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Real algebraic links in S^3 and simple branched covers

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1 Introduction

In contrast to the set of algebraic links, the real algebraic links in S^3 are not classified yet. In this paper we aim to give an overview of work on this open problem. There are various instructive surveys on closely related topics such as Milnor fibrations and more general aspects regarding the topology of isolated critical points of polynomials [15, 17, 38, 39]. What this survey aims to add to the discussion is on the one hand an accessible treatment of the work that has already been done on the classification of real algebraic links without assuming any previous knowledge on the topic (apart from basic knot theory) and without any results that are not directly relevant for this question and might complicate the theory. This means that we are going to ignore otherwise very worthwile work on higher dimensions [3] and non-isolated critical points [16, 19].

On the other hand we would like to point out connections to the theory of simple branched coverings of S^3 over itself and the role of fibred links in this setting, in particular work by Montesinos and Morton [30]. This connection has not been spelled out explicitly before and we hope that it inspires future work on the classification of real algebraic links.

The remainder of this paper is structured as follows. Section 2 gives the necessary definitions and background to state a conjecture by Benedetti and Shiota on the set of real algebraic links in S^3. Section 3 discusses the known constructions of real algebraic links, their similarities and limitations. In Section 4 we summarize results on branched coverings of S^3 over itself and how these could lead to more general constructions.

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2 Preliminaries

The central objects of this paper are polynomial maps whose vanishing sets contain knots and links in a very specific way. Before we give the definitions of algebraic and real algebraic links, we give the best-known example due to Brauner [11].

Example: Consider the polynomial $f:\mathbb{C}^2\to\mathbb{C}$ given by $f(u,v)=u^p-v^q$ with p, $q\in\mathbb{N}$. More precisely, we are interested in its vanishing set on a 3-sphere $S^3_\rho=\{(u,v)\in\mathbb{C}^2:|u|^2+|v|^2=\rho^2\}$ of a given radius ρ around the origin $(0,0)\in\mathbb{C}^2$. We can solve the polynomial equation or simply plot $f^{-1}(0)\cap S_\rho$ for different values of ρ, but with either method we find that it is $T_{p,q}$, the (p,q)-torus link. Remarkably, the link type does not depend on the radius ρ

at all. This is because the origin $(0,0)$ is the only point $(u,v) \in \mathbb{C}^2$ where f vanishes and where $\nabla f(u,v) = \lambda(u,v)$ for some real λ. In this sense, the (p,q)-torus link is a very dominating feature of the vanishing set of f.

We could now try to get rid of the property that the topology of $f^{-1}(0) \cap S^3_\rho$ does not depend on ρ. We could simply multiply f by another complex polynomial g that does not vanish at the origin. Then for large ρ the link $fg^{-1}(0) \cap S^3_\rho$ might be different from the (p,q)-torus link, but for small enough radii it is still $T_{p,q}$. The following definition captures the essential properties of f and fg.

Definition 2.1. *A link L is algebraic if there exists a polynomial* $f : \mathbb{C}^2 \to \mathbb{C}$ *such that*

- f *has an isolated singularity at the origin, i.e.,* $f((0,0)) = 0$, $\nabla f((0,0)) = (0,0)$ *and there is a neighbourhood B of* $(0,0) \in \mathbb{C}^2$ *such that* $(0,0)$ *is the only point in B where the rank of* ∇f *is not full,*
- $f^{-1}(0) \cap S^3_\rho = L$ *for all small enough radii* ρ.

The polynomial $u^p - v^q$ in the example has an isolated singularity at the origin. The fact that the singularity is isolated guarantees that the link type of the intersection $f^{-1}(0) \cap S^3_\rho$ does not depend on ρ if it is small enough. Hence the (p,q)-torus link is algebraic.

The algebraic links are completely classified and they turn out to be iterated cables of torus links, whose cabling coefficients satisfy an additional positivity condition. The precise result is not that important in this context. We recommend the excellent book by Eisenbud and Neumann [18] on the subject. For us it is for now enough to know that the algebraic links are classified and that they are a very small subset of the set of all links.

The original construction of the polynomials in the example is due to Brauner [11], who also gave a description of all algebraic links in terms of their cabling coefficients. Work by Burau [12, 13, 14] then established that the links that one obtains from this description are actually distinct. The term 'algebraic link' is due to Lê [27]. Another excellent source for many interesting results in the theory of algebraic links is Milnor's book [29], where he proves (among many other things) that if f has an isolated singularity at the origin, then the argument of f is a locally trivial fibration map over the circle, $\arg f : S^3_\rho \backslash f^{-1}(0) \to S^1$ for small enough radii ρ.

Definition 2.2. *A link L is called fibred if its complement* $S^3 \backslash L$ *is admits a locally trivial fibration over the circle* S^1 *and the closures of the fibres are compact surfaces (Seifert surfaces) that intersect precisely in their common boundary L.*

Even if we didn't know it from the classification of algebraic links, we would now know from Milnor's result that algebraic links are fibred.

In the last chapter of his seminal work [29], Milnor investigates properties of the real analogue of links of isolated singularities of complex plane curves, which were later termed *real algebraic links*. We use this name in the sense of Perron [33] as links of isolated critical points of polynomials $f : \mathbb{R}^4 \to \mathbb{R}^2$. This should not be confused with knotted algebraic varieties in $\mathbb{RP}^3$ as they were introduced by Viro [42], which are also called real algebraic links.

