, y, z, w \in X$,

(i)$(x, y, z, w) \in M \Rightarrow (w, z, y, x) \in M$;
(ii)$(x, y, z, w) \in M \Rightarrow (F(x, y), F(y, x), F(z, w), F(w, z)) \in M$.

Theorem 5. *(Samet and Vetro [6]) Let (X, d) be a complete metric space, $F : X \times X \to X$ be a continuous mapping and M be a nonempty subset of X. We assume that*

(i) *M is F-invariant;*
(ii) *there exists $(x_0, y_0) \in X$ such that $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M$;*
(iii) *for all $(x, y, u, v) \in M$, we have*

$$\begin{aligned} d(F(x,y),F(u,v)) &\le \frac{\alpha}{2}[d(x,F(x,y)) + d(y,F(y,x))] \\ &+ \frac{\beta}{2}[d(u,F(u,v)) + d(v,F(v,u))] + \frac{\theta}{2}[d(x,F(u,v)) + d(y,F(v,u))] \\ &+ \frac{\gamma}{2}[d(u,F(x,y)) + d(v,F(y,x))] + \frac{\delta}{2}[d(x,u) + d(y,v)], \end{aligned}$$

where $\alpha, \beta, \theta, \gamma, \delta$ are nonnegative constants such that $\alpha+\beta+\theta+\gamma+\delta < 1$.

Then F has a coupled fixed point, i.e., there exists $(x, y) \in X \times X$ such that $F(x, y) = x$ and $F(y, x) = y$.

Let (X, d) be a metric space and M be a subset of X^4. We say that M satisfies the transitive property if, for all $x, y, z, w, a, b \in X$, $(x, y, z, w) \in M$ and $(z, w, a, b) \in M \Rightarrow (x, y, a, b) \in M$.

Theorem 6. *(Sintunavarat et al. [7]) Suppose that either*

(a) *F is continuous or*

(b) *if for any two sequences x_m, y_m with $(x_{m+1}, y_{m+1}, x_m, y_m) \in M$, $\{x_m\} \to x$, $\{y_m\} \to y$, for all $m \geq 1$, then $(x, y, x_m, y_m) \in M$ for all $m \geq 1$.*

If there exists $(x_0, y_0) \in X \times X$ such that $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M$ and M is an F-invariant set which satisfies the transitive property, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point.

Definition 7. Let (X, d) be a complete metric space endowed with a partial order $\preceq$. We say that

(i) $(X, d, \preceq)$ is nondecreasing-regular (nd-M-regular) if a nondecreasing sequence $\{x_n\} \subset X$ with $(x_n, x_{n+1}) \in M$ converges to x, then $(x_n, x) \in M$ for all n;

(ii) $(X, d, \preceq)$ is nonincreasing-regular (ni-M–regular) if a nonincreasing sequence $\{x_n\} \subset X$ with $(x_n, x_{n+1}) \in M$ converges to x, then $(x, x_n) \in M$ for all n.

Definition 8. (Sintunavarat et al. [7]) Let (X, d) be a metric space and $F : X \times X \to X$ be a given mapping and M be a subset of X^4. We say that M is an F-closed subset of X^4 if, for all $x, y, u, v \in X$, $(x, y, u, v) \in M \Rightarrow (F(x, y), F(y, x), F(u, v), F(v, u)) \in M$. Obviously, every F-invariant set is an F-closed set. In particular, $\emptyset$ and X are F-closed sets.

The definition of F-closed is obtained to the mapping $f : X \to X$.

Definition 9. Let (X, d) be a metric space and $f : X \to X$ be a given mapping and M be a subset of $X \times X$. We say that M is an f-closed subset of $X \times X$ if, for all $x, y \in X$, $(x, y) \in M \Rightarrow (F(x), F(y)) \in M$.

Then we have the following fixed point theorem.

Theorem 10. *Let (X, d) be a complete metric space, let $f : X \to X$ be a continuous mapping, and let M be a subset of $X \times X$. Assume that:*

(i) *M is f-closed;*

(ii) *there exists $x_0 \in X$ such that $(f(x_0), x_0) \in M$;*

(iii) *there exists $k \in [0, 1)$ such that for all $(x, y) \in M$, we have*

$$d(f(x), f(y)) \leq kd(x, y).$$

Then f has a fixed point x^* and $\{f^n(x)\}$ converges to x^*.

