Local Moves Generating Writhe Polynomials of Virtual Knots

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1 Introduction

This paper is a résumé of the paper [8]. The odd writhe $J(K)$ is one of fundamental invariants of a virtual knot $K$ (cf. [5]). In [10], Taniguchi and the third author introduced local moves called $\Xi$-moves and proved that $\Xi$-moves correspond to the odd writhe; namely, two virtual knots have the same odd writhe if and only if they are related by a finite sequence of $\Xi$-moves.

On the other hand, there is another invariant of a virtual knot $K$ called the writhe polynomial $W_K(t)$ (cf. [1, 6, 10]). The writhe polynomial is stronger than the odd writhe in the sense that $J(K) = -W_K(-1)/2$ holds. Therefore, it is natural to ask which local moves correspond to the writhe polynomial.

In Section 2, we introduce local moves for virtual knots and links called shell moves. In Section 3, we prove that the writhe polynomial correspond to shell moves. In Section 4, we study which invariants of a 2-component virtual link correspond to shell moves.

2 Gauss diagrams

A Gauss diagram consists of a finite number of oriented circles and oriented, signed chords equipped with signs of endpoints as shown in Figure 1.

![Figure 1: Self- and nonself-chords of a Gauss diagram](image)

A $\mu$-component virtual link is an equivalence class of Gauss diagrams with $\mu$ circles up to Reidemeister moves R1–R3 (cf. [4, 5]). See Figure 2. If $\mu = 1$, it is called a virtual knot.
The shells for a chord $\gamma$ are parallel self-chords which are oriented with respect to the sign of the endpoint of $\gamma$ as shown in Figure 3. We remark that a shell in this paper is a special case of an anklet in [7].

The shell moves $S_1$ and $S_2$ are local moves defined by using Gauss diagrams as shown in Figure 4. Precisely, an $S_1$-move slides a shell along a chord to the opposite side with the same sign, and an $S_2$-move changes the position of the adjacent endpoints of chords with making a pair of shells with respect to the signs of the endpoints.

We say that two Gauss diagram are $S$-equivalent if they are related by a finite sequence of Reidemeister moves and shell moves, and two virtual links are $S$-equivalent if their Gauss diagrams are S-equivalent.
Lemma 2.1. If two Gauss diagrams are related by a deformation (1) or (2) as shown in Figure 5, then they are $S$-equivalent. □

Figure 5: $S$-equivalent Gauss diagrams in Lemma 2.1

Lemma 2.2. If two Gauss diagrams are related by a deformation (1)-(4) as shown in Figure 6, then they are $S$-equivalent. Here, $P$ and $Q$ are portions of whole chords. □

Figure 6: $S$-equivalent Gauss diagrams in Lemma 2.2

Let $G$ be a Gauss diagram with $\mu$ circles $C_1, \ldots, C_\mu$. For $n \in \mathbb{Z}$ and $1 \leq i \neq j \leq \mu$, we define the $n$-snail of type $i$ and the $n$-snail of type $(i,j)$ to be the portion of chords as shown in Figure 7. Then we have the following.

Lemma 2.3. If two Gauss diagrams are related by a deformation (1)-(4) as shown in Figure 8, then they are $S$-equivalent. □

By using Lemmas 2.1–2.3, we have the following standard form of a Gauss diagram up to $S$-moves.
Proposition 2.4. Any Gauss diagram of an oriented $\mu$-component virtual link is $S$-equivalent to a Gauss diagram $G$ with $\mu$ circles $C_1, \ldots, C_\mu$ which satisfies the following conditions. Figure 9 shows the case $\mu = 3$.

(i) The chords of $G$ form a finite number of snails.
(ii) There is an arc $\alpha_i$ on each $C_i$ such that all snails of type $i$ spans $\alpha_i$.
(iii) All snails of type $(i, j)$ spans $(C_i \setminus \alpha_i) \cup (C_j \setminus \alpha_j)$ in parallel.
(iv) There is no snails $\pm S_i(0)$ or $\pm S_i(1)$ for any $i$.
(v) There is no pair of snails $+S_i(n)$ and $-S_i(n)$ for any $i$ and $n$.
(vi) There is no pair of snails $+S_{ij}(n)$ and $-S_{ij}(n)$ for any $i \neq j$ and $n$. $\square$

3 The case $\mu = 1$

In this section, we consider an oriented virtual knot $K$ and its Gauss diagram $G$ with a circle $C$. For portions $P_1, \ldots, P_k$ of chords, we denote by $\left( \sum_{i=1}^k P_i \right)$ the Gauss diagram as shown in Figure 10. For integers $a, n \in \mathbb{Z}$, $aS(n)$ denotes the concatenation of $|a|$ copies of $\varepsilon S(n)$, where $\varepsilon$ is the sign of $a$.

