Title

MAXIMAL ATTRACTOR AND INERTIAL SET FOR
EGUCHI-OKI-MATSUMURA EQUATION
(Evolution Equations and Asymptotic Analysis of Solutions)

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MAXIMAL ATTRACTOR AND INERTIAL SET FOR EGUCHI-OKI-MATSUMURA EQUATION

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1. INTRODUCTION

This is a joint work with Naoto Tanaka (Fukuoka University) and Atsusi Tani (Keio University). We consider following system of equations which was proposed by Eguchi-Oki-Matsumura ([7]):

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta (-\Delta u + 2u + uv^2), \quad (x, t) \in Q_T \equiv \Omega \times (0, T), \\
\frac{\partial v}{\partial t} &= \beta \Delta v + \alpha v(a^2 - u^2 - b^2v^2), \quad (x, t) \in Q_T, \\
\frac{\partial u}{\partial n} &= 0, \quad \frac{\partial v}{\partial n} = 0, \quad (x, t) \in \Gamma_T \equiv \Gamma \times (0, T), \\
u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega,
\end{aligned}
\]

(1.1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with smooth boundary \( \partial \Omega \equiv \Gamma \) and \( T > 0 \). Here \( u(x, t) \) is the local concentration of the solute atoms, \( v(x, t) \) is the local degree of order, respectively. \( \alpha, \beta, a, b \) are positive constants and \( \frac{\partial}{\partial n} \) is exterior normal derivative to \( \Gamma \).

It is well known that phase separation is described by so-called Cahn-Hilliard equation which is fourth order parabolic type [5], while the order-disorder transition is described by Allen-Cahn equation [1]. The system (1.1) is a model of simultaneous order-disorder and phase separation in binary alloys. T. Eguchi, K. Oki and S. Matsumura derived the system (1.1) assuming that the order-disorder transformation is second order and that phase separation can not take place in the disorder state, but can in the ordered state.

In our previous work [11], it was proved that there exist a unique local and global solution to problem (1.1). Many authors studied the dynamics of equations describing phase transition (for example, [2], [3], [4], [10], [12]). In this talk we show the existence of a maximal attractor and of an inertial set to problem (1.1). The main theorems are as follows:

**Theorem 1.** Let \( H_\delta = \{(u, v) \in (H^1(\Omega))^2 \times L^2(\Omega); \|u\| = \|v\| = 1 \} \). For any \( \delta \geq 0 \), the semigroup \( S(t) \) associated with problem (1.1) possesses in \( H_\delta = \bigcup_{|u| \leq \delta} H_\delta \) a maximal attractor \( A_\delta \) that is connected.

**Theorem 2.** Let \( B_\delta \) be the absorbing set in \( (H^1(\Omega))^2 \times L^2(\Omega) \) and \( X_\delta = \bigcup_{t \geq t_0} S(t)B_\delta \). Then there exists an inertial set \( M_\delta \) for \( (S(t))_{t \geq 0}, X_\delta \) which has fractal dimension.

2. PRELIMINARIES

We shall summarize the results of [11]. First of all the existence theorem is as follows:
Theorem 3. For any \((u_0, v_0) \in (H^2(\Omega))^2\) satisfying the compatibility conditions \(\frac{\partial u_0}{\partial n} |_{\Gamma} = \frac{\partial v_0}{\partial n} |_{\Gamma} = 0\), problem (1.1) has a unique local solution \((u, v)\) defined on \(Q_{T'}\) for some \(T' \in (0, T)\) such that
\[
 u \in H^{4,1}(Q_{T'}) \cap C(0, T'; H^2(\Omega)),
\]
\[
 v \in L^2(0, T'; H^3(\Omega)) \cap H^1(0, T'; L^2(\Omega)) \cap C(0, T'; H^2(\Omega)).
\]
Here \(H^{4,1}(Q_T) = H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^4(\Omega))\).

Theorem 4. Under the same assumptions of Theorem 3, problem (1.1) admits a unique global solution \((u, v)\) on \(Q_T\) for any \(T > 0\).

Problem (1.1) is a gradient flow and it has the Lyapunov functional
\[
 J(u, v) = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + \frac{\beta}{2\alpha} |\nabla v|^2 - jv^2 + b\mathbf{7}v^4 + u^2 + tu^2v^2 \right) dx,
\]
which satisfies
\[
 \frac{d}{dt} J(u, v) + \int_{\Omega} (|\nabla K(u, v)|^2 + \frac{1}{\alpha} |v_t|^2) dx = 0,
\]
where \(K(u, v) \equiv -Au + 2u + uv^2\).

