数理解析研究所講究録 第2129巻 2019年 83-96

Divisibility of Lee's class and its relation with Rasmussen's invariant

Taketo Sano
Graduate School of Mathematical Sciences, The University of Tokyo

1 Introduction

In this article we introduce a knot invariant $\bar{s}_c$ derived from the *divisibility* of Lee's class, and summarize its relation with Rasmussen's s-invariant. A more detailed discussion can be found in the author's preprint [13].

Rasmussen [12] introduced the *s-invariant* and he proved that: (i) s defines a homomorphism from the knot concordance group in S^3 to $2\mathbb{Z}$, (ii) it provides a lower bound for the slice genus g_* of knots, and (iii) s and g_* are equal for positive knots. Then the Milnor conjecture [10] follows as a corollary. The conjecture was originally proved by Kronheimer and Mrowka in [4] using gauge theory, but Rasmussen's result was notable since it provides a purely combinatorial proof of an important fact of four-dimensional topology.

The definition of s is based on Lee homology [6] (a variant of Khovanov homology [2]). The above stated results are obtained from the invariance of the "canonical generators" of Lee homology over $\mathbb{Q}$. For a knot diagram D, there are two distinct cycles α, β constructed combinatorially from D, and those homology classes form a basis of $H_{Lee}(D; \mathbb{Q})$. Lee originally introduced these classes, and Rasmussen proved that they are knot invariants (up to multiplication by units), hence called them the canonical generators.

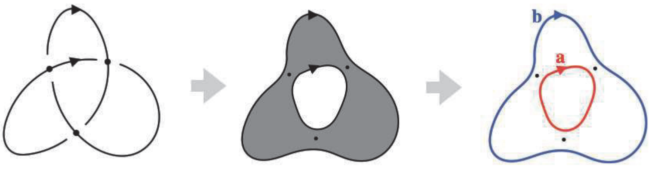

Figure 1: Construction of the cycle α.

The construction of Lee's classes can be done over $\mathbb{Z}$, so one may expect that they also generate the homology group over $\mathbb{Z}$. However this is not the case. In fact by direct computation, we observe that many of these classes are divisible by 2-powers. The aim of our research is to explore what information we can obtain from the 2-divisibility of these classes.

We consider this situation in a more generalized setting. By following the arguments given by Khovanov in [3] and by Mackaay, Turner, Vaz in [9], we define a family of

Khovanov-type link homology theories $H_c(-;R)_{c\in R}$ over an arbitrary commutative ring R parameterized by $c \in R$. Here Khovanov's original theory corresponds to $c = 0$, and Lee's theory corresponds to $c = 2$. For each $c \in R$, Lee's classes $[\alpha], [\beta]$ of a knot diagram D can be defined in $H_c(D;R)$.

Proposition 1. *With the above setting:*

1. *If c is invertible in R, then $[\alpha], [\beta]$ form a basis of $H_c(D;R)$.*
2. *If D, D' are two diagrams related by a single Reidemeister move, then the 'ratio' of the corresponding two classes is given by $\pm c^j$ for some $j \in \{0, \pm 1\}$.*

Thus the situation is completely analogous to $\mathbb{Q}$-Lee theory when c is invertible. As for $\mathbb{Z}$-Lee theory $c = 2$ is not invertible, so our concern is when c is *not* invertible in R. In the following we assume R is an integral domain and c is non-zero, non-invertible. We define the *c-divisibility* of $[\alpha]$ (or $[\beta]$) by the exponent of its c-power factor (modulo torsions), and denote it by $k_c(D)$. From following proposition we may regard k_c as measuring the "non-positivity" of the diagram.

Proposition 2. *If D is positive, then $k_c(D) = 0$.*

From Proposition 1 (2.) we see that k_c varies by the exponent j of the ratio under the Reidemeister moves, hence k_c is *not* a knot invariant. Here j can be calculated without computing the homology group or the homology class:

Proposition 3. *The exponent j is given by the difference of $\frac{r-w}{2}$ between the two diagrams, where w is the writhe and r is number of Seifert circles.*

If we subtract $\frac{r-w}{2}$ from k_c, then the value becomes constant under the Reidemeister moves. Hence we obtain a knot invariant:

Theorem 1. *Let K be a knot. For any diagram D of K, the value*

$$\bar{s}_c(K) := 2k_c(D) + w(D) - r(D) + 1$$

is an invariant of K.