Lemma 3.1. Any Gauss diagram of $K$ is $S$-equivalent to $\left( \sum_{n \neq 0, 1} a_n S(n) \right)$ for some $a_n \in \mathbb{Z}$. $\square$
A chord $\gamma$ divides the circle $C$ into two arcs. Let $\alpha$ be the one oriented from the initial to the terminal endpoint of $\gamma$. We define the index of $\gamma$ to be the sum of signs of endpoints of chords on $\alpha$. For each $n \neq 0$, the sum of signs of all chords whose index is equal to $n$ defines an invariant of $K$. It is called the $n$-writhe of $K$ and denoted by $J_n(K)$. The writhe polynomial is defined by

$$W_K(t) = \sum_{n \neq 0} J_n(K) (t^n - 1) \in \mathbb{Z}[t, t^{-1}].$$

Refer to [1, 2, 6, 10] for more details.

**Lemma 3.2.** Let $K$ be an oriented virtual knot.

(i) The writhe polynomial $W_K(t)$ is invariant under $S$-moves.

(ii) If $K$ is presented by a Gauss diagram given in Lemma 3.1, then we have

$$W_K(t) = \sum_{n \neq 0, 1} a_n t^n - \left( \sum_{n \neq 0, 1} na_n \right) t + \sum_{n \neq 0, 1} (n - 1) a_n.$$

By Lemma 3.1 and Lemma 3.2(ii), we have the following.

**Proposition 3.3.** Let $K$ and $K'$ be oriented virtual knots. If $W_K(t) = W_{K'}(t)$ holds, then $K$ and $K'$ are $S$-equivalent.
Therefore the following holds by Lemma 3.2(i) and Proposition 3.3.

**Theorem 3.4.** For two oriented virtual knots \( K \) and \( K' \), the following are equivalent.

(i) \( W_K(t) = W_{K'}(t) \).

(ii) \( K \) and \( K' \) are related by a finite sequence of shell moves.

\( \square \)

4 The case \( \mu = 2 \)

In this section, we consider an oriented 2-component virtual link \( L = K_1 \cup K_2 \) and its Gauss diagram \( G \) with a pair of circles \( C_1 \) and \( C_2 \). By Proposition 2.4, we have the following.

**Lemma 4.1.** Any Gauss diagram of \( L \) is \( S \)-equivalent to a Gauss diagram

\[
\left( \sum_{n \neq 0,1} a_n S_1(n), \sum_{n \neq 0,1} b_n S_2(n); \sum_{m \in \mathbb{Z}} c_m S_{12}(m), \sum_{m \in \mathbb{Z}} d_m S_{21}(m) \right)
\]

for some integers \( a_n, b_n \) (\( n \neq 0,1 \)) and \( c_m, d_m \) (\( m \in \mathbb{Z} \)) as shown in Figure 11. Here, the entries present the concatenations of snails of type 1, 2, (1, 2), and (2, 1), respectively. \( \square \)

![Figure 11: A Gauss diagram of an oriented 2-component virtual link](image)

For \((i, j) = (1, 2) \) or \((2, 1)\), the \((i, j)\)-linking number of \( L \), denoted by \( \text{Lk}(K_i, K_j) \), is defined to be the sum of signs of all nonself-chords oriented from \( C_i \) to \( C_j \). The virtual linking number of \( L \) is defined by \( \lambda(L) = \text{Lk}(K_1, K_2) - \text{Lk}(K_2, K_1) \) [9] (cf. [3]). It is easy to see that \( \text{Lk}(K_1, K_2), \text{Lk}(K_2, K_1) \), and \( \lambda(L) \) are invariant under \( S \)-moves.

If \( \lambda(L) < 0 \), then by switching the roles of \( K_1 \) and \( K_2 \), the case reduces to \( \lambda(L) > 0 \). In what follows, we may assume that \( \lambda(L) \geq 0 \). We denote \( \lambda(L) \) by \( \lambda \) for simplicity. The following propositions give standard forms of \( L \) up to \( S \)-equivalence.

**Proposition 4.2.** Let \( G \) be a Gauss diagram of \( L \).

(i) If \( \lambda \geq 1 \), then

\[
G \sim \left( \sum_{n \neq 0,1,-\lambda,-\lambda+1} a_n S_1(n), \sum_{n \neq 0,1,\lambda,\lambda+1} b_n S_2(n) \right)
\]
\[ \sum_{m=0}^{\lambda-1} c_m S_{12}(p+m), \sum_{m=0}^{\lambda-1} d_m S_{21}(-p-m) \]

for some integers \( a_n (n \neq 0, 1, -\lambda, -\lambda + 1), b_n (n \neq 0, 1, \lambda, \lambda + 1), c_m, d_m (0 \leq m \leq \lambda - 1) \), and \( p \).

(ii) In particular, if \( \lambda = 1 \), then

\[
G \sim \left( \sum_{n \neq 0, 1, -1} a_n S_1(n), \sum_{n \neq 0, 1, 2} b_n S_2(n); c_0 S_{12}(0), d_0 S_{21}(0) \right)
\]

for some integers \( a_n (n \neq 0, 1, -1), b_n (n \neq 0, 1, 2), c_0, \) and \( d_0 \).