Lemma 5. If \((u, v)\) satisfies problem (1.1), then
\[
 \frac{1}{2} \|\nabla u\|^2 + \|u\|^2 + \frac{\beta}{2\alpha} \|\nabla v\|^2 + \frac{b^2}{8} \|v\|_{L^4}^4 + \|uv\|^2
\]
\[
 + \int_0^t ds \int_{\Omega} \left( |\nabla K(u, v)(s)|^2 + \frac{1}{\alpha} |v_t(s)|^2 \right) dx \leq J(u_0, v_0) + \frac{a^4}{2b^2} |\Omega|.
\]

Moreover we can obtain the boundedness of the norm \(\|v(t)\|_{L^\infty}\).

Lemma 6. The estimate
\[
 \sup_{t>0} \|v(t)\|_{L^\infty} \leq C \max \left\{ \|v_0\|_{L^\infty}, \sup_{t>0} \|v(t)\| \right\}
\]
is valid for the solution \((u, v)\) to problem (1.1).

3. The Maximal Attractor

Let \(H = (H^1(\Omega))' \times L^2(\Omega)\). We define semigroup \(S(t)\) associated to problem (1.1) by \((u(t), v(t)) = S(t)(u_0, v_0)\). Theorems 3 and 4 yield that \((u, v) = S(\cdot)(u_0, v_0) \in C(0, \infty; H)\), and that the mapping \((u_0, v_0) \mapsto (u, v)\) is a continuous operator from \(H\) to \(H\). For \(u \in (H^1(\Omega))'\) let \(Nu\) be the solution of boundary value problem
\[
 \begin{cases}
 -\Delta \psi = u - \bar{u}, & x \in \Omega, \\
 \frac{\partial \psi}{\partial n} = 0, & x \in \Gamma, \\
 \int_{\Omega} \psi(x) dx = 0
\end{cases}
\]
and put
\[
 \|u\|_{-1}^2 = \int_{\Omega} |\nabla \psi|^2 dx + |\Omega|u^2.
\]
To apply theorem I.1.1 in [13], it is necessary to show

Theorem 7. For any \(\delta \geq 0\), there exist absorbing sets in \(\mathcal{H}_s\) and in \((H^2(\Omega))^2 \cap \mathcal{H}_s\) for semigroup \(S(t)\) associated to problem (1.1).
Proof of Theorem 7. We first consider the existence of an absorbing set in \( H_\delta \). Multiplying the equation for \( u \) by \( \psi \) and the equation for \( v \) by \( v \) and integrating by parts respectively, we get
\[
\frac{1}{2} \frac{d}{dt} (\|u\|_{-1}^2 + \frac{2\overline{u}^2}{ab^2} \|v\|^2) + \|\nabla u\|^2 + \frac{2\beta\overline{u}^2}{ab^2} \|
abla v\|^2 + \|u\|^2 \\
+ (1 + \frac{2\overline{u}^2}{b^2}) \int_\Omega u^2 v^2 dx + \frac{3}{4} u^2 \|v\|_{L^4}^4 \leq (2 + \frac{a^4}{b^4}) \bar{u}^2 |\Omega|.
\]
And we use the inequalities
\[
\frac{\lambda_2}{2} \|u\|_{-1}^2 \leq \frac{1}{2} \|u\|^2 + \frac{\lambda_2}{2} \bar{u}^2 |\Omega|,
\]
\[
\frac{\lambda_2}{2} \frac{2\overline{u}^2}{ab^2} \|v\|^2 \leq \frac{\overline{u}^2}{4} |\Omega| + (\frac{\lambda_2}{ab^2})^2 \bar{u}^2 |\Omega|,
\]
where \( \lambda_2 \) is the least positive eigenvalue of the \(-\Delta\) with homogeneous Neumann boundary condition to obtain
\[
\frac{1}{2} \frac{d}{dt} (\|u\|_{-1}^2 + \frac{2\overline{u}^2}{ab^2} \|v\|^2) + \frac{\lambda_2}{2} (\|u\|_{-1}^2 + \frac{2\overline{u}^2}{ab^2} \|v\|^2) + \|\nabla u\|^2 + \frac{2\beta\overline{u}^2}{ab^2} \|
abla v\|^2 + \frac{1}{2} \|u\|^2 \\
+ (1 + \frac{2\overline{u}^2}{b^2}) \int_\Omega u^2 v^2 dx + \frac{1}{2} \bar{u}^2 \|v\|_{L^4}^4 \leq (2 + \frac{a^4}{b^4} + \frac{\lambda_2}{2} + (\frac{\lambda_2}{ab^2})^2) \bar{u}^2 |\Omega| \equiv C_1.
\]
And we have
\[
(3.2) \quad \frac{d}{dt} (\|u\|_{-1}^2 + \frac{2\overline{u}^2}{ab^2} \|v\|^2) + \lambda_2 (\|u\|_{-1}^2 + \frac{2\overline{u}^2}{ab^2} \|v\|^2) \leq C_1.
\]
Applying Gronwall's lemma to (3.2) we deduce for all
\[
\|u\|_{-1}^2 + \frac{2\overline{u}^2}{ab^2} \|v\|^2 \leq \left( \|u_0\|_{-1}^2 + \frac{2\overline{u}^2}{ab^2} \|v_0\|^2 \right) e^{-\lambda_2 t} + \frac{C_1}{\lambda_2} (1 - e^{-\lambda_2 t}).
\]
We have obtained an absorbing set in \( H_\delta \).