Again by computational experiments done for $(R, c) = (\mathbb{Z}, 2)$, we observed that $\bar{s}_c$ coincides with s for all prime knots of crossing number up to 11. Here we pose the main question:

Question 1. *Does $\bar{s}_c$ coincide with s for any (R, c)?*

If this is true, then s can be given a description in terms of the divisibility of Lee's class. If not, then we obtain some potentially new invariants. We give several facts that support the affirmative answer.

Theorem 2.

1. *$\bar{s}_c$ is a $2\mathbb{Z}$-valued knot concordance invariant in S^3,*
2. *$|\bar{s}_c(K)| \leq 2g_*(K)$ for any knot K, and*

3. $\bar{s}_c(K) = 2g_*(K) = 2g(K)$ *if K is positive.*

These properties are common to s (except that s also possesses the homomorphism property), and as a corollary, each $\bar{s}_c$ gives an alternative proof for the Milnor conjecture. Next, we show that there exists a pair (R, c) such that $\bar{s}_c$ coincides with s.

Theorem 3. *If $(R, c) = (\mathbb{Q}[h], h)$,*

$$s(K) = \bar{s}_h(K)$$

for any knot K.

More generally, we can prove that for any field F of $\operatorname{char} F \neq 2$, the s-invariant over F and $\bar{s}_c$ for $(R, c) = (F[h], h)$ coincides. There is a famous open question whether there exists any F such that $s(-; F)$ is distinct from $s = s(-; \mathbb{Q})$ ([8, Question 6.1]). If Question 1 is solved affirmatively, then it follows that $s(-; F)$ are equal among all fields F of $\operatorname{char} F \neq 2$.

Viewing the s-invariant from the perspective of divisibility has been suggested by Kronheimer and Mrowka in [5], and by Collari in [1], both based on the alternative definition of s given by Khovanov in [3]. We expect that our approach would also lead to a better understanding of s.

2 Preliminaries

In this section we briefly review the construction of Khovanov's chain complex in the generalized form, as given in [3]. Let R be a commutative ring with unity. Given a Frobenius algebra over R and a link diagram D, we obtain a chain complex $C_A(D; R)$ and its homology $H_A(D; R)$ by following the construction of the original version of Khovanov homology, except that the algebra $R[X]/(X^2)$ is replaced with A. The construction is given as follows: a choice of a simultaneous resolutions for all crossings of D (which is called a *state*) yields a diagram consisting of disjoint circles. Consider all possible states of D and place them on the vertices of an n-dimensional cube, so that each edge corresponds to a cobordism between two resolved diagrams. The Frobenius algebra A determines a TQFT (i.e. a functor from the category of 2-dimensional cobordisms to the category of R-modules), and by applying it to the cube we obtain a commutative cubic diagram of R-modules and R-homomorphisms. After some adjustment of signs of the maps, the cube is folded to form the (unnormalized) chain complex (see Figure 2). Finally the normalized version is obtained after some degree shift.

Here we consider the algebra of the following special form $A_{h,t} = R[X]/(X^2 - hX - t)$ with $h, t \in R$. We denote the corresponding chain complex and its homology by $C_{h,t}$ and $H_{h,t}$ respectively. Khovanov's original theory [2] and Lee's theory [6] are given by

$$H_{Kh} = H_{0,0}, \quad H_{Lee} = H_{0,1}.$$

The chain complex is also given a secondary grading called the q-degree, and under some condition $H_{h,t}(D; R)$ becomes bigraded or filtered. The following theorem assures that any $H_{h,t}$ gives a link invariant:

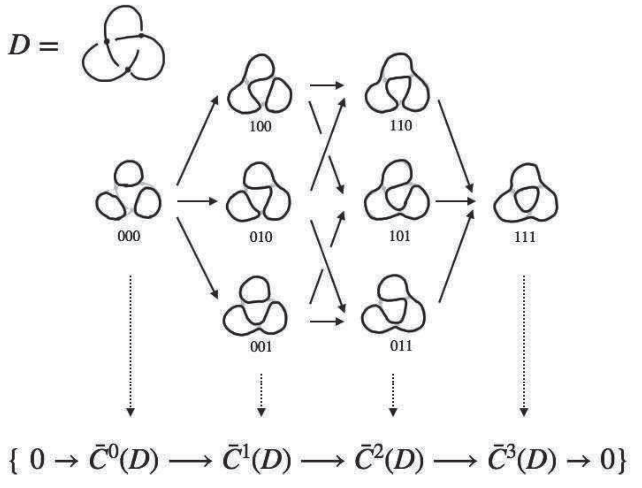

Figure 2: The (unnormalized) chain complex for the left hand trefoil.