Definition 2.3. *A link L is real algebraic if there exists a polynomial* $f : \mathbb{R}^4 \to \mathbb{R}^2$ *such that*

- *f has an isolated singularity at the origin, i.e., $f((0,0,0,0)) = (0,0)$, $\nabla f((0,0,0,0)) = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$ and there is a neighbourhood B of the origin $(0,0,0,0)$ such that $(0,0,0,0)$ is the only point in B where the rank of ∇f is not full,*
- *$f^{-1}((0,0)) \cap S^3_\rho = L$ for all small enough radii ρ.*

From now on we will usually write 0 for the origin in $\mathbb{R}^4$, the origin in $\mathbb{R}^2$ and a matrix whose entries are all equal to zero. It should be clear from the context which of these we are referring to. In the literature one often also encounters the term "isolated critical point". In this scenario the two terms can be used interchangeably. A critical point is a point, where the gradient does not have full rank, while a singularity refers to a point where the gradient is the matrix with 0-entries. Via a change of coordinates, any critical point can be turned into a singularity. In Definition 2.3 the origin is a singularity, but is isolated in the sense of a critical point (it is the only point in a neighbourhood where the gradient does not have full rank).

Definition 2.3 differs from Definition 2.1 only in that we replaced every instance of $\mathbb{C}$ by $\mathbb{R}^2$. This might seem like a small change, but note that it enlargens the set of polynomials considerably. While every complex polynomial from Definition 2.1 can be written as a real polynomial as in Definition 2.3 by considering its real and imaginary parts, the converse is not true. It should be obvious that most real polynomials are not holomorphic for example. Hence, every algebraic link is real algebraic, but not vice versa. At first (around the time of Milnor's book), it was not clear at all if there were any real algebraic links that are not algebraic, but even though the real algebraic links are not classified yet, we now have plenty of examples for this (cf. Section 3).

Both Definition 2.1 and Definition 2.3 can be generalized to higher dimensions and many of the results (such as Milnor's fibration theorem) remain true.

One difference between the complex and the real polynomials is that in general the argument of a real polynomial as in Definition 2.3 ($\arg f : S^3_\rho \to S^1$) is not a fibration. However, Milnor established that the following is still true.

Theorem 2.4 (Milnor [29])**.** *If a link L is real algebraic, then L is fibred.*

According to Benedetti and Shiota this implication should be an equivalence.

Conjecture 2.5 (Benedetti-Shiota [5])**.** *A link L is real algebraic if and only if L is fibred.*

Akbulut and King showed that every link can be constructed around a *weakly isolated* singularity [2] meaning that the origin is allowed to lie on a component of the critical set of positive dimension as long as the origin is the only point in a neighbourhood where the critical set intersects the vanishing set $f^{-1}(0)$.

The idea of Benedetti and Shiota is based on a procedure of blow-ups and blow-downs starting from a polynomial f with the properties from Akbulut and King. At this point however, an appropriate blow-down technique is not made precise yet and Conjecture 2.5 remains conjectural.

Another result that might be interpreted as encouraging regarding Conjecture 2.5 is due to Kauffman and Neumann [26]. They showed that if you allow real analytic functions with so called *tame singularities* rather than only polynomials, then the links around these singularities are precisely the fibred links.

It should also be noted that real polynomial maps $\mathbb{R}^4 \to \mathbb{R}^2$ with isolated singularities are rare, in the sense that a 'generic' polynomial map between these spaces has a critical set of dimension 1.

3 Constructions of real algebraic links

In this section we review the different constructions of real algebraic links that have been developed over the years. If this list is incomplete, it is only due to the author's own limitations.

At the moment the set of links that are known to be real algebraic is still comparatively small. There are of course the algebraic links, but until Looijenga showed that the connected sum $K\#K$ of any fibred knot K with itself is real algebraic [28], it was not known if there would be any other examples. Looijenga in fact proved that every fibred knot K that is *odd*, i.e. can be realized as an invariant set under the inversion $i : S^3 \to S^3$, $i((u,v)) = (-u,-v)$ such that the fibration map is equivariant under i, is real algebraic.

The basic idea of his construction is as follows. Use the fibration to define a map from a neighbourhood U of S^3 to $\mathbb{R}^2 \cong \mathbb{C}$, which vanishes on K and whose argument on S^3 is equal to the fibration map. We can approximate this map by a polynomial map $\Phi = (\Phi_1, \Phi_2) : U \to \mathbb{R}^2$ up to their first derivatives, which because of the symmetry of the fibration we can construct such that it consists only of terms with odd degrees. Therefore the rescaled function $f = (f_1, f_2) : \mathbb{R}^4 \to \mathbb{R}^2$, $f_i(x) = \Phi(x/|x|)|x|^{\deg \Phi_i}$ is a polynomial. By construction this map satisfies the conditions from Definition 2.3. The singularity at the origin is isolated essentially because the fibration map does not have any critical points.

Pichon showed that if $f, g : \mathbb{C}^2 \to \mathbb{C}$ are holomorphic maps, then the function $f\overline{g}$ has an isolated singularity at the origin if and only if $L_{f\overline{g}} := f\overline{g}^{-1}(0) \cap S_\varepsilon^3$ is a fibred link [34]. Note that the link of this singularity $f\overline{g} \cap S_\varepsilon^3$ is the union of $L_f = f^{-1}(0) \cap S_\varepsilon^3$ and the $L_{\overline{g}} = L_g := g^{-1}(0) \cap S_\varepsilon^3$. In particular, if f and g are complex polynomials, then L_f and L_g are algebraic links. That in general the resulting link $L_{f\overline{g}}$ is not algebraic, can be seen for example from [25]. Functions of the form $f\overline{g}$ with holomorphic polynomials f and g have taken a prominent role in the construction of isolated critical points since A'Campo's first example in this context [1].