Next we show the existence of an absorbing set in \((H^2(\Omega))^2 \cap H_\delta \). Multiplying the equation for \( u \) by \( \Delta^2 u \) and integrating by parts yield
\[
(3.3) \quad \frac{1}{2} \frac{d}{dt} \|\Delta u\|^2 + \frac{1}{2} \|\Delta^2 u\|^2 + 2 \|\nabla \Delta u\|^2 \leq \frac{1}{2} \|\Delta (uv^2)\|^2.
\]
Multiplying the equation for \( v \) by \( \Delta^2 v \) and integrating by parts, we have
\[
\frac{1}{2} \frac{d}{dt} \|\Delta v\|^2 + \beta \|\nabla \Delta u\|^2 = -\alpha \int_\Omega \nabla [v(a^2 - u^2 - b^2 v^2)] \cdot \nabla \Delta u dx \\
\leq \frac{\beta}{2} \|\Delta v\|^2 + \frac{3\alpha^2}{2\beta} (a^4 \|\nabla v\|^2 + \|\nabla (uv^2)\|^2 + 9b^4 \|v^2 \nabla v\|^2).
\]
Using Lemmas 5 and 6, the right hand side of (3.3) leads to
\[
\|\Delta (uv^2)\|^2 = \|u v^2 \Delta u + 2u v \nabla \Delta u + 4v \nabla u \cdot \nabla v + 2u \|v\|^2 \|^2 \\
\leq C(\|u\|_{L^\infty}^2 \|\Delta u\|^2 + \|u\|_{L^\infty}^2 \|v\|_{L^\infty}^2 \|\Delta u\|^2 + \|u\|_{L^\infty}^2 \|\nabla u\|_{L^4}^2 \|\nabla v\|_{L^4}^4 + \|u\|_{L^\infty}^2 \|v\|_{L^4}^2 + \|u\|_{L^\infty}^2 \|v\|_{L^4}^4 \|\nabla v\|_{L^4}^4) \\
(3.5) \quad \leq C(\|\Delta u\|^2 + \|u\|_{L^\infty}^2 \|\Delta u\|^2 + \|u\|_{L^\infty}^2 \|\nabla v\|_{L^4}^4 + \|u\|_{L^\infty}^2 \|\nabla v\|_{L^4}^4)
\]
and
\[
\|\nabla (uv^2)\|^2 = \|u^2 \nabla v + 2uv \nabla u\|^2 \\
(3.6) \quad \leq C(\|u\|_{L^\infty}^2 \|\nabla v\|_{L^4}^4 + \|u\|_{L^\infty}^2 \|\nabla v\|_{L^4}^4)
\]
and
\[
\|\nabla (uv^2)\|^2 \leq \|u^2 \nabla v + 2uv \nabla u\|^2 \\
(3.6) \quad \leq C(\|u\|_{L^\infty}^2 \|\nabla v\|_{L^4}^4 + \|u\|_{L^\infty}^2 \|\nabla v\|_{L^4}^4).
\]
Using the inequalities (see, [13] p.161 and p.52)
\[ \|\nabla v\|_{L^4} \leq C\|\nabla v\|_{H^{1}}^{\frac{1}{4}}\|\nabla^2 v\|^{\frac{3}{4}}, \]
\[ \|u\|_{L^{\infty}} \leq C\|u\|_{H^{1}}^{\frac{1}{2}}\|u\|_{H^{2}}^{\frac{1}{2}}, \]
we find, for example,
\[ \|u\|_{L^{\infty}}^{2}\|\nabla v\|_{L^{4}}^{2} \leq C(\|u\|_{H^{1}}^{\frac{1}{2}}\|u\|_{H^{2}}^{\frac{1}{2}})^2(\|\nabla v\|_{H^{1}}^{\frac{1}{2}}\|\nabla^2 v\|^{\frac{3}{4}})^4 \]
\[ \leq C\|\Delta u\|\|\Delta v\|^{3} + C' \]
\[ \leq C(\|\Delta u\|^{4} + \|\Delta v\|^{4}) + C'. \]