Theorem 2.1 ([3, Proposition 6])**.** *Let L be a link. For any diagram D of L, the isomorphism class of $H_{h,t}(D;R)$ as a (graded / bigraded / filtered) R-module is an invariant of L.*

3 Generalizing Lee's class

For simplicity we focus on knots, but the following discussion can be extended to links otherwise stated. We assume that $h, t \in R$ satisfies the following:

Condition 3.1. $X^2 - hX - t$ factors into linear polynomials.

Fix one $c = \sqrt{h^2 + 4t}$, and take the two roots:

$$u = (h - c)/2, \quad v = (h + c)/2 \ \in \ R.$$

Then $X^2 - hX - t = (X - u)(X - v)$ in $R[X]$. Define

$$\mathbf{a} = X - u, \quad \mathbf{b} = X - v \ \in \ A.$$

Then obviously $\mathbf{ab} = \mathbf{ba} = 0$. Also with $\mathbf{a} - \mathbf{b} = v - u = c$, we have:

$$\begin{aligned} m(\mathbf{a} \otimes \mathbf{a}) &= c\mathbf{a}, & \Delta(\mathbf{a}) &= \mathbf{a} \otimes \mathbf{a}, \\ m(\mathbf{a} \otimes \mathbf{b}) &= 0, & \Delta(\mathbf{b}) &= \mathbf{b} \otimes \mathbf{b} \\ m(\mathbf{b} \otimes \mathbf{a}) &= 0 & & \\ m(\mathbf{b} \otimes \mathbf{b}) &= -c\mathbf{b} & & \end{aligned}$$

where m, Δ is the multiplication and the comultiplication of A respectively. In the case of Lee's theory (where $(R, h, t) = (\mathbb{Q}, 0, 1)$), the corresponding values are $c = 2$, $\mathbf{a} = X + 1$

and $\mathbf{b} = X - 1$. We call $\mathbf{a}$ and $\mathbf{b}$ *colors*. For any state s, a coloring on the resolved circles defines an element in $C_{h,t}(D;R)$ by the tensor product of factors in $\mathbf{a}$ and $\mathbf{b}$. Recall that any oriented knot diagram possesses a unique *orientation preserving state* s, where every state circle admits an orientation coherent with the given orientation of D. From the standard procedure of the bicoloring of the Seifert circles (see Figure 1), we obtain an element $\alpha \in C_{h,t}(D;R)$, which is in fact a cycle. By reversing the orientation of D, we obtain another cycle β. (In the case of an ℓ-component link diagram D, there are 2^ℓ distinct cycles, one for each alternative orientation on D.)

Lee proved in [6] that $H_{Lee}(D;\mathbb{Q})$ is freely generated by these classes, so in particular $H_{Lee}(D;\mathbb{Q}) \cong \mathbb{Q}^2$. This generalizes as follows:

Proposition 3.2. *If $c = \sqrt{h^2 + 4t}$ is invertible in R, then $H_{h,t}(D;R)$ is freely generated by $[\alpha], [\beta]$ over R. In particular $H_{h,t}(D;R) \cong R^2$.*

Lee's proof cannot be applied directly, since it uses Hodge theory and requires that R is a field. However there is an alternative proof that can be applied to our case, that is the *admissible coloring decomposition* of $C_{h,t}(D;R)$ proposed by Wehrli in [15, Remark 5.4]. Briefly, if c is invertible, then the colored states form a basis of $C_{h,t}(D;R)$, and there is a decomposition of $C_{h,t}(D;R)$ by the set of admissible colorings of D. The subcomplex spanned by α, β gives the homology, while the remaining part is acyclic. A detailed proof can be found in Lewark's paper [7, Lemma I.14].

The following proposition is a generalization of [12, Proposition 2.3], and is essential for the definition of the invariant $\bar{s}_c$ in the next section.

Proposition 3.3. *Suppose D, D' are two diagrams related by a Reidemeister move. There is an isomorphism $\rho : H_{h,t}(D;R) \to H_{h,t}(D';R)$ such that the α, β-classes correspond as:*

$$[\alpha'] = \varepsilon c^j \rho[\alpha],$$
$$[\beta'] = \varepsilon' c^j \rho[\beta]$$

with some $j \in \{0, \pm 1\}$ and $\varepsilon, \varepsilon' \in \{\pm 1\}$ satisfying $\varepsilon\varepsilon' = (-1)^j$. (Here c is not necessarily invertible, so the equation $z = c^j w$ is to be understood as $c^{-j} z = w$ when $j < 0$.) Moreover j is determined as in Table 1 by the type of the move and the difference of the numbers of Seifert circles.

