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Automorphisms on the ring of symmetric functions and stable and dual stable Grothendieck polynomials

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The stable Grothendieck polynomials G_λ and the dual stable Grothendieck polynomials g_λ are certain families of inhomogeneous symmetric functions parametrized by interger partitions λ. They are certain K-theoretic deformations of the Schur functions and dual to each other via the Hall inner product.

Historically the stable Grothendieck polynomials (parametrized by permutations) were introduced by Fomin and Kirillov [FK96] as a stable limit of the Grothendieck polynomials of Lascoux–Schützenberger [LS82]. In [Buc02] Buch gave a combinatorial formula for the stable Grothendieck polynomials G_λ for partitions using so-called set-valued tableaux, and showed that their span $\bigoplus_{\lambda\in\mathcal{P}}\mathbb{Z}G_\lambda$ is a bialgebra and its certain quotient ring is isomorphic to the K-theory of the Grassmannian $\mathrm{Gr}=\mathrm{Gr}(k,\mathbb{C}^n)$.

The dual stable Grothendieck polynomials g_λ were introduced by Lam and Pylyavskyy [LP07] as generating functions of reverse plane partitions, and shown to be the dual basis for G_λ via the Hall inner product. They also showed there that g_λ represent the K-homology classes of ideal sheaves of the boundaries of Schubert varieties in the Grassmannians.

In this article we give the following properties of g_λ and G_λ:

(A) The linear map I given by

$$g_\lambda \mapsto \sum_{\mu\subset\lambda} g_\mu$$

is an algebra automorphism.

(B) The Pieri formulas for G_λ (resp. g_λ) can be written as alternating sums of joins (resp. meets) of the leading terms (i.e. the terms appearing in the Pieri formula for the Schur functions s_λ).

In Section 2 we explain that the ring automorphism in (A) is written as both

(a) the substitution $f(x)\mapsto f(1,x)$, (that is, $f(x_1,x_2,\cdots)\mapsto f(1,x_1,x_2,\cdots)$), and

(b) the map $H(1)^\perp$, where $H(1)=\sum_i h_i$,

where the linear map $F^\perp$ is the adjoint of the multiplication map $(F\cdot)$. The equivalence of two maps in (a) and (b) is previously known (more generally, $H(t)^\perp(f(x))=f(t,x)$ where $H(t)=\sum_i t^i h_i$). The key observation to show $I(f(x))=f(1,x)$ is that the substitution $f\mapsto f(1,0,0,\cdots)$ maps $g_{\lambda/\mu}$ to 1 for any skew shape λ/μ; then since I is a certain composition of this map and the coproduct on Λ it follows that $I=(f(x)\mapsto f(1,x))$.

In Section 3 we give an exposition for (B) without technical details of the proofs.

1 Stable and dual stable Grothendieck polynomials

For basic definitions for symmetric functions, see for instance [Mac95, Chapter I].

Let Λ $(=\Lambda(x)=\Lambda_K=\Lambda_K(x))$ be the ring of symmetric functions, namely the set of all symmetric formal power series of bounded degree in variable $x=(x_1,x_2,\dots)$ with coefficients in K. We omit the variable x when no confusion arise. Let $\widehat{\Lambda}$ be its completion, consisting of all symmetric formal power series (with possibly unbounded degree). Let $\mathcal{P}$ be the set of partitions. The Schur functions s_λ $(\lambda\in\mathcal{P})$ are a family of homogeneous symmetric functions satisfying $\Lambda = \bigoplus_{\lambda\in\mathcal{P}} Ks_\lambda$ and $\widehat{\Lambda} = \prod_{\lambda\in\mathcal{P}} Ks_\lambda$. The *Hall inner product* $(\,,)$ is a bilinear form on Λ for which $(s_\lambda, s_\mu) = \delta_{\lambda\mu}$. This is naturally extended to $(\,,)\colon \widehat{\Lambda}\times\Lambda \longrightarrow K$.