After some similar calculation, we obtain that

\[ \frac{d}{dt}(\|\Delta u(t)\|^{2} + \beta\|\Delta v(t)\|^{2}) \leq C_{2}(\|\Delta u\|^{2} + \beta\|\Delta v\|^{2})(\|\Delta u\|^{2} + \beta\|\Delta v\|^{2}) + C_{3}. \]

Multiplying the equations for \( u \) by \( u \) and the equations for \( v \) by \( \Delta v \) and integrating by parts respectively, we have

\[ \frac{d}{dt}(\|u\|^{2} + \|\nabla v\|^{2}) + \beta\|\Delta v\|^{2} \leq C_{4}. \]

We note that \( X_{\delta} \) is bounded in \((C(\bar{\Omega}))^{2}\).

**Lemma 8.** The semigroup \( S(t) : X_{\delta} \rightarrow X_{\delta} \) is Lipschitz continuous, i.e.,

\[ \|(u_1 - u_2, v_1 - v_2)\|_{H^2} \leq \|(u_{01} - u_{02}, v_{01} - v_{02})\|_{H^2} e^{2\alpha t}, \]

where \((u_i, v_i)\) is the solutions of (1.1) with the initial conditions \((u_{0i}, v_{0i}), i = 1, 2\), and \(\|u, v\|_{H}^{2} = \|u\|_{-1}^{2} + \|v\|^{2}\).

**Proof of Lemma 8.** The difference of solution \((u_1 - u_2, v_1 - v_2)\) satisfies

\[ \left\{ \begin{array}{l}
\frac{\partial(u_1 - u_2)}{\partial t} = \Delta(-\Delta(u_1 - u_2) + 2(u_1 - u_2) + u_1 v_1^2 - u_2 v_2^2), \\
\frac{\partial(v_1 - v_2)}{\partial t} = \beta\Delta(v_1 - v_2) + \alpha(v_1 - v_2)(a^2 - u_1^2 - b^2 v_1^2) - \alpha_2 \{(u_1^2 - u_2^2) + b^2(v_1^2 - v_2^2)\}, \\
\frac{\partial(u_1 - u_2)}{\partial n} = \frac{\partial(v_1 - v_2)}{\partial n} = 0, \quad (x, t) \in \Gamma_T, \\
u_1(x, 0) - u_2(x, 0) = u_{01}(x) - u_{02}(x), \quad v_1(x, 0) - v_2(x, 0) = v_{01}(x) - v_{02}(x), \quad x \in \Omega,
\end{array} \right. \]

Let \( \psi \) be the solution of (3.1) with replacing \( u - \bar{u} \) by \( u_i - \bar{u}, i = 1, 2 \). Multiplying the first equation of (4.2) by \( \psi = \psi_1 - \psi_2 \) and multiplying the second equation of (4.2) by \( v_1 - v_2 \) and
integrating by parts respectively, we get
\[ \frac{1}{2} \frac{d}{dt} \| (u_1 - u_2, v_1 - v_2) \|^2_{H} + \| \nabla (u_1 - u_2) \|^2 + \| u_1 - u_2 \|^2 + \beta \| \nabla (v_1 - v_2) \|^2 \]
\[ + \int_{\Omega} v_1^2 (u_1 - u_2)^2 \, dx + \alpha \int_{\Omega} (v_1 - v_2)^2 (u_1^2 + b^2 v_1^2) \, dx \leq d \| v_1 - v_2 \|^2, \]
where \( d = \alpha a^2 + 2 \alpha b^2 M^2 + \frac{1}{4} M^2 (\alpha + 4) (1 + 4 \alpha) \) and \( M \) is a constant such that \( \| (u_i, v_i) \|_{C(\Omega \times [0, \infty))} \leq M \).
Applying Gronwall lemma leads to (4.1).