Type	Δr	j
$\mathrm{RM1}_L$	1	0
$\mathrm{RM1}_R$	1	1
RM2	0	0
	2	1
RM3	0	0
	2	1
	-2	-1

Table 1: The exponent j corresponding to the Reidemeister moves

This is proved by considering all possible patterns that occur under each move. In particular if c is not invertible, then $[\alpha], [\beta]$ are *not* invariant under the Reidemeister moves. From Table 1 we see that j is given by a single equation:

Corollary 3.4. *The exponent j is given by*

$$j = \frac{\Delta r - \Delta w}{2}$$

where w denotes the writhe, r denotes the number of Seifert circles, and the prefixed Δ denotes the difference of the corresponding values of D and D'.

Finally we state that under Condition 3.1 the two parameters $h, t \in R$ can be reduced to a single parameter $c \in R$.

Proposition 3.5. *Let (h, t), (h', t') be two pairs satisfying $\sqrt{h^2 + 4t} = \sqrt{h'^2 + 4t'}$. For any link diagram D, there is an isomorphism $\sigma : H_{h,t}(D; R) \to H_{h',t'}(D; R)$ that commutes with the isomorphism ρ of Proposition 3.3, and that the α, β-classes correspond one-to-one.*

We denote the isomorphism class of $H_{h,t}(D; R)$ by $H_c(D; R)$. Lee's classes $[\alpha], [\beta]$ are well-defined under the identification. Figure 3 depicts the (h, t)-parameter space, where each point (h, t) corresponds to $H_{h,t}(D; R)$ and the parabola $h^2 + 4t = c^2$ corresponds to the isomorphism class $H_c(D; R)$.

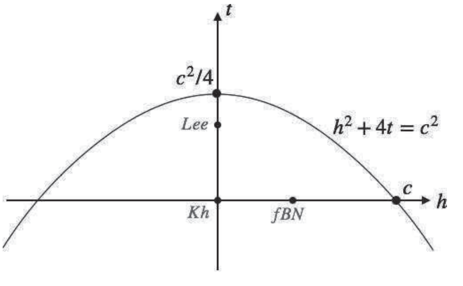

Figure 3: The (h, t)-parameter space.

From the above propositions, we conclude that the situation is completely analogous to $\mathbb{Q}$-Lee theory when c is invertible: $[\alpha]$ and $[\beta]$ form a basis of $H_c(D; R)$ and are invariant (up to unit) under the Reidemeister moves. Now our main concern is when c is *not* invertible in R.

4 Divisibility of Lee's class

In the remaining we assume that R is an integral domain and c is non-zero, non-invertible. Denote by $H_c(D; R)_f$ the *free part* of $H_c(D; R)$, i.e. the quotient of $H_c(D; R)$ by its torsion submodule. By abuse of notation, we denote the image of an element $[z] \in H_c(D; R)$ by the same symbol $[z] \in H_c(D; R)_f$. We define the *c-divisibility* of $[\alpha]$ by the exponent of its c-power factor (modulo torsions). More formally,

Definition 4.1. For any knot diagram D, define

$$k_c(D) = \max\{\ k \geq 0 \mid [\alpha] \in c^k \cdot H_c(D; R)_f\ \} \ \in\ [0, \infty].$$

Example 4.2. $k_c(\bigcirc) = 0$ since $H(\bigcirc) = C(\bigcirc) = R\langle 1, X\rangle$.

Example 4.3. $D =$ (unknot with one negative crossing). Let $(R, h, t) = (\mathbb{Z}, 0, 1)$ and $c = 2$. We have

$$C(D) = \{\ A \xrightarrow{\Delta} A^{\otimes 2}\ \}, \quad \alpha(D) = (X - 1) \otimes (X + 1).$$

From $\Delta(X) = X \otimes X + 1 \otimes 1$ and $\Delta(1) = 1 \otimes X + X \otimes 1$, we see that $\alpha(D)$ is homologous to $2(1 \otimes X - 1 \otimes 1)$. Since $\{[1 \otimes X], [1 \otimes 1]\}$ form a basis of $H(D)_f$, we have $k_2(D) = 1$.

These examples show that $k_c(D)$ is *not* a link invariant. In fact from Proposition 3.3, the difference of k_c between the two diagrams can be given without even computing the homology groups.

Proposition 4.4. *Let D, D' be two diagrams of the same knot. Then*

$$\Delta k_c = \frac{\Delta r - \Delta w}{2},$$

where w denotes the writhe, r denotes the number of Seifert circles, and the prefixed Δ denotes the difference of the corresponding values of D, D'.