In [Buc02, Theorem 3.1] Buch gave a combinatorial description of the *stable Grothendieck polynomial* G_λ as a generating function of so-called *set-valued tableaux*. We do not review the detail here and just recall some of its properties: $G_\lambda \in \widehat{\Lambda}$ (although $G_\lambda \notin \Lambda$ if $\lambda \neq \varnothing$), G_λ is an infinite linear combination of $\{s_\mu\}_{\mu\in\mathcal{P}}$ whose lowest degree component is s_λ. Hence $\widehat{\Lambda} = \prod_{\lambda\in\mathcal{P}} KG_\lambda$, i.e. every element in $\widehat{\Lambda}$ is uniquely written as an infinite linear combination of G_λ. Moreover the span $\bigoplus_\lambda KG_\lambda$ $(\subset \widehat{\Lambda})$ is a bialgebra, in particular the expansion of the product

$$G_\mu G_\nu = \sum_\lambda c^\lambda_{\mu\nu} G_\lambda$$

and the coproduct

$$\Delta(G_\lambda) = \sum_{\mu,\nu} d^\lambda_{\mu\nu} G_\mu \otimes G_\nu$$

are finite.

Next we recall the dual stable Grothendieck polynomial $g_{\lambda/\mu}$. For a skew shape λ/μ, a *reverse plane partition* of shape λ/μ is a filling of the boxes in λ/μ with positive integers such that the numbers are weakly increasing in every row and column.

Definition 1.1 ([LP07])**.** For a skew shape λ/μ, the *dual stable Grothendieck polynomial* $g_{\lambda/\mu}$ is defined by

$$g_{\lambda/\mu} = \sum_T x^T, \tag{1}$$

summed over reverse plane partitions T of shape λ/μ, where $x^T = \prod_i x_i^{T(i)}$ where $T(i)$ is the number of columns of T that contain i.

When $\mu = \varnothing$ we write $g_\lambda = g_{\lambda/\varnothing}$. It is shown in [LP07] that $g_{\lambda/\mu} \in \Lambda$ and g_λ has the highest degree component s_λ and forms a basis of Λ that is dual to G_λ via the Hall inner product:

$$(G_\lambda, g_\mu) = \delta_{\lambda\mu}. \tag{2}$$

Hence the product (resp. coproduct) structure constants for $\{G_\lambda\}$ coincide with the coproduct (resp. product) structure constants for $\{g_\lambda\}$:

$$g_\mu g_\nu = \sum_\lambda d^\lambda_{\mu\nu} g_\lambda \qquad \text{and} \qquad \Delta(g_\lambda) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} g_\mu \otimes g_\nu.$$

2 On the automorphism

2.1 Hopf structure of Λ

The ring Λ is a self-dual Hopf algebra with a coproduct $\Delta\colon \Lambda = \Lambda(x) \longrightarrow \Lambda(x,y) \hookrightarrow \Lambda(x)\otimes\Lambda(y)\,;\ f(x)\mapsto f(x,y)$, a counit $\epsilon\colon \Lambda \longrightarrow K\,;\ f \mapsto f(0,0,\dots)$, i.e. $\epsilon(s_\lambda) = \delta_{\lambda\varnothing}$, and an antipode $S\colon \Lambda\longrightarrow\Lambda\,;\ s_\lambda \mapsto (-1)^{|\lambda|}s_{\lambda'}$. Here λ' denotes the transpose of $\lambda\in\mathcal{P}$.

For $F\in\widehat{\Lambda}$, we have linear maps

- $(F,-)\colon \Lambda\longrightarrow K\,;\ f\mapsto (F,f)$, and
- $F^\perp\colon \Lambda\longrightarrow\Lambda\,;\ f\mapsto \sum (F,f_1)f_2$

where we put $\Delta(f) = \sum f_1\otimes f_2$ for $f\in\Lambda$ by the Sweedler notation. It is known that the multiplication map $(F\cdot)$ and the map $F^\perp$ are adjoint, i.e. $(FG,f) = (G, F^\perp(f))$ for $\forall F, G\in\widehat{\Lambda}$ and $\forall f\in\Lambda$.