Perron [33] and Rudolph [36] independently constructed polynomials for the figure-eight knot 4_1, which is neither odd nor algebraic. This construction is also based on a particularly symmetric parametrisation. Perron considers a parametrisation of a braid that closes to 4_1 in terms of trigonometric functions. This leads him to a parametrisation of 4_1 itself in a 3-sphere of a given radius and consequentially to a polynomial $(f_1, f_2) : \mathbb{R}^4 \to \mathbb{R}^2$ whose nodal set on that 3-sphere is 4_1. The elegance of this lies in the simplicity of the resulting polynomial. It requires only 6 terms in total and is of degree 3, while the polynomials in Looijenga's construction for example can have arbitrarily large degree. Furthermore and like in Looijenga's construction, the symmetries of the original braid parametrisation mean that all terms of (f_1, f_2) have odd degree. The polynomial map with an isolated singularity is then obtained (exactly like in Looijenga's construction) by considering the radially rescaled function $|x|^3 \left(f_1(x/|x|), f_2(x/|x|)\right)$. Showing that the singularity at the origin is indeed isolated requires some tedious calculations. In this regard, Rudolph's construction is more satisfying as his functions can be written as a polynomial in complex variables u, v and the complex conjugate $\overline{v}$, which makes the proof of the isolated singularity significantly easier.

Perron's construction can be generalized to a much larger class of braids as we pointed out in [9].

Definition 3.1. *A braid B on s strands is called homogeneous if for every $i = 1,2,\ldots,s-1$ the generator σ_i appears in the word B if and only if σ_i^{-1} does not appear.*

We showed in [9] that all closures of squares of homogeneous braids are real algebraic. Like in Perron's construction we start with a braid parametrisation that involves trigonometric functions. Let B be a homogeneous braid and let s_C be the number of strands that form the component C of the closure of B. Hence the number of strands s of B equals $\sum_C s_C$. We require parametrisations of the form

$$\bigcup_C \bigcup_{j=1}^{s_C} \left(F_C\left(\frac{t+2\pi j}{s_C}\right), G_C\left(\frac{t+2\pi j}{s_C}\right), t\right), \quad t \in [0,2\pi], \tag{1}$$

where F_C, $G_C : [0,2\pi] \to \mathbb{R}$ are trigonometric polynomials.

We then define $g_\lambda : \mathbb{C} \times S^1 \to \mathbb{C}$.

$$g_\lambda(u,t) = \prod_C \prod_{j=1}^{s_C} \left(u - \lambda\left(F_C\left(\frac{t+2\pi j}{s_C}\right) + \mathrm{i}G_C\left(\frac{t+2\pi j}{s_C}\right)\right)\right), \tag{2}$$

which has the closure of B in $\mathbb{C} \times S^1$ as its vanishing set for all $\lambda > 0$. We show in [8] that we can find for every homogeneous braid a parametrisation as in Eq. (1) such that $\arg g_\lambda : (\mathbb{C} \times S^1) \backslash g^{-1}(0) \to S^1$ is a fibration map.

Note that the fibration property of $\arg g_\lambda$ can be phrased entirely in terms of the critical values of g_λ. For a fixed value of t the critical values of $g_\lambda(u,t)$ are given by $v_i(t) = g_\lambda(c_i,t)$ with $\frac{\partial g_\lambda}{\partial u}(c_i,t) = 0$, $i = 1,2,\ldots,s-1$. Then $\arg g_\lambda$ is a fibration if and only if for all i the derivative $\frac{\partial \arg v_i(t)}{\partial t}$ never vanishes. This has a nice geometric interpretation in terms of the movements of the critical values in the complex plane as t varies. Note that they are always non-zero and the non-vanishing derivatives of their arguments mean that they never change the orientation in which they twist around 0 as t varies between 0 and 2π. For every $v_i(t)$ this twisting is either always clockwise $\left(\frac{\partial \arg v_i(t)}{\partial t} < 0\right)$ or always anti-clockwise $\left(\frac{\partial \arg v_i(t)}{\partial t} > 0\right)$.

The union of the critical values $(v_i(t),t) \subset \mathbb{C} \times [0,2\pi]$ of $g_\lambda(u,t)$ and the curve $(0,t) \subset \mathbb{C} \times [0,2\pi]$ form a braid A on $s = \deg(g_\lambda)$ strands. It turns out that we can take $g_\lambda(u,t)$ such that there is a close relation between this braid and the braid B that is formed by the roots of $g_\lambda(u,t)$. Namely, if we write $B = \prod_{j=1}^{\ell} \sigma_{i_j}^{\varepsilon_j}$, then A can be taken to be a conjugate of

$$A = \prod_{j=1}^{\ell} A_{i_j}^{\varepsilon_j}, \tag{3}$$

where

$$\begin{aligned} X_i &= \sigma_i^{-1}\sigma_{i-1}^{-1}\ldots\sigma_2^{-1}\sigma_1^2\sigma_2\ldots\sigma_{i-1}\sigma_2, \\ A_i &= \begin{cases} X_{\frac{i+1}{2}} & \text{if } i \text{ is odd,} \\ X_{\frac{i}{2}+\lfloor\frac{s}{2}\rfloor} & \text{if } i \text{ is even.} \end{cases} \end{aligned} \tag{4}$$

This is shown in [9] and stems from an idea in [37]. Note that if B is homogeneous, then A can be parametrised such that $\frac{\partial \arg v_i(t)}{\partial t}$ never vanishes. The example of the homogeneous braid $B=\sigma_1\sigma_2\sigma_1\sigma_3^{-2}\sigma_2$ is illustrated in Figure 1.