Thus we obtain a knot invariant:

Theorem 1. *For any knot K,*

$$\bar{s}_c(K) = 2k_c(D) + w(D) - r(D) + 1,$$

gives an invariant of K, where D is any diagram of K.

First we state some basic properties of k_c. The following ones can be proved by observations of the diagram with some elementary algebraic arguments.

Proposition 4.5. *For any knot diagram D*

$$0 \leq k_c(D) \leq n^-(D),$$

where $n^-(D)$ denotes the number of negative crossings. In particular if D is positive, then $k_c(D) = 0$.

Thus we may regard k_c as measuring the "non-positivity" of the diagram.

Proposition 4.6.

$$k_c(D) = k_c(-D).$$

The above proposition tells us that we may either use $[\alpha]$ or $[\beta]$ for the definition of $k_c(D)$.

Proposition 4.7. *Let D, D' be knot diagrams.*

1. $k_c(D \sqcup D') \geq k_c(D) + k_c(D')$.

2. $k_c(D\#D') \leq k_c(D \sqcup D') \leq k_c(D\#D') + 1$.

Moreover, if R is a PID and c is prime in R, then instead we have:

1' $k_c(D \sqcup D') = k_c(D) + k_c(D')$.

2' $k_c(D\#D') = k_c(D) + k_c(D')$ *or* $k_c(D) + k_c(D') - 1$.

Next we state some basic properties of $\bar{s}_c$. The following can be obtained immediately from the definition and the corresponding results for k_c.

Proposition 4.8. *Let K, K' be any two knot diagrams.*

1. $\bar{s}_c(\bigcirc) = 0$.

2. $\bar{s}_c(K) = \bar{s}_c(-K)$.

3. $\bar{s}_c(K \sqcup K') \geq \bar{s}_c(K) + \bar{s}_c(K') - 1$.

4. $\bar{s}_c(K\#K') = \bar{s}_c(K \sqcup K') \pm 1$.

Moreover if R is a PID and c is prime in R, then we have

3'. $\bar{s}_c(K \sqcup K') = \bar{s}_c(K) + \bar{s}_c(K') - 1$.

4'. $\bar{s}_c(K\#K') = \bar{s}_c(K) + \bar{s}_c(K')$ *or* $\bar{s}_c(K) + \bar{s}_c(K') - 2$.

Proposition 4.9.

$$\bar{s}_c(K) \equiv 0 \bmod 2.$$

Proposition 4.10. *Let K be a positive knot, and D be a positive diagram of L. Let S be the Seifert surface of K obtained by applying the Seifert's algorithm to D. Then*

$$\bar{s}_c(K) = 2g(S).$$

The above two propositions easily follows from

$$\chi(S) = 1 - 2g(S) = r(D) - n(D).$$

Remark 4.11. We have modded out the torsions so that Proposition 4.4 holds. In general, if there is a torsion, then c-divisibility is not additive under multiplying a power of c. Consider the case $(R, c) = (\mathbb{Z}, 2)$, $H = \mathbb{Z} \oplus \mathbb{Z}_2$, and $z = (2, 1)$. In this case $k_2(z) = 0$, but $2z = (4, 0)$ so $k_2(2^1 z) = 2$. The *c-invariant* of a transverse link is introduced by Collari in [1], whose definition is similar to $k_c(D)$ but it is defined without modding out the torsions.

5 Behavior of $\bar{s}_c$ under cobordisms

Let K, K' be two knots in $\mathbb{R}^3$ and D, D' be diagrams of K, K' respectively. Let $S \subset \mathbb{R}^3 \times [0,1]$ be an (oriented smooth) cobordism between K and K' with $\partial S = (-K) \times \{0\} \cup K' \times \{1\}$. By following the arguments of [2, Section 6.3] and [12, Section 4], we define a homomorphism

$$\phi : H_c(D; R) \to H_c(D'; R)$$

as follows. By modifying S by a small isotopy, we may assume that S decomposes as a union of elementary cobordisms, such that for each $t \in [0,1]$ the section $S \cap (\mathbb{R}^3 \times \{t\})$ is a link and its projection onto the plane is regular, except for finitely many t's. Decompose $S = \bigcup_{i=0}^{N-1} T_i$ so that each $T_i = S \cap (\mathbb{R}^3 \times [t_i, t_{i+1}])$ corresponds to a Reidemeister move or a Morse move. Denote by D_i the projection of the link $S \cap (\mathbb{R}^3 \times \{t_i\})$. Each T_i gives a homomorphism $\phi_i : H_c(D_i) \to H_c(D_{i+1})$, namely if T_i corresponds to a Reidemeister move then ϕ_i is the isomorphism ρ given in Proposition 3.3, and if T_i corresponds to a Morse move, then ϕ_i is induced from the chain map given by the corresponding operation of the Frobenius algebra A. Define ϕ by the composition of all ϕ_i's. The following is a generalization of [12, Proposition 4.1] and [11, Proposition 3.2].