Note that

$$F^{\perp} = ((F,-)\otimes \mathrm{id})\circ\Delta = (\mathrm{id}\otimes(F,-))\circ\Delta \tag{3}$$

where the second equality is by cocommutativity. We also have

$$(F,-) = \epsilon\circ F^{\perp} \tag{4}$$

since $\epsilon\circ F^{\perp} = \epsilon\circ((F,-)\otimes\mathrm{id})\circ\Delta = ((F,-)\otimes\epsilon)\circ\Delta = (F,-)*\epsilon = (F,-)$. The following lemma is standard:

Lemma 2.1. *For $F, G \in \widehat{\Lambda}$,*

(1) $(FG,-) = (F,-)*(G,-)$ *where $*$ denotes the convolution product on* $\mathrm{Hom}(\Lambda, K)$.
(2) $(FG)^{\perp} = G^{\perp}\circ F^{\perp}$ $(= F^{\perp}\circ G^{\perp})$.

2.2 The maps $H(t)^{\perp}$ and $E(t)^{\perp}$

There are well-known generating functions

$$H(t) = \sum_{i\geq 0} t^i h_i, \qquad E(t) = \sum_{i\geq 0} t^i e_i$$

where $t\in K$ (hence $H(t), E(t)\in\widehat{\Lambda}$). Let

$$H^{\perp}(t) := H(t)^{\perp} = \sum_{i\geq 0} t^i h_i^{\perp}, \qquad E^{\perp}(t) := E(t)^{\perp} = \sum_{i\geq 0} t^i e_i^{\perp}.$$

It is known (see [Mac95, Chapter 1.5, Example 29]) that

$$H^{\perp}(t),\ E^{\perp}(t)\colon \Lambda \longrightarrow \Lambda \text{ are ring automorphisms,} \tag{5}$$

$$H^{\perp}(t)(f(x_1,x_2,\cdots)) = f(t,x_1,x_2,\cdots) \qquad \text{for } f\in\Lambda. \tag{6}$$

The proof of (5) was as follows: for $F\in\widehat{\Lambda}$, we can see that the map $F^{\perp}\colon \Lambda\longrightarrow\Lambda$ is an algebra automorphism if and only if $F(x,y) = F(x)F(y)$ and $F(0)=1$, and it is easy to see that $H(t)$ and $E(t)$ satisfy them.

To show (6), it then suffices to show it when $f = h_n$, which is straightforward.

From (5), (6) and (4) we have

$$(H(t),-),\ (E(t),-)\colon \Lambda\longrightarrow K \text{ are ring homomorphisms,} \tag{7}$$

$$(H(t), f) = f(t,0,0,\cdots). \tag{8}$$

Since $H(t)E(-t) = 1$, by Lemma 2.1 and the fact that the counit is the identity with respect to the convolution product we have

Lemma 2.2. (1) $(H(t),-)*(E(-t),-) = \epsilon$, *where $\epsilon\colon\Lambda\longrightarrow K$ is the counit.*
(2) $H(t)^{\perp}\circ E(-t)^{\perp} = \mathrm{id}_{\Lambda}$.

2.3 Descriptions of $H(t)$, $(H(t),-)$ and $H(t)^{\perp}$

Let $c(\lambda/\mu)$ denote the number of columns in the skew shape λ/μ.