In the context of Section 4 it is an important observation that not only does the union of the critical values $(v_i(t),t)\subset\mathbb{C}\times[0,2\pi]$ of $g_\lambda(u,t)$ and the curve $(0,t)\subset\mathbb{C}\times[0,2\pi]$ form a braid on s strands, but also $(0,t)$ is a braid axis for the union of the critical values $(v_i(t),t)\subset\mathbb{C}\times S^1$. The closure of this braid is an $s-1$-component unlink.

When we expand the product in Eq. (2) we obtain a polynomial in the complex variable u, but also in $\mathrm{e}^{\mathrm{i}t}$ and $\mathrm{e}^{-\mathrm{i}t}$.

The function $g_\lambda(u,2t)$ then also gives a fibration map via its argument and has the closure of B^2 as its vanishing set. Furthermore, expanding the product results in a polynomial in u, $\mathrm{e}^{\mathrm{i}t}$ and $\mathrm{e}^{-\mathrm{i}t}$, where all exponents of $\mathrm{e}^{\mathrm{i}t}$ and $\mathrm{e}^{-\mathrm{i}t}$ are even.

We define a real polynomial in $\mathbb{C}^2\cong\mathbb{R}^4\to\mathbb{R}^2\cong\mathbb{C}$ by rescaling g_λ:

$$f_\lambda(u,v)=\begin{cases} r^{2k\sum_C s_C} g_\lambda\left(\frac{u}{r^{2k}},2t\right), & \text{if } v=r\mathrm{e}^{\mathrm{i}t}\\ u^s & \text{if } v=0,\end{cases} \tag{5}$$

with $k\geq\max\{\deg_{\mathrm{e}^{\mathrm{i}t}} g_\lambda,\deg_{\mathrm{e}^{-\mathrm{i}t}} g_\lambda\}/2s$. Then f_λ can be written as a polynomial in u, v and $\overline{v}$. This is because the exponents of $\mathrm{e}^{\mathrm{i}t}=\frac{v}{\sqrt{v\overline{v}}}$ and $\mathrm{e}^{-\mathrm{i}t}=\frac{\overline{v}}{\sqrt{v\overline{v}}}$ in g_λ are even and therefore cancel the square roots. Moreover, f_λ has an isolated singularity at the origin and if λ is small enough, the link around that singularity is the closure of B^2 by construction.

The details of these constructions go beyond the scope of this short overview. What we would like to highlight are the striking similarities between the different constructions. All of them (apart from the ones based directly on algebraic links such as Pichon's [34]) start with a particular parametrisation of the desired knot or link L. Using this parametrisation we then define a function whose vanishing set is in some sense L (either on a 3-sphere or on $\mathbb{C}\times S^1$) and then perform some sort of rescaling argument to obtain the desired polynomial (eliminating the dependence on either the modulus $|x|=\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}$ or $|v|=\sqrt{x_3^3+x_4^2}$). There are two main obstacles to look out for. Firstly, rescaling, whether radially or by the modulus of v, introduces square root terms. We can therefore in general not expect to obtain a polynomial. This is only achieved if the parametrisation we started with has some particular symmetries. In the case of Looijenga's construction, this means that we have to limit ourselves to odd knots. In the construction in [9], it requires us to only consider closures of 2-periodic braids. It is not a coincidence at all that both of these symmetries are related to the number 2, the exponent that cancels a square-root term.

The second obstacle is that we need to make sure that we have an isolated singularity. In the previous constructions this is closely linked to explicit fibration maps over the circle. Looijenga's construction starts with the fibration, so it is clear that we need a fibred link to start with. Having Milnor's theorem in mind this turns out not to be a restriction at all. In [9] we have to limit ourselves to homogeneous braids because these are the ones, for which we know how to construct polynomial fibrations that are holomorphic in u.

In both our [9] and Rudolph's construction the resulting polynomial is *semiholomorphic*: It can be written as a polynomial in complex variables u, v and $\overline{v}$. This makes it a lot easier to