Proposition 5.1. *If S is a connected cobordism between K and K', then ϕ maps*

$$\phi[\alpha(D)] = \pm c^l[\alpha(D')], \quad l = \frac{1}{2}(-\Delta r + \Delta w - \chi(S)),$$

where the prefixed Δ denotes the difference of the corresponding values of D, D'.

Proof. This is proved by carefully inspecting the successive images of $[\alpha(D)]$ under the maps ϕ_i. Figure 4 depicts the schematic picture. Here we may assume that c is invertible, since the canonical map $H_c(D; R)_f \to H_c(D; R)_f \otimes R[c^{-1}] = H_c(D; R[c^{-1}])$ is injective.

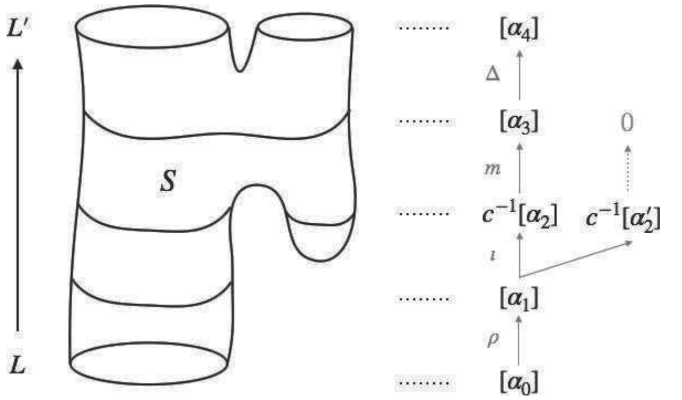

Figure 4: The cobordism map

□

We immediately obtain:

Proposition 5.2. *Proposition 5.1,*

$$|\bar{s}_c(K') - \bar{s}_c(K)| \leq -\chi(S).$$

From Proposition 5.2 we obtain properties of $\bar{s}_c$ that are common to s. From these properties, the Milnor conjecture can be reproved using each of $\bar{s}_c$.

Theorem 2. *1. $\bar{s}_c$ is a knot concordance invariant,*

2. $|\bar{s}_c(K)| \leq 2g_*(K)$ for any knot K, and

3. $\bar{s}_c(K) = 2g_*(K) = 2g(K)$ if K is positive,
where g_* is the slice genus, and g is the ordinary genus of a knot.

Corollary 5.3 (The Milnor Conjecture [10])**.** *The slice genus and the unknotting number of the (p, q) torus knot are both equal to $(p-1)(q-1)/2$.*

6 Coincidence with s

Finally we prove that $\bar{s}_c$ coincides with s for the case $(R, c) = (\mathbb{Q}[h], h)$. We declare that h is an indeterminate of degree -2, so that the corresponding homology group $H_h(D; R)$ becomes bigraded. Recall that in the case of $(R, c) = (\mathbb{Q}, 2)$, Rasmussen called the corresponding classes $[\alpha], [\beta]$ the "canonical generators", from the fact that they form a basis and that they are invariant (up to unit) under the Reidemeister moves. We have seen that this does not hold for a general (R, c). However for $(R, c) = (\mathbb{Q}[h], h)$, by "normalizing" the two classes we obtain a pair of classes $[\zeta], [\zeta']$ that are reasonable to be called the "canonical generators" of $H_h(D; \mathbb{Q}[h])_f$.

Proposition 6.1. *There is a unique pair of classes $[\zeta], [\zeta'] \in H_h(D; \mathbb{Q}[h])_f$ such that:*

- *they form a basis of $H_h(D; \mathbb{Q}[h])_f$,*
- *they are (strictly) invariant under the Reidemeister moves, and,*
- *$[\alpha], [\beta]$ can be written as*

$$\begin{aligned} [\alpha] &= \quad h^k(\quad (h/2)[\zeta] + [\zeta']\ ) \\ [\beta] &= (-h)^k(-(h/2)[\zeta] + [\zeta']\ ) \end{aligned}$$

where $k = k_h(D)$.

Note that from the descriptions of $[\alpha], [\beta]$, it is obvious that those h-divisibility is given by k. Here we only state that the proof is non-constructive, and it depends on the algebraic property of $R = \mathbb{Q}[h]$. Also this result is only obtained for knots at the time of writing.