Proposition 2.3. $(H(t), g_{\lambda/\mu}) = t^{c(\lambda/\mu)}$ *for any skew shape λ/μ.*

Proof. By (8) we have $(H(t), g_{\lambda/\mu}) = g_{\lambda/\mu}(t,0,0,\cdots)$. By (1), it is the generating function of reverse plane partitions on λ/μ filled with one alphabet 1. Clearly there is exactly one such filling, whose weight is $x_1^{c(\lambda/\mu)}$. Hence $g_{\lambda/\mu}(t,0,0,\cdots) = t^{c(\lambda/\mu)}$. □

Next we give another description of the map $I\colon g_\lambda \mapsto \sum_{\mu\subset\lambda} g_\mu$.

For a skew shape λ/μ and a totally ordered set X called *alphabets* (most commonly $\{1,2,3,\dots\}$), we shall denote by $\mathrm{RPP}(\lambda/\mu, X)$ the set of reverse plane partition of shape λ/μ where each box is filled with an element of X. The expression (1) of $g_{\lambda/\mu}$ as a generating function of reverse plane partitions implies

$$\Delta(g_{\lambda/\mu}) = \sum_{\mu\subset\nu\subset\lambda} g_{\lambda/\nu}\otimes g_{\nu/\mu}, \tag{9}$$

since we have a natural bijection between $\mathrm{RPP}(\lambda/\mu, \{1,2,\cdots,1',2',\dots\})$ and $\bigsqcup_{\mu\subset\nu\subset\lambda}\mathrm{RPP}(\nu/\mu,\{1,2,\cdots\})\times \mathrm{RPP}(\lambda/\nu,\{1',2',\cdots\})$ where $1<2<\cdots<1'<2'<\cdots$.

By (3) and Proposition 2.3, we apply $(H(t),-)\otimes \mathrm{id}$ and $\mathrm{id}\otimes(H(t),-)$ to (9) and obtain

Proposition 2.4. *The algebra automorphism $H(t)^\perp\colon \Lambda\longrightarrow\Lambda$ satisfies*

$$H(t)^\perp(g_{\lambda/\mu}) = \sum_{\mu\subset\nu\subset\lambda} t^{c(\lambda/\nu)} g_{\nu/\mu} = \sum_{\mu\subset\nu\subset\lambda} t^{c(\nu/\mu)} g_{\lambda/\nu} \tag{10}$$

for any $\mu\subset\lambda$.

In particular, setting $\mu=\varnothing$ and $t=1$ in (10), for any $\lambda\in\mathcal{P}$ we have

$$H^\perp(1)(g_\lambda) = \sum_{\nu\subset\lambda} g_\nu,$$

hence

$$I = H^\perp(1) = \big(f(x)\mapsto f(1,x)\big). \tag{11}$$

In particular (11) recovers that $I\colon\Lambda\longrightarrow\Lambda$ is a ring automorphism. Moreover, (10) and (11) imply

$$I(g_{\lambda/\mu}) = \sum_{\mu\subset\nu\subset\lambda} g_{\nu/\mu} = \sum_{\mu\subset\nu\subset\lambda} g_{\lambda/\nu}. \tag{12}$$

2.3.1 Dual map

Next we recall that $H^\perp(t)\colon\Lambda\longrightarrow\Lambda$ and $(H(t)\cdot)\colon\widehat{\Lambda}\longrightarrow\widehat{\Lambda}$ are adjoint. By (2) and $H(t)^\perp(g_\mu)=\sum_{\lambda\subset\mu} t^{c(\mu/\lambda)} g_\lambda$ (by setting $\mu=\varnothing$ in (10)) we have

$$H(t)G_\lambda = \sum_{\lambda\subset\mu} t^{c(\mu/\lambda)} G_\mu. \tag{13}$$

Setting $\lambda=\varnothing$ in (13) we get $H(t)=\sum_{\lambda\in\mathcal{P}} t^{c(\lambda)}G_\lambda$, and by plugging it into (13) we have

$$\Big(\sum_{\mu\in\mathcal{P}} t^{c(\mu)}G_\mu\Big)G_\lambda = \sum_{\lambda\subset\mu} t^{c(\mu/\lambda)} G_\mu. \tag{14}$$

Remark 2.5. Since $I=H^\perp(1)$ it follows that $I^*=(H(1)\cdot)=\big((\sum_\lambda G_\lambda)\,\cdot\,\big)$, and (14) specializes to

$$\Big(I^*(G_\lambda)=\Big)\quad (\sum_{\mu\in\mathcal{P}} G_\mu)G_\lambda = \sum_{\lambda\subset\mu} G_\mu \tag{15}$$

which appeared in [Buc02, Section 8].