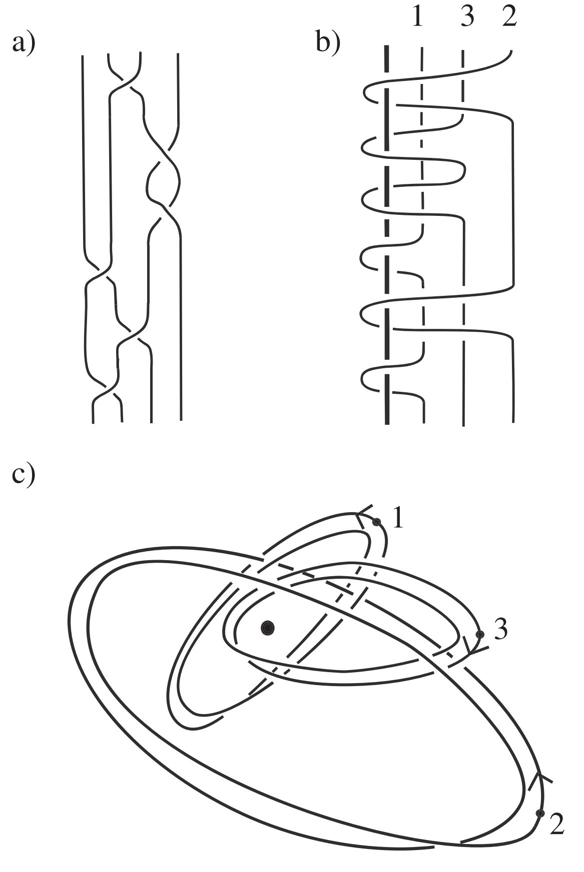

Figure 1: A simple relation between the braid that is formed by the critical values of a loop in the space of polynomials and the braid that is formed by the roots of the same polynomials. a) The braid $B=\sigma_1\sigma_2\sigma_1\sigma_3^{-2}\sigma_2$. b) B can be parametrised such that it is given by the roots of a loop in the space of complex polynomials, whose critical values and the 0-strand $(0,t)\subset\mathbb{C}\times[0,2\pi]$ form the braid $A=A_1A_2A_1A_3^{-2}A_2=X_1X_3X_1X_2^{-2}X_3$. The number i above a strand gives a correspondence between the Artin generator σ_i and one twist of that strand around the 0-strand. c) The top view of the braid in b). The critical values (i.e., A without the 0-strand) can be parametrised such that $\frac{\partial \arg v_i(t)}{\partial t}$ never vanishes. They form a $s-1$-component unlink and the 0-strand is a braid axis for it.

check that the singularity is isolated. Semiholomorphic polynomials are special cases of *mixed polynomials* that are studied in the context of singularities in [32, 35, 38]. In lack of a better term we call a link *semiholomorphic* if it is the link of an isolated singularity of a semiholomorphic polynomial $\mathbb{R}^4 \to \mathbb{R}^2$. It is a natural question to ask if the set of semiholomorphic links is the same as the set of real algebraic links.

Theorem 3.2 (Looijenga [28], Perron [33], Pichon [34], Rudolph [36], Bode [9])**.** *The links that are known to be real algebraic are as follows:*

- *algebraic links,*
- $L_{f\overline{g}}$*, the link of the singularity of* $f\overline{g}$*, where* f *and* g *are holomorphic polynomials with isolated singularities, i.e.* $L_f \sqcup L_g$ *if it is fibred*
- *odd fibred links (including K#K for K a fibred knot),*
- *the closure of* B^2 *if* B *is a homogeneous braid (including* 4_1*).*

Out of these, the algebraic links and closures of squares of homogeneous braids are known to be semiholomorphic as well.

Recently, the techniques from [9] have been further generalised to constructions of families of links that might not be closures of homogeneous braids [10]. The underlying principle of symmetric parametrisations and rescaling remains the same though. It remains challenging to check whether the newly constructed links in [10] are already known to be real algebraic from Theorem 3.2.

4 Branched coverings of S^3 over itself and Hopf plumbings

So far, all proofs of real algebraicity of links are constructive and all constructions are either based on algebraic links (as in Pichon's construction) or on a rescaling of a polynomial, whose vanishing set on a subset of $\mathbb{R}^4$ is the desired link. This rescaling introduces square root terms and naturally restricts the class of links for which the resulting function is a polynomial to those that satisfy certain symmetry constraints. This not only means that it is doubtful that this approach can prove the conjecture by Benedetti and Shiota, but also makes it very hard to check if a link falls into that class as it might not be straightforward to investigate if it possesses the necessary symmetries.

A proof of the conjecture of Benedetti and Shiota should therefore not involve an exact radial rescaling. In Looijenga's construction for example, the argument map $S^3_\rho \backslash L \to S^1$ given by the argument of the polynomial does not depend on the radius $\rho = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$. In the construction in [9] the argument of $f_\lambda(u, r\mathrm{e}^{\mathrm{i}t})$ does not depend on $r = |v| = \sqrt{x_3^2 + x_4^2}$. Instead it is probably worth looking for a way to construct a polynomial map $f : \mathbb{R}^4 \to \mathbb{R}^2$ with an isolated singularity from a fibration $\chi : S^3 \backslash L \to S^1$, where $\arg f$ does depend on the radius ρ or the modulus r. Instead of having $\frac{\partial \arg f}{\partial \rho} = 0$ we could demand that it is very small; small enough to ensure that the constructed singularity is still isolated. Of course, similar considerations apply

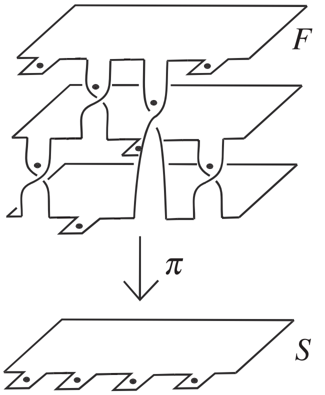

Figure 2: A 3-sheeted simple branched cover of a disc. The branch set Q consists of 4 points.

to the dependence on r. This is all very vague at the moment and it is not clear at all how such a construction could be achieved.

If such a rescaling could be defined, it would be easier to check whether the resulting function has an isolated singularity if it is a semiholomorphic polynomial. Hence, while we still lack a proper rescaling mechanism, we can investigate for which fibred links the fibration map can be taken to be the argument of a semiholomorphic polynomial. From [8] we know this for closures of homogeneous braids, but recently there have been constructions of links that appear not to be homogeneous braid closures [10].