Denote by D^* the mirror image of D. There is a unimodular pairing

$$\langle -, - \rangle : C_h(D) \otimes C_h(D^*) \longrightarrow R$$

defined by the composition of the natural identification $T : C(D^*) \to C(D)^*$ (the dual complex of $C(D)$) and the standard pairing between $C(D)$ and $C(D)^*$. From a general argument of homological algebra, this descends to

$$\langle -, - \rangle : H_h(D)_f \otimes H_h(D^*)_f \longrightarrow R,$$

and is unimodular since $R = \mathbb{Q}[h]$ is a PID.

Lemma 6.2. *Let* $(\alpha, \beta) = (\alpha(D), \beta(D))$ *and* $(\alpha^*, \beta^*) = (\alpha(D^*), \beta(D^*))$. *Then*

$$\left\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \right\rangle = h^{r(D)} \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

Proof. From

$$\begin{aligned} \mathbf{a} &= X + (h/2)1, \quad & \mathbf{b} &= X - (h/2)1 \\ T(\mathbf{a}) &= 1^* + (h/2)X^*, \quad & T(\mathbf{b}) &= 1^* - (h/2)X^* \end{aligned}$$

we have:

$$\left\langle \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}, \begin{pmatrix} T(\mathbf{a}) & T(\mathbf{b}) \end{pmatrix} \right\rangle = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}$$

The result follows since the Seifert circles of D and D^* are identical, and from the construction of the cycles α, β. □

Proposition 6.3 (Mirror formula)**.**

$$k_h(D) + k_h(D^*) = r(D) - 1.$$

Proof. With the description of Proposition 6.1,

$$\begin{aligned} &\left\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \right\rangle \\ &= h^{k+k'} \begin{pmatrix} h/2 & 1 \\ \mp h/2 & \pm 1 \end{pmatrix} \left\langle \begin{pmatrix} \zeta \\ \zeta' \end{pmatrix}, \begin{pmatrix} \zeta^*, \zeta'^* \end{pmatrix} \right\rangle \begin{pmatrix} h/2 & \mp h/2 \\ 1 & \pm 1 \end{pmatrix} \end{aligned}$$

Since the pairing is unimodular, the middle matrix on the right hand side must have unital determinant. By comparing the determinants on both sides we have

$$2r(D) = 2(k + k') + 2.$$

□

Proposition 6.4. *Let* D, D' *be knot diagrams.*

$$k_h(D \# D') = k_h(D) + k_h(D').$$

Proof. From Proposition 4.7 we have

$$k_h(D \# D') \leq k_h(D) + k_h(D').$$

Together with Proposition 6.3 we obtain the reverse inequality:

$$\begin{aligned} k_h(D\#D') &= -k_h((D\#D')^*) + r((D\#D')^*) - 1 \\ &\geq -(k_h(D^*) + k_h(D'^*)) + (r(D) + r(D') - 1) - 1 \\ &= k_h(D) + k_h(D'). \end{aligned}$$

□

Proposition 6.5. $\bar{s}_h$ *defines a homomorphism from the concordance group of knots in* S^3 *to* $2\mathbb{Z}$*.*

Proof. Immediate from Proposition 6.3, 6.4. □

Now we are ready to prove the main statement:

Theorem 3. *For any knot* K*,*

$$s(K) = \bar{s}_h(K; \mathbb{Q}[h]).$$

Proof. Since both s and $\bar{s}_h$ changes sign by mirroring the knot, it suffices to prove the inequality

$$s(K) \geq \bar{s}_h(K).$$

Let α, α_h be the α-cycles of D in $C_{Lee}(D;\mathbb{Q}) = C_2(D;\mathbb{Q})$ and in $C_h(D;\mathbb{Q}[h])$ respectively. Then

$$\pi : h \mapsto 2$$

maps $\pi(\alpha_h) = \alpha$. Let $[\alpha_h] = h^k[\alpha'_h]$ with maximal k. Obviously

$$\mathrm{qdeg}_h[\alpha'_h] = 2k + \mathrm{qdeg}_h[\alpha_h] = 2k + w(D) - r(D).$$

Recall that $H_h(D;\mathbb{Q}[h])$ is bigraded, whereas $H_{Lee}(D;\mathbb{Q})$ is only filtered. Since π is q-degree non decreasing, we have

$$\begin{aligned} s(K) &= \mathrm{qdeg}[\alpha] + 1 \\ &= \mathrm{qdeg}(\pi_*[\alpha_h]) + 1 \\ &= \mathrm{qdeg}(\pi_*[\alpha'_h]) + 1 \\ &\geq \mathrm{qdeg}_h[\alpha'_h] + 1 \\ &= \bar{s}_h(K; \mathbb{Q}[h]). \end{aligned}$$

□

Remark 6.6. There is a well known lower bound for s ([14, Lemma 1.3])

$$s(K) \geq w(D) - r(D) + 1,$$

so we see that $2k_h(D)$ gives the correction term of the inequality.