2.4 Description of $E(t)$, $(E(t), -)$ and $E(t)^{\perp}$

In this section we give descriptions using G_λ and g_λ for the element $E(t)$ and maps $(E(t), -)$ and $E^{\perp}(t)$. Note that by $I = H^{\perp}(1)$ and $I^* = (H(1)\cdot)$ it follows that $I^{-1} = E^{\perp}(-1)$ and $(I^*)^{-1} = (E(-1)\cdot)$.

By a tour-de-force combinatorial argument we can prove

Proposition 2.6. *The ring homomorphism $(E(t), -)\colon \Lambda \longrightarrow K$ satisfies*

$$(E(t), g_{\lambda/\mu}) = \begin{cases} t^{c(\lambda/\mu)}(t+1)^{|\lambda/\mu|-c(\lambda/\mu)} & \text{if } \lambda/\mu \text{ is a vertical strip,} \\ 0 & \text{otherwise} \end{cases}$$

for any skew shape λ/μ. In particular, for any $\lambda \in \mathcal{P}$,

$$(E(t), g_\lambda) = \begin{cases} 1 & \text{if } \lambda = \varnothing, \\ t(t+1)^{n-1} & \text{if } \lambda = (1^n)\ (n \geq 1), \\ 0 & \text{otherwise.} \end{cases}$$

Later We give a sketch of the proof of Proposition 2.6, and beforehand give as its corollaries descriptions for $E(t)$ and $E(t)^{\perp}$.

Proposition 2.7. *The ring automorphism $E(t)^{\perp}\colon \Lambda \longrightarrow \Lambda$ satisfies*

$$\begin{aligned} E(t)^{\perp}(g_{\lambda/\mu}) &= \sum_{\substack{\mu\subset\nu\subset\lambda \\ \lambda/\nu:\ \text{vertical strip}}} t^{c(\lambda/\nu)}(t+1)^{|\lambda/\nu|-c(\lambda/\nu)} g_{\nu/\mu} \\ &= \sum_{\substack{\mu\subset\nu\subset\lambda \\ \nu/\mu:\ \text{vertical strip}}} t^{c(\nu/\mu)}(t+1)^{|\nu/\mu|-c(\nu/\mu)} g_{\lambda/\nu} \end{aligned}$$

for any skew shape λ/μ. In particular, for any $\lambda \in \mathcal{P}$,

$$\begin{aligned} E(t)^{\perp}(g_\lambda) &= \sum_{\substack{\nu\subset\lambda \\ \lambda/\nu:\ \text{vertical strip}}} t^{c(\lambda/\nu)}(t+1)^{|\lambda/\nu|-c(\lambda/\nu)} g_\nu \\ &= \begin{cases} g_\lambda + \sum_{k=1}^{l(\lambda)} t(t+1)^{k-1} g_{\lambda/(1^k)} & \text{if } \lambda \neq \varnothing, \\ g_\varnothing & \text{if } \lambda = \varnothing. \end{cases} \end{aligned} \tag{16}$$

Proof. Proved similarly to Proposition 2.4, with Proposition 2.6 in hand. □

Now we have a description of $E(-1)^{\perp} = I^{-1}$ by setting $t = -1$ in the proposition above.