3. Application

As an application of Theorem 2.8, we consider the following fractional boundary value problems of cantilever beam type equations.

$$\begin{cases} D_{0+}^{\alpha} u(t) = f(t, u(t), D_{0+}^{\alpha-3} u(t), D_{0+} D_{0+}^{\alpha-3} u(t)), \\ \quad 0 < t < 1, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases} \tag{3.1}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative and f is a function of $[0, 1] \times \mathbb{R}$ into $\mathbb{R}$. Let $\alpha > 0$. The Riemann-Liouville fractional derivative of order α of a function u of $(0, \infty)$ into $\mathbb{R}$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t (t - s)^{n-\alpha-1} u(s) ds,$$

where $n = [\alpha] + 1$ ($[\alpha]$ denotes the integer part of α) and $\Gamma(\alpha)$ denotes the gamma function; see [3, 8].

We denote by $\mathbb{R}$ the set of all real numbers, N by natural numbers and $N_0 = N \cup \{0\}$. Let $AC[0,1]$ be the space of functions which are absolutely continuous on $[0,1]$,

$$AC^n[0,1] = \left\{ y : [0,1] \to \mathbb{R} \text{ and } D^{n-1}y(t) \in AC[0,1], D = \frac{d}{dt} \right\}.$$

First we have the following lemma, see Lemma 2.22 of [3]

Lemma 11. *Let $\alpha > 0$. If $u(t) \in AC^n[0,1]$ or $y(t) \in C^n[0,1]$, then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} y(t) = y(t).$$

Lemma 12. *Let $g \in C^n(0,1)$. Then the unique solution to problem $D^\alpha y(t) = g(t)$ together with the boundary conditions in (3.1) is*

$$u(t) = \int_0^t G(t,s)g(s)ds,$$

where

$$G(t,s) = \begin{cases} G_1(t,s) & (0 \le s \le t < 1), \\ G_2(t,s) & (0 \le t \le s < 1). \end{cases} \tag{3.2}$$

In this case

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)} \left((t-s)^{\alpha-1} + t^{\alpha-1}((4-\alpha)s - 1)(1-s)^{\alpha-4} \right.$$
$$\left. + t^{\alpha-2}(\alpha-1)(1-s)^{\alpha-4}s \right),$$

and

$$G_2(t,s) = \frac{1}{\Gamma(\alpha)} \left(t^{\alpha-1}(1-s)^{\alpha-4}((4-\alpha)s - 1) \right.$$
$$\left. + (\alpha-1)t^{\alpha-2}(1-s)^{\alpha-4}s \right).$$

Put $F_1(\alpha,t,s) = G_1(t,s)$. Then $F_1(\alpha,t,s)$ is continuous with respect to α. There exists t_0 and s_0 such that $F_1(3,t_0,s_0) < 0$, $F_1(4,t_0,s_0) > 0$. In fact $F_1(3,1/4,1/8) = -5/192$, $F_1(4,1/4,1/8) = 5/112$. Moreover there exists α^*, t^*, s^* with $3 < \alpha^* \le 4$ $t^*, s^* \in [0,1]$ such that $F_1(\alpha^*, t^*, s^*) = 0$. Let

$$N = \left\{ (\alpha,t,s) \mid G(t,s) < 0 \text{ or } D_{0+}^{\alpha-3}G(t,s) < 0 \text{ or } D_{0+}D_{0+}^{\alpha-3}G(t,s) < 0 \right\}.$$

Thus if $(\alpha,t,s) \notin N$, then for any $f \in C^+[0,1]$, $\int_0^1 G^\alpha(t,s)f(s)ds \ge 0$. The following argument we assume that;

(A0) $(\alpha,t,s) \notin N$.

Next we consider the following assumptions $(A1)$ and $(A2)$.

(A1) There exists $\omega \in \Omega$ such that for all $t \in I$ and for all $(a_1,a_2,a_3), (b_1,b_2,b_3) \in \mathbb{R}^3$, with $(a_1,a_2,a_3) \ll (b_1,b_2,b_3)$,

$$0 \le f(t,a_1,a_2,a_3) - f(t,b_1,b_2,b_3) \le \omega(a_1 - b_1) + \omega(a_2 - b_2) + \omega(a_3 - b_3). \tag{3.3}$$

(A2) There exist $\alpha, \beta, \gamma \in C(I, \mathbb{R})$ which are solutions of

$$\alpha(t) \leq \int_0^1 G(t,s) f(s, \alpha(s), \beta(s), \gamma(s)) ds, t \in I, \tag{3.4}$$

$$\beta(t) \leq \int_0^1 D_{0+}^{\alpha-3} f(s, \alpha(s), \beta(s), \gamma(s)) ds, t \in I, \tag{3.5}$$

$$\gamma(t) \leq \int_0^1 D_{0+} D_{0+}^{\alpha-3} f(s, \alpha(s), \beta(s), \gamma(s)) ds, t \in I, \tag{3.6}$$

We define the subset M of $C[0,1]$ by

$$M = \{(f, g) \in C[0,1] \times C[0,1] \mid f \geq g \text{ or } f \leq g\}.$$

Consider the natural partial order relation $\preceq$ on $X = C(I, R)$, that is,

$$u, v \in X, u \preceq v \Leftrightarrow u(t) \leq v(t) \text{ for all } t \in I.$$