It would obviously be desirable if the construction of isolated singularities (or the construction of semiholomorphic polynomial fibrations) did not start with some particular parametrisation or symmetry requirement, but rather with a property that is satisfied by all fibred links. While there are several at hand, for some of them, such as properties of the commutator subgroup of the link group [40] or knot Floer homology [31], it might not be easy to establish a connection to the topic of polynomial maps. There are however two properties where such a connection might exist, the first being the study of branched coverings of S^3 over itself. The following exposition follows Montesinos and Morton [30].

Definition 4.1. *Let F and S be surfaces and let $\pi : F \to S$ be a continuous surjective map. Then π is a simple branched cover with d sheets if there is a finite branch set $Q \subset int(S)$, where $int(S)$ denotes the interior of S, such that:*

- $\pi|_{F\setminus\pi^{-1}(Q)}$ *is a d-sheeted covering map,*
- *for every $q \in Q$ there is a neighbourhood U such that $\pi^{-1}(U)$ has $d-1$ components, one of which is a disc that is projecting to U as a double cover branched over q, while the others are discs that are projecting homeomorphically.*

The concept of a simple branched cover is illustrated in Figure 2.

Definition 4.2. *Let M and N be closed 3-manifolds. A map $\pi : M \to N$ is called a simple d-sheeted cover with branch set $C \subset N$ if it is locally homeomorphic to the product of an interval with a simple d-sheeted cover of a disc, and the branch points in the product form the set C.*

Now let L_{branch} be a link in S^3 and O be an unknot in S^3 such that O is a braid axis for L_{branch}, i.e., there is a fibration of $S^3\backslash O$ over the circle such that each fibre surface intersects L_{branch} the same number of times and the intersections are transverse. Let $\pi : S^3 \to S^3$ be a simple branched cover of S^3 over itself, with branching set L_{branch}. Then $\pi^{-1}(O)$ is a fibred link in S^3 and conversely, for every fibred link there are such links L_{branch}, braid axes O and covers π. The relation between fibred links and simple branched covers has been studied extensively by Birman [6], Goldsmith [22], Hilden [24] and Montesinos with Morton [30]. This relates to the construction of semiholomorphic fibrations as follows.

Consider a semiholomorphic polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ constructed as in [9]. In particular, $\arg f : (\mathbb{C} \times rS^1)\backslash f^{-1}(0) \to S^1$ is a fibration for all $r > 0$. Since $\arg f(u, r\mathrm{e}^{\mathrm{i}t})$ goes to $\arg u^s$ and $|f(u, r\mathrm{e}^{\mathrm{i}t})|$ goes to infinity for all $r\mathrm{e}^{\mathrm{i}t}$ as $|u|$ goes to infinity, we can compactify $\mathbb{C} \times S^1$ to S^3. This way the map

$$\begin{aligned} \left(\frac{u}{1+|u|}, \mathrm{e}^{\mathrm{i}t}\right) &\mapsto \left(\frac{f(u, r\mathrm{e}^{\mathrm{i}t})}{1+|f(u, r\mathrm{e}^{\mathrm{i}t})|}, \mathrm{e}^{\mathrm{i}t}\right), \\ (\mathrm{e}^{\mathrm{i}\chi}, 0) &\mapsto (\mathrm{e}^{\mathrm{i}s\chi}, 0) \end{aligned} \tag{6}$$

is a branched covering of S^3 over itself for every value of r. Here we have identified S^3 with $(\mathbb{D} \times S^1)$ modulo $(\mathrm{e}^{\mathrm{i}\chi}, \mathrm{e}^{\mathrm{i}t_1}) = (\mathrm{e}^{\mathrm{i}\chi}, \mathrm{e}^{\mathrm{i}t_2})$ for all χ, t_1, t_2, where $\mathbb{D}$ is the closed unit disc in $\mathbb{C}$.

The maps in (6) are branched over the set

$$\left(\frac{v_i(t)}{1+|v_i(t)|}, r\mathrm{e}^{\mathrm{i}t}\right) = \left(\frac{f(u, r\mathrm{e}^{\mathrm{i}t})}{1+|f(u, r\mathrm{e}^{\mathrm{i}t})|}, \mathrm{e}^{\mathrm{i}t}\right), \tag{7}$$

where $i = 1, 2, \ldots, s-1$, $\frac{\partial f}{\partial u}(u, r\mathrm{e}^{\mathrm{i}t}) = 0$. Recall from the remark in Section 3 that this set is a $s-1$-component unlink and that $\arg f$ being a fibration implies that the derivative $\frac{\partial \arg v_i(t)}{\partial t}$ never vanishes. This implies that $(0, \mathrm{e}^{\mathrm{i}t})$ is a braid axis for $\bigcup_i (v_i(t)/(1+|v_i(t)|), \mathrm{e}^{\mathrm{i}t})$. Since $f^{-1}(0) \cap (\mathbb{C} \times rS^1)$ is the constructed fibred link L, the functions that are constructed in [9] can be explained in the context of [30].

There is a second obvious braid axis for $\bigcup_i (v_i(t)/(1+|v_i(t)|), \mathrm{e}^{\mathrm{i}t})$, namely $(\mathrm{e}^{\mathrm{i}\chi}, 0)$, $\chi \in [0, 2\pi]$, and its preimage under branched covering map is the unknot $(\mathrm{e}^{\mathrm{i}\chi}, 0)$, $\chi = [0, 2\pi]$.