Corollary 2.8. *The ring automorphism $E(-1)^{\perp} = I^{-1}\colon \Lambda \longrightarrow \Lambda$ satisfies*

$$I^{-1}(g_{\lambda/\mu}) = \sum_{\substack{\mu\subset\nu\subset\lambda \\ \lambda/\nu:\ \text{rook strip}}} (-1)^{|\lambda/\nu|} g_{\nu/\mu} = \sum_{\substack{\mu\subset\nu\subset\lambda \\ \nu/\mu:\ \text{rook strip}}} (-1)^{|\nu/\mu|} g_{\lambda/\mu}.$$

In particular, when $\mu = \varnothing$ we have

$$I^{-1}(g_\lambda) = \sum_{\lambda/\nu:\ \text{rook strip}} (-1)^{|\lambda/\nu|} g_\nu = \begin{cases} g_\lambda - g_{\lambda/(1)} & \text{if } \lambda \neq \varnothing, \\ 1 & \text{if } \lambda = \varnothing. \end{cases} \tag{17}$$

Since $E^{\perp}(t)$ and $(E(t)\cdot)$ are adjoint, by (16) and (2) we have the following:

Proposition 2.9. *The element* $E(t) = \sum_{i\geq 0} t^i e_i \in \widehat{\Lambda}$ *satisfies*

$$E(t)G_\lambda = \sum_{\mu/\lambda:\ \textit{vertical strip}} t^{c(\mu/\lambda)}(t+1)^{|\mu/\lambda|-c(\mu/\lambda)}G_\mu. \tag{18}$$

In particular, setting $\lambda = \varnothing$ *we have*

$$E(t) = 1 + \sum_{n\geq 1} t(t+1)^{n-1}G_{(1^n)},$$

and hence

$$\Big(1 + \sum_{n\geq 1} t(t+1)^{n-1}G_{(1^n)}\Big)G_\lambda = \sum_{\mu/\lambda:\ \textit{vertical strip}} t^{c(\mu/\lambda)}(t+1)^{|\mu/\lambda|-c(\mu/\lambda)}G_\mu. \tag{19}$$

2.5 Sketch of the proof of Proposition 2.6

We recall the *incidence algebras* (see [Sta12, Chapter 3.6] for details). Let $\mathrm{Int}(\mathcal{P}) = \{(\mu,\lambda) \in \mathcal{P}\times\mathcal{P} \mid \mu \subset \lambda\}$, consisting of all comparable (ordered) pairs in $\mathcal{P}$ (or equivalently all skew shapes, by identifying (μ,λ) with λ/μ). The *incidence algebra* $I(\mathcal{P}) = I(\mathcal{P}, K)$ is the algebra of all functions $f\colon \mathrm{Int}(\mathcal{P}) \longrightarrow K$ where multiplication is defined by the convolution

$$(fg)(\mu,\lambda) = \sum_{\mu\subset\nu\subset\lambda} f(\mu,\nu)g(\nu,\lambda). \tag{20}$$

Then $I(\mathcal{P},K)$ is an associative algebra with two-sided identity $\delta := ((\mu,\lambda) \mapsto \delta_{\mu\lambda})$.

A linear function $f\colon \Lambda \longrightarrow K$ can be considered as an element of $I(\mathcal{P},K)$ by setting $f(\mu,\lambda) = f(g_{\lambda/\mu})$. Then the convolution product $*$ on $\mathrm{Hom}(\Lambda,K)$ coincides with the multiplication on $I(\mathcal{P})$ due to (9), i.e. this inclusion $\mathrm{Hom}(\Lambda,K) \longrightarrow I(\mathcal{P})$ is as algebras. Note that the counit $\epsilon \in \mathrm{Hom}(\Lambda,K)$ is mapped to $\delta \in I(\mathcal{P})$.