Moreover we find from (4.3)

**Corollary 9.**

\[ \int_{0}^{t} \| u_1 - u_2 \|^2_{H^1} e^{Ds} \, ds \leq \frac{1}{2} \| (u_{10} - u_{20}, v_{10} - v_{20}) \|^2_{H}, \]
where \( D \) is a constant.

**Proof of Corollary 9.** From (4.3), we have
\[ \frac{1}{2} \frac{d}{dt} \| (u_1 - u_2, v_1 - v_2) \|^2_{H} + \| u_1 - u_2 \|^2 \leq d \| v_1 - v_2 \|. \]
Multiplying (4.5) by \( e^{Ds} \) and integrating, we have
\[ \int_{0}^{t} \| u_1 - u_2 \|^2_{H^1} e^{Ds} \, ds \leq \frac{1}{2} \| (u_{10} - u_{20}, v_{10} - v_{20}) \|^2_{H} + \left( \frac{1}{2} D + d \right) \int_{0}^{t} \| (u_1 - u_2, v_1 - v_2) \|^2_{H} e^{Ds} \, ds. \]

By using (4.1),
\[ \int_{0}^{t} \| u_1 - u_2 \|^2_{H^1} e^{Ds} \, ds \leq \frac{1}{2} \| (u_{10} - u_{20}, v_{10} - v_{20}) \|^2_{H} \]
\[ + \left( \frac{1}{2} D + d \right) \| (u_{10} - u_{20}, v_{10} - v_{20}) \|^2_{H} \int_{0}^{t} e^{(D+2d)s} \, ds \]
\[ \leq \frac{1}{2} \left( 1 + e^{(D+2d)t} \right) \| (u_{10} - u_{20}, v_{10} - v_{20}) \|^2_{H}. \]

Next we shall show the squeezing property of \( S(t) \). We denote by \( \lambda_i, (0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_l \leq \cdots) \) the eigenvalue of the operator \( -\Delta \) with homogeneous Neumann boundary conditions and \( u_i \) the corresponding eigenfunctions such that \( \| u_i \|_{L^2} = 1, i = 1, 2, \ldots \). It is well-known that \( \{ u_i \}_{i=1}^{\infty} \) are a complete orthogonal basis in \( L^2(\Omega) \). Let \( H_n = \text{span}\{ u_1, \ldots, u_n \} \) and the operator \( p_n : (H^1(\Omega))^\prime \rightarrow H_n \) be orthogonal projection and \( q_n = I - p_n \), where \( I \) is identity on \( (H^1(\Omega))^\prime \). Then it holds that
\[ \| \varphi \|^2_{-1} \leq \frac{1}{\lambda_{n+1}} \| \varphi \|^2 \leq \frac{1}{\lambda_{n+1}^2} \| \nabla \varphi \|^2 \]
for any \( \varphi \in q_n((H^1(\Omega))^\prime) \). Furthermore we define the corresponding product projection \( P_n(u, v) \), \( Q_n(u, v) \) on \( H \) such that \( P_n(u, v) = (p_n u, p_n v), Q_n = I - P_n \).

**Theorem 10.** Semigroup \( S(t) \) for problem (1.1) possesses the squeezing property, i.e., for any \( t^* > 0 \) there exists number \( n_0 = n_0(t^*) \) such that for any \( \Psi_1 = (u_1, v_1), \Psi_2 = (u_2, v_2) \in X_\delta \) satisfying that if
\[ \| p_{n_0}(S(t^*)\Psi_1 - S(t^*)\Psi_2) \|_H \leq \| (I - p_{n_0})(S(t^*)\Psi_1 - S(t^*)\Psi_2) \|_H, \]
then
\[ \| S(t^*)\Psi_1 - S(t^*)\Psi_2 \|_H \leq \frac{1}{8} \| \varphi_1 - \varphi_2 \|_H. \]
Proof of Theorem 10. We set \((U, V) \equiv Q_n(u_1 - u_2, v_1 - v_2)\). Operating the equation (4.2) by \(Q_n\), it holds that

\[
\begin{align*}
\frac{\partial U}{\partial t} &= \Delta (-\Delta U + 2U + q_n(u_1v_1^2 - u_2v_2^2)) \\
\frac{\partial V}{\partial t} &= \beta \Delta V + \alpha a^2 V - \alpha q_n(v_1(u_1^2 + b^2v_1^2) - v_2(u_2^2 + b^2v_2^2)).
\end{align*}
\]