Therefore every semiholomorphic link L that is constructed as in [9] gives rise to a simple s-sheeted cover $\pi : S^3 \to S^3$ with branch set L_{branch}, which is an $s-1$-component unlink, such that $L = \pi^{-1}(O)$ for some braid axis O for L_{branch}. Furthermore, there is another braid axis O' for L_{branch} whose preimage set under π is an unknot. In fact, O' is a braid axis for $O \cup L_{branch}$ and $O \cup L_{branch}$ is an s-strand braid with respect to O'.

We will revisit this idea once we have introduced the second property of fibred links that could be useful in the context of polynomial singularities.

We call an unknotted annulus with a positive or negative full twist a positive or negative *Hopf band*. Let F be a fibre surface and γ an arc, i.e., a simple path in $\mathrm{int}(F)$ except for its endpoints, which lie in ∂F. A neighbourhood U of γ in F is then a properly embedded square

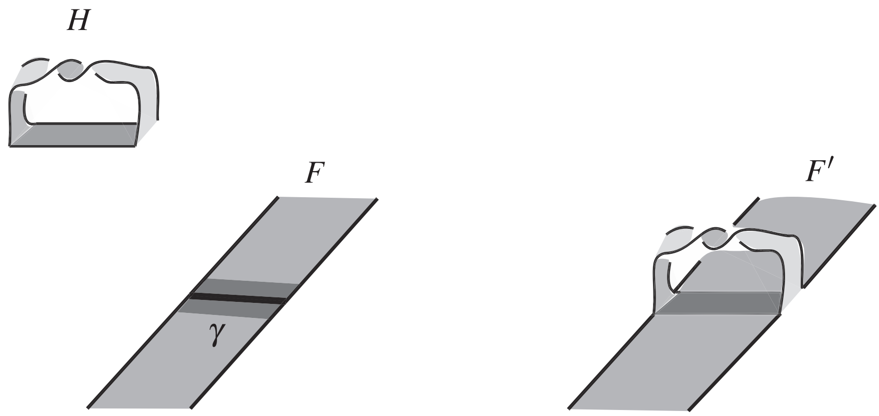

Figure 3: A Hopf band H is plumbed to a surface F along a path γ.

which has two opposite sides in ∂F. Similarly, we consider a square $U' \subset H$ with two opposite sides on ∂H. A Hopf plumbing along γ is obtained from F by glueing H to F by identifying U and U' such that the two sides of U' in ∂H run parallel to γ. We also say a surface F' is obtained from F by deplumbing a (positive or negative) Hopf band if F is obtained from F' by Hopf plumbing.

Analogously, we say a fibred link $L' = \partial F'$ is obtained from $L = \partial F$ by Hopf plumbing if the corresponding fibre surface F' is obtained from F by Hopf plumbing.

An illustration of Hopf plumbing is shown in Figure 3.

If L is a fibred link, then so are all links that are obtained from it by Hopf plumbing and deplumbing [20, 41]. Conversely, Giroux and Goodman showed the following theorem, which was originally conjectured by Harer [23].

Theorem 4.3 (Giroux-Goodman [21])**.** *Every fibred link can be obtained from the unknot through a sequence of Hopf plumbings and deplumbings.*

Montesinos and Morton established a connection between these two concepts, branched coverings of S^3 over itself and Hopf plumbings, and thereby also a connection between sequences of Hopf plumbings and polynomial fibrations. At the time of their work, Harer's conjecture was still unproven and it is not far-fetched to believe that they hoped to make progress on Harer's conjecture using their idea of branched coverings.

Let $\pi : S^3 \to S^3$ be simple branched covering with branch set L_{branch}. Let O be a braid axis for L_{branch} and $\phi : S^3 \backslash O \to S^1$ be a fibration such that each fibre intersects L_{branch} transversally in precisely $s-1$ points. Let D be a fibre of ϕ and γ be a simple path in D starting at one of the points in $D \cap L_{branch}$, ending at the boundary ∂D and avoiding all points in $D \cap L_{branch}$ in between. Let U be an open neighbourhood of γ in S^3. Now we push $D \cap U$ in the normal direction of D and connect this push-off with ∂D. This means we have attached a disc to D that in some sense lies directly over the path γ. Near the starting point of γ which is in $D \cap L_{branch}$, this disc intersects L_{branch}. We apply a positive or negative half-twist to the attached disc before this intersection. The result should look similar to Figure 4.

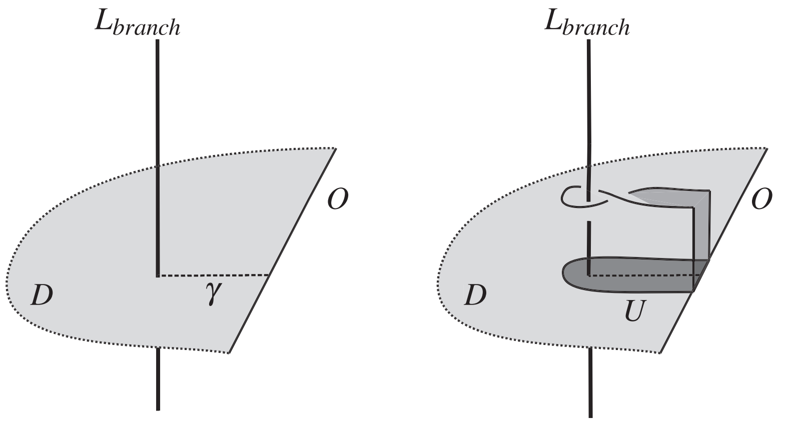

Figure 4: The dotted line is a path γ from the intersection of L_{branch} with D to $O = \partial D$. A neighbourhood U of that path is pushed off D. Applying a half-twist to this pushed-off disc results in this figure. The disc D' is the union of D and the pushed-off disc with a half-twist.