Define $i_t, j_t \in I(\mathcal{P})$ by

$$i_t(\mu,\lambda) = t^{c(\lambda/\mu)}$$

and

$$j_t(\mu,\lambda) = \begin{cases} (-1)^{|\lambda/\mu|}t^{c(\lambda/\mu)}(t-1)^{|\lambda/\mu|-c(\lambda/\mu)} & \text{if } \lambda/\mu \text{ is a vertical strip,}\\ 0 & \text{otherwise.}\end{cases}$$

By Proposition 2.3 $(H(t),-) \in \mathrm{Hom}(\Lambda,K)$ corresponds to $i_t \in I(\mathcal{P})$. Since $(H(t),-) * (E(-t),-) = \epsilon$, it suffices to show that $i_t j_t = \delta$ in order to prove that $(E(-t),-)$ corresponds to j_t, whence Proposition 2.6 follows by replacing t with $-t$.

By the definitions of i_t and j_t and (20)

$$(i_t j_t)(\mu,\lambda) = \sum_{\substack{\mu\subset\nu\subset\lambda\\ \lambda/\nu:\ \text{vertical strip}}} t^{c(\nu/\mu)}(-1)^{|\lambda/\nu|}t^{c(\lambda/\nu)}(t-1)^{|\lambda/\nu|-c(\lambda/\nu)}. \tag{21}$$

Now it suffices to show that the value of the right-hand side of (21) is $\delta_{\mu\lambda}$, which is not hard.

3 On the Pieri rules for G_λ and g_λ

The (row) Pieri formula for G_λ was given by Lenart [Len00, Theorem 3.2]: for any partition $\lambda \in \mathcal{P}$ and integer $a \geq 0$,

$$G_{(a)}G_\lambda = \sum_{\mu/\lambda:\ \text{horizontal strip}} (-1)^{|\mu/\lambda|-a}\binom{r(\mu/\lambda)-1}{|\mu/\lambda|-a}G_\mu, \tag{22}$$

where $r(\mu/\lambda)$ denotes the number of the rows in the skew shape μ/λ. Namely,

$$c^{\mu}_{(a),\lambda} = (-1)^{|\mu/\lambda|-a}\binom{r(\mu/\lambda)-1}{|\mu/\lambda|-a}.$$

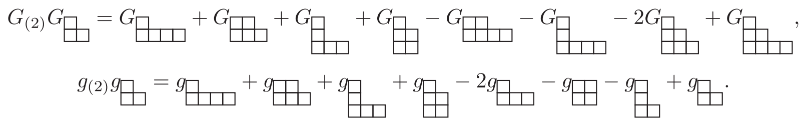
Subsequently, the (row) Pieri formula for g_λ is given in [Buc02, Corollary 7.1] (as a formula for $d^{\mu}_{\lambda,(a)}$, the coproduct structure constants for G_λ):

$$g_{(a)}g_\lambda = \sum_{\mu/\lambda:\ \text{horizontal strip}} (-1)^{a-|\mu/\lambda|}\binom{r(\lambda/\bar{\mu})}{a-|\mu/\lambda|} g_\mu, \tag{23}$$

where $\bar{\mu} = (\mu_2, \mu_3, \dots)$. Namely,

$$d^{\mu}_{(a),\lambda} = (-1)^{a-|\mu/\lambda|}\binom{r(\lambda/\bar{\mu})}{a-|\mu/\lambda|}.$$

Example 3.1. For $\lambda = (2, 1)$ and $a = 2$,



By the example above we can observe

$$\sum_{\nu\subset\mu} c^{\nu}_{(a),\lambda} = 1 \tag{24}$$

for each μ such that μ/λ is a horizontal strip of size $\geq a$, and

$$\sum_{\nu\supset\mu} d^{\nu}_{(a),\lambda} = 1 \tag{25}$$

for each μ such that μ/λ is a horizontal strip of size $\leq a$.

(24) and (25) can be shown through a tour de force argument, which we omit here.