Multiplying the first equation of (4.11) by \(NU \in q_n(H^2(\Omega))\) and the second equation of (4.11) by \(V\) and integrating, we have

\[
\frac{1}{2} \frac{d}{dt} \|U\|_{H}^2 + \|\nabla U\|^2 + 2\|U\|^2 + \beta \|\nabla V\|^2 \leq \|U\|^2 + \alpha (a^2 + 1) \|V\|^2 + \frac{1}{4} \int_{\Omega} (u_1v_1^2 - u_2v_2^2)^2 dx 
+ \frac{\alpha}{4} \int_{\Omega} [v_1(u_1^2 + b^2v_1^2) - v_2(u_2^2 + b^2v_2^2)]^2 dx.
\]

Using the inequalities

\[
\begin{align*}
\left\{ \lambda_{n+1}^2 + \lambda_{n+1} \right\} \|U\|_{-1}^2 &\leq \|\nabla U\|^2 + \|U\|^2 \\
\beta \lambda_{n+1} \|V\|^2 &\leq \beta \|\nabla V\|^2,
\end{align*}
\]

it yields that

\[
\frac{d}{dt} \|(U, V)\|_H^2 + \beta \|\nabla V\|^2 \leq C(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2),
\]

where \(D_1 = 2 \min\{1, \beta\}, D_2 = 2\alpha(a^2 + 1)\). Applying Gronwall’s lemma leads to

\[
||U(0)||_H^2 \leq ||U(t^*)||_H^2 e^{-D_1 t^*} + \frac{\epsilon}{2} (e^{-D_1 t^*} + e^{2dt^*}) + C_{\epsilon} \frac{e^{2dt^*}}{D_3 + 2d},
\]

where \(D = D_1 \lambda_{n+1} - D_2\). Here we use the inequality \(\|u_1 - u_2\|^2 \leq C\|u_1 - u_2\|_{-1} \|u_1 - u_2\|_{H^1}\).

From Lemma 8 and Corollary 9, it holds that

\[
||U(t)||_H^2 \leq ||U(t_0) - u_{20} , v_{10} - v_{20}||_H^2 \left\{ e^{-D_1 t} + \frac{\epsilon}{2} (e^{-D_1 t} + e^{2dt}) + C_{\epsilon} \frac{e^{2dt}}{D_3 + 2d} \right\}.
\]

Now assume that

\[
||q_{n0}(u_1 - u_2, v_1 - v_2)(t^*)||_H \leq ||Q_{n0}(u_1 - u_2, v_1 - v_2)(t^*)||_H
\]

for \(t^* > 0\), then by using (4.16)

\[
||u(t_0) - u_{20}, v_{10} - v_{20}||_H^2 \leq 2||Q_{n0}(u_1 - u_2, v_1 - v_2)(t^*)||_H^2 \left\{ e^{-D_1 t} + \frac{\epsilon}{2} (e^{-D_1 t} + e^{2dt}) + C_{\epsilon} \frac{e^{2dt}}{D_3 + 2d} \right\},
\]

where \(D_3 = D_1 \lambda_{n0+1} - D_2\). Taking \(\epsilon > 0\) so small that

\[
\epsilon (e^{-D_1 t} + e^{2dt}) \leq \frac{1}{128}
\]

and choosing a number \(n_0\) sufficiently large so as to satisfy

\[
2(e^{-D_1 t} + C_{\epsilon} \frac{e^{2dt}}{D_3 + 2d}) \leq \frac{1}{128},
\]
then we conclude

\begin{align}
(4.21) \quad \|(u_1 - u_2, v_1 - v_2)(t^*)\|_{H}^2 \leq \left(\frac{1}{8}\right)^2 \|(u_{10} - u_{20}, v_{10} - v_{20})\|_{L}^2.
\end{align}

Therefore the proof of Theorem 2 is completed if we apply Theorem 2.1 in [6].

REFERENCES