Note that by attaching this twisted disc to D we have constructed a new disc D' that intersects L_{branch} in s points, one more than D. Furthermore, the boundary of D' is also a braid axis for L_{branch}. The braid word of L_{branch} with respect to $\partial D'$ can be obtained from the braid with respect to ∂D by a conjugation that depends on the path γ and Markov stabilization $\sigma_{s-1}^{\pm 1}$, where the sign depends on the sign of the half-twist of the attached disc. Since $\partial D'$ is a braid axis for L_{branch}, its preimage under the same covering map π is a fibred link L' just like $L = \pi^{-1}(\partial D)$ is fibred.

Theorem 4.4 (Montesinos-Morton [30]). *Let the situation be as described above. Let $F = \pi^{-1}(D)$ be the fibre surface of L. Then there is a path $\tilde{\gamma}$ in F, starting and ending at ∂F with $\pi(\tilde{\gamma}) = \gamma$ such that $F' = \pi^{-1}(D')$, the fibre surface of L', is up to isotopy obtained from F by Hopf plumbing along $\tilde{\gamma}$, where the sign of the Hopf plumbing depends on the sign of the half-twist in the attached disc or equivalently the sign of the Markov stabilization that converts the braid word of L_{branch} with respect to ∂D into the braid word of L_{branch} with respect to $\partial D'$.*

Corollary 4.5 (Montesinos-Morton [30]). *If O and O' are braid axes for the same link L_{branch} in S^3, which is the branch set of a simple branched covering $\pi : S^3 \to S^3$, then $\pi^{-1}(O)$ and $\pi^{-1}(O')$ are related by a sequence of Hopf plumbings and deplumbings.*

This corollary is nowadays obvious because Giroux's and Goodman's work tells us that all fibred links are related by such sequences. The remarkable insight is the close connecting between Hopf plumbings and changes of the braid axis. It is also important to note that the converse of Theorem 4.4 and Corollary 4.5 is not known. It is not clear if we can always arrange that the path along which the plumbing or deplumbing happens can be arranged to lie over γ as in the Theorem. If a Hopf plumbing (or deplumbing) can be obtained as above, then we say that it *corresponds to a Markov move*.

Montesinos and Morton raised the following question.

Question 4.6 (Montesinos-Morton [30])**.** *Can every fibred link be obtained as the preimage of a braid axis of an $s-1$-component unlink L_{branch} under a simple s-sheeted branched covering $\pi: S^3 \to S^3$, whose branch set is L_{branch}?*

By the earlier remark this would imply that every fibred link can be obtained from the unknot via a sequence of Hopf plumbings and deplumbings, all of which correspond to Markov moves on the branch set of the branched covering π. At the time a positive answer to this question would have been the first proof of Harer's conjecture. However, even now that Harer's conjecture is proven, it is still not known if this question by Montesinos and Morton has a positive answer.

For the construction of more real algebraic links we would like to go beyond Montesinos and Morton's question.

Question 4.7. *Can every fibred link L be obtained as a the preimage of a braid axis O of an $s-1$-component unlink L_{branch} under a simple s-sheeted branched covering $\pi: S^3 \to S^3$, whose branch set is L_{branch} and the braid index of $O \cup L_{branch}$ is s?*

The braid index of $O \cup L_{branch}$ being equal to s is equivalent to saying that there is a braid axis O' for $O \cup L_{branch}$ with $\pi^{-1}(O')$ being an unknot. Note that this question can just like Montesinos and Morton's question be interpreted as a question on whether it is possible for every link to find a sequence of Hopf plumbings and deplumbings that has certain additional properties.

Recall that the situation described in this question is precisely what we encounter for the simple branched covering from a semiholomorphic link as in [9].

Suppose that the answer to the question above is positive. Then maybe there is a way to construct from the simple branched covering π a polynomial map $g_\lambda : \mathbb{C} \times S^1 \to \mathbb{C}$, whose argument is a fibration. Whether or not this is possible depends on the answer to the following question.

Question 4.8. *Let X_s denote the space of simple s-sheeted branched covers of the disc $\mathbb{D}$ over itself such that $0 \in \mathbb{D}$ is not in the branch set. Is every loop in X_s homotopic to a loop in the space V_s of complex polynomials of degree s with distinct non-zero critical values?*

Note that V_s is a subset of X_s and using results from [4] and [7] a positive answer would imply that for every fibred link there is a semiholomorphic polynomial $f: \mathbb{C}^2 \to \mathbb{C}$ such that $f^{-1}(0) \cap S^3 = L$ (and $f^{-1}(0) \cap (\mathbb{C} \times S^1) = L$) and $\arg f|_{S^3 \setminus L}$ is a fibration.

Subject to an appropriate rescaling mechanism that does not rely on particular symmetries to yield polynomials (which admittedly comes with its own problems and difficulties), this could result in a construction of polynomial isolated singularities for any fibred link and hence in a proof of the conjecture by Benedetti and Shiota. At the moment this is still highly speculative, but we hope that this exposition inspires future work on the classification of real algebraic links.

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