Corollary 6.7.

$$s(K) = \mathrm{qdeg}_h[\zeta] - 1 = \mathrm{qdeg}_h[\zeta'] + 1.$$

Proof. Obvious from Theorem 3 and the description of $[\alpha]$ given in Proposition 6.1. □

Compared to the definition of s given by the filtration of $\mathbb{Q}$-Lee homology, this gives a direct characterization of s by the canonical generators $[\zeta], [\zeta']$.

Remark 6.8. The above discussion works verbatim for any $(R, c) = (F[h], h)$ when F is a field of $\operatorname{char} F \neq 2$. On the other hand there is a generalization of s over an arbitrary field F [8]. Theorem 3 generalizes as

$$s(K; F) = \bar{s}_h(K; F[h]), \quad \operatorname{char} F \neq 2.$$

We end this article with some open questions.

Question 6.9. Is there any (R, c) such that $\bar{s}_c(-; R)$ is distinct from s? Or are they all equal?

It is a famous open question whether there exists any F such that $s(-; F)$ is distinct from $s = s(-; \mathbb{Q})$ ([8, Question 6.1]). Theorem 3 implies that if Question 6.9 is solved affirmatively, then all $s(-; F)$ are equal among fields F of $\operatorname{char} F \neq 2$.

Remark 6.10. As for $F = \mathbb{F}_2$, Seed showed by direct computation that $K = K14n19265$ has $s(K; \mathbb{Q}) = 0$ but $s(K; \mathbb{F}_2) = -2$ (see [8, Remark 6.1]).

Question 6.11. Does the (refined) canonical generators $[\zeta], [\zeta']$ exist for a general (R, c)? Is there any geometric explanation for these classes?

References

[1] Carlo Collari. Transverse invariants from Khovanov-type homologies. *J. Knot Theory Ramifications*, 28(1):1950012, 37, 2019.

[2] Mikhail Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000.

[3] Mikhail Khovanov. Link homology and Frobenius extensions. *Fund. Math.*, 190:179–190, 2006.

[4] P. B. Kronheimer and T. S. Mrowka. Gauge theory for embedded surfaces. I. *Topology*, 32(4):773–826, 1993.

[5] P. B. Kronheimer and T. S. Mrowka. Gauge theory and Rasmussen's invariant. *J. Topol.*, 6(3):659–674, 2013.

[6] Eun Soo Lee. An endomorphism of the Khovanov invariant. *Adv. Math.*, 197(2):554–586, 2005.

[7] Lukas Lewark. *The Rasmussen invariant of arborescent and of mutant links.* PhD thesis, Master thesis, ETH Zürich, 2009.

[8] Robert Lipshitz and Sucharit Sarkar. A refinement of Rasmussen's S-invariant. *Duke Math. J.*, 163(5):923–952, 2014.

[9] Marco Mackaay, Paul Turner, and Pedro Vaz. A remark on Rasmussen's invariant of knots. *J. Knot Theory Ramifications*, 16(3):333–344, 2007.

[10] John Milnor. *Singular points of complex hypersurfaces.* Annals of Mathematics Studies, No. 61. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968.

[11] Jacob Rasmussen. Khovanov's invariant for closed surfaces. *arXiv Mathematics e-prints*, page math/0502527, February 2005.

[12] Jacob Rasmussen. Khovanov homology and the slice genus. *Invent. Math.*, 182(2):419–447, 2010.

[13] Taketo Sano. Divisibility of Lee's class and its relation with Rasmussen's invariant. *arXiv e-prints*, page arXiv:1812.10258, Dec 2018.

[14] Alexander N. Shumakovitch. Rasmussen invariant, slice-Bennequin inequality, and sliceness of knots. *J. Knot Theory Ramifications*, 16(10):1403–1412, 2007.

[15] S. Wehrli. A spanning tree model for Khovanov homology. *J. Knot Theory Ramifications*, 17(12):1561–1574, 2008.

Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba Meguro-ku Tokyo 153-8914
JAPAN
E-mail address: `tsano@ms.u-tokyo.ac.jp`

東京大学・数理科学研究科　佐野 岳人