It is well known that X is a complete metric space with respect to the metric

$$d(u, v) = max_{t \in I} |u(t) - v(t)| := \| u - v \|_\infty, u, v \in C(I, \mathbb{R}).$$

It is easy to show that $(X, d, \preceq)$ is nondecreasing-regular and nonincreasing-regular ($\uparrow\downarrow$-regular), and that every pair of elements in $X \times X$ has either a lower bound or an upper bound. Let $(X, d, \preceq)$ is an ordered complete metric space. Moreover in X^3 define the metric D by

$$D((x, y, z), (u, v, w)) = \frac{1}{3}(d(x, u) + d(y, v) + d(z, w)).$$

Also define the order $\ll$ in X^3 by

$$(x, y, z) \ll (u, v, w) \text{ iff } x \preceq u, y \preceq v, z \preceq w$$

Then $(X^3, D, \ll)$ is an ordered complete metric space.

The boundary problem (3.1) is equivalent to the following integral equation form.

$$\begin{cases} u(t) = \int_0^1 G(t,s) f(s, u(s), v(s), w(s)) ds, \\ v(t) = \int_0^1 D_{0+}^{\alpha-3} G(t,s) f(s, u(s), v(s), w(s)) ds, \\ w(t) = \int_0^1 D_{0+} D_{0+}^{\alpha-3} G(t,s) f(s, u(s), v(s), w(s)) ds, \end{cases}$$

where the green function G is given by (3.2) . We define the operator F_1, F_2 and F_3 by

$$\begin{cases} F_1(u(t), v(t), w(t)) = \int_0^1 G(t,s) f(s, u(s), v(s), w(s)) ds, \\ F_2(u(t), v(t), w(t)) = \int_0^1 D_{0+}^{\alpha-3} G(t,s) f(s, u(s), v(s), w(s)) ds, \\ F_3(u(t), v(t), w(t)) = \int_0^1 D_{0+} D_{0+}^{\alpha-3} G(t,s) f(s, u(s), v(s), w(s)) ds, \end{cases}$$

where $v(t) = D_{0+}^{\alpha-3} u(t)$ and $w(t) = D_{0+} D_{0+}^{\alpha-3} u(t)$. We define the operator $A : X^3 \to X^3$ by

$$\begin{aligned} &A((u(t), v(t), w(t))) \\ &= (F_1(u(t), v(t), w(t)), F_2(u(t), v(t), w(t)), F_3(u(t), v(t), w(t))). \end{aligned}$$

Then M is A-closed. In fact let $u \leq v$, we have $D_{0+}^{\alpha-3} u(t) \leq D_{0+}^{\alpha-3} v(t)$ and $D_{0+} D_{0+}^{\alpha-3} u(t) \leq D_{0+} D_{0+}^{\alpha-3} v(t)$. Thus

$$U = (u, D_{0+}^{\alpha-3} u, D_{0+} D_{0+}^{\alpha-3} u), V = (v, D_{0+}^{\alpha-3} v, D_{0+} D_{0+}^{\alpha-3} v) \in M.$$

Then by assumption $(A.1)$,

$$f(t,u(t),D_{0+}^{\alpha-3}u(t),D_{0+}D_{0+}^{\alpha-3}u(t))$$
$$\leq f(t,v(t),D_{0+}^{\alpha-3}v(t),D_{0+}D_{0+}^{\alpha-3}v(t)).$$

Then by assumption $(A.0)$, that is, for any $(\alpha,t,s)\notin N$, we have

$$F_1U(t)=\int_0^1 G^\alpha(t,s)f(t,u(t),D_{0+}^{\alpha-3}u(t),D_{0+}D_{0+}^{\alpha-3}u(t))ds$$
$$\leq \int_0^1 G^\alpha(t,s)f(t,v(t),D_{0+}^{\alpha-3}v(t),D_{0+}D_{0+}^{\alpha-3}v(t))ds=F_1v(t).$$

Also $F_2U\leq F_2V$ and $F_3U\leq F_3V$. Then we have

$$AU=(F_1U,F_2U,F_3U)\ll(F_1V,F_2V,F_3V)=AV.$$

We have the following:

Theorem 13. *Under the assumptions* $(A0)$, $(A1)$ *and* $(A2)$, *the fourth-order two-point boundary value problem (3.1) has a solution.*

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(Toshikazu Watanabe) School of Interdisciplinary Mathematical Sciences, Meiji University, 4-21-1 Nakano, Nakano-ku, Tokyo, Japan 164-8525

Email address: wa-toshi@mti.biglobe.ne.jp, twatana@edu.tuis.ac.jp