Letting $\widetilde{G}_\kappa = \sum_{\kappa\subset\eta} G_\eta$ and $\widetilde{g}_\kappa = \sum_{\eta\subset\kappa} g_\eta$, we see (24) and (25) are equivalent to

$$\sum_{\mu} c^{\mu}_{(a),\lambda}\widetilde{G}_\mu = \sum_{\mu\supset\exists(\text{h.s.}/\lambda\text{ of size }a)} G_\mu, \tag{26}$$

$$\sum_{\mu} d^{\mu}_{(a),\lambda}\widetilde{g}_\mu = \sum_{\mu\subset\exists(\text{h.s.}/\lambda\text{ of size }a)} g_\mu. \tag{27}$$

Since $H(1)G_\lambda = \widetilde{G}_\lambda$ and $H(1)^\perp(g_\lambda) = \widetilde{g}_\lambda$ (shown in Section 2),

$$\text{since}\quad G_{(a)}G_\lambda = \sum_\mu c^{\mu}_{(a),\lambda}G_\mu \qquad \text{we have} \qquad G_{(a)}\widetilde{G}_\lambda = \sum_\mu c^{\mu}_{(a),\lambda}\widetilde{G}_\mu, \tag{28}$$

$$\text{since}\quad g_{(a)}g_\lambda = \sum_\mu d^{\mu}_{(a),\lambda}g_\mu \qquad \text{we have} \qquad \widetilde{g}_{(a)}\widetilde{g}_\lambda = \sum_\mu d^{\mu}_{(a),\lambda}\widetilde{g}_\mu. \tag{29}$$

Let $\lambda^{(1)}, \lambda^{(2)}, \cdots$ be the list of all horizontal strips over λ of size a. Combining (26) and (28), we have

Proposition 3.2. *We have*

$$G_{(a)}\widetilde{G}_\lambda = \sum_{\mu\supset\lambda^{(i)}\ \text{for}\ \exists i} G_\mu \tag{30}$$

$$= \sum_i \widetilde{G}_{\lambda^{(i)}} - \sum_{i<j}\widetilde{G}_{\lambda^{(i)}\cup\lambda^{(j)}} + \sum_{i<j<k}\widetilde{G}_{\lambda^{(i)}\cup\lambda^{(j)}\cup\lambda^{(k)}} - \cdots, \tag{31}$$

and

$$G_{(a)}G_\lambda = \sum_i G_{\lambda^{(i)}} - \sum_{i<j} G_{\lambda^{(i)}\cup\lambda^{(j)}} + \sum_{i<j<k} G_{\lambda^{(i)}\cup\lambda^{(j)}\cup\lambda^{(k)}} - \cdots \tag{32}$$

Note that the right-hand sides of (30) and (31) are equal by the Inclusion-Exclusion Principle, and the equivalence of (31) and (32) follows from that $H(1)G_\lambda = \widetilde{G}_\lambda$.

Similarly, by (27) and (29) we have

Proposition 3.3. *We have*

$$\widetilde{g}_{(a)}\widetilde{g}_\lambda = \sum_{\mu\subset\lambda^{(i)} \textit{ for } \exists i} g_\mu \tag{33}$$

$$= \sum_i \widetilde{g}_{\mu^{(i)}} - \sum_{i<j} \widetilde{g}_{\mu^{(i)}\cap\mu^{(j)}} + \sum_{i<j<k} \widetilde{g}_{\mu^{(i)}\cap\mu^{(j)}\cap\mu^{(k)}} - \cdots, \tag{34}$$

and

$$g_{(a)}g_\lambda = \sum_i g_{\lambda^{(i)}} - \sum_{i<j} g_{\lambda^{(i)}\cap\lambda^{(j)}} + \sum_{i<j<k} g_{\lambda^{(i)}\cap\lambda^{(j)}\cap\lambda^{(k)}} - \cdots. \tag{35}$$

Similarly, the right-hand sides of (33) and (34) are equal by the Inclusion-Exclusion Principle, and the equivalence of (34) and (35) follows from that $H(1)^\perp\colon g_\lambda \mapsto \widetilde{g}_\lambda$ is an algebra morphism.

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