**Proposition 4.3.** We have the following \( S \)-equivalent Gauss diagrams.

(i) If \( \lambda = 0 \), then

\[
\left( P, Q; \sum_{m \in \mathbb{Z}} c_m S_{12}(m), \sum_{m \in \mathbb{Z}} d_m S_{21}(m) \right) \sim \left( P, Q; \sum_{m \in \mathbb{Z}} c_m S_{12}(m+k), \sum_{m \in \mathbb{Z}} d_m S_{21}(m-k) \right)
\]

for any \( k \in \mathbb{Z} \).

(ii) If \( \lambda \geq 2 \), then

\[
\left( P, Q; \sum_{m=0}^{\lambda-1} c_m S_{12}(p+m), \sum_{m=0}^{\lambda-1} d_m S_{21}(-p-m) \right) \sim \left( P, Q; \sum_{m=0}^{\lambda-1} c'_m S_{12}(p'+m), \sum_{m=0}^{\lambda-1} d'_m S_{21}(-p'-m) \right),
\]

where

\[
\begin{array}{l}
\{(c'_0, \ldots, c'_{\lambda-k-1}, c'_{\lambda-k}, \ldots, c'_{\lambda-1}) = (c_k, \ldots, c_{\lambda-1}, c_0, \ldots, c_{k-1}),
\\
(d'_0, \ldots, d'_{\lambda-k-1}, d'_{\lambda-k}, \ldots, d'_{\lambda-1}) = (d_k, \ldots, d_{\lambda-1}, d_0, \ldots, d_{k-1}),
\end{array}
\]

and \( p' = p + k - \sum_{i=0}^{k-1} (c_i - d_i) \) for any \( k \) with \( 1 \leq k \leq \lambda - 1 \). \( \square \)

A self-chord \( \gamma \) spanning \( C_i \) divides \( C_i \) into two arcs. Let \( \alpha \) be the one oriented from the initial to the terminal endpoint of \( \gamma \) in \( C_i \). We define the index of \( \gamma \) in \( G \) to be the sum of signs of endpoints of self- and nonself-chords on \( \alpha \). For \( n \in \mathbb{Z} \) and \( i = 1, 2 \), the sum of signs of all self-chords whose indices are equal to \( n \) defines the invariants \( J_n(K_1; L) \) for \( n \neq 0, -\lambda \) and \( J_n(K_2; L) \) for \( n \neq 0, \lambda \). These are called the \( n \)-writhes of \( K_1 \) and \( K_2 \) in \( L \), respectively.
On the other hand, for nonself-chords $\gamma$ and $\gamma_0$, the relative index of $\gamma$ with respect to $\gamma_0$ to be the index of $\gamma$ in the Gauss diagram obtained from $G$ by surgery along $\gamma_0$. For $n \in \mathbb{Z}$ and $(i, j) = (1, 2), (2, 1)$, let $J_n^{ij}(G; \gamma_0)$ denote the sum of signs of nonself-chords $\gamma$ oriented from $C_i$ to $C_j$ whose relative index with respect to $\gamma_0$ are equal to $n$. Put

$$F_{ij}(t; \gamma_0) = \sum_{n \in \mathbb{Z}} J_n^{ij}(G; \gamma_0) t^n.$$  

For an integer $s \geq 0$, let $\Lambda_s$ denote the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]/(t^s - 1)$. In particular, we have $\Lambda_0 = \mathbb{Z}[t, t^{-1}]$ and $\Lambda_1 = \mathbb{Z}$. We consider an equivalence relation on $\Lambda_s \times \Lambda_s$ such that $(f_1(t), g_1(t))$ and $(f_2(t), g_2(t))$ are equivalent if there is an integer $k$ with

$$f_2(t) = t^k f_1(t)$$

and

$$g_2(t) = t^{-k} g_1(t).$$

We denote by $[f(t), g(t)]$ the equivalence class represented by $\binom{f(t), g(t)}$, and by $\Gamma(s)$ the set of such equivalence classes.

Then the equivalence class $[F_{12}(t; \gamma_0), F_{21}(t; \gamma_0)] \in \Gamma(\lambda)$ defines the invariant of $L$ (cf. [2]). We call it the linking class of $L$ and denote it by $F(L)$. In particular, $F(L) = (\text{Lk}(K_1, K_2), \text{Lk}(K_2, K_1)) \in \mathbb{Z} \times \mathbb{Z}$ for $\lambda = 1$. Then by Propositions 4.2 and 4.3, we have the following.

**Theorem 4.4.** Let $L = K_1 \cup K_2$ and $L' = K'_1 \cup K'_2$ be oriented 2-component virtual links with $\lambda = \lambda' = 0$. Then $L$ and $L'$ are related by a finite sequence of shell moves if and only if

(i) $J_n(K_1; L) = J_n(K'_1; L')$ for any $n \neq 0, 1$,

(ii) $J_n(K_2; L) = J_n(K'_2; L')$ for any $n \neq 0, 1$,

(iii) $F(L) = F(L')$.

$\square$

**Theorem 4.5.** Let $L = K_1 \cup K_2$ and $L' = K'_1 \cup K'_2$ be oriented 2-component virtual links with $\lambda = \lambda' = 1$. Then $L$ and $L'$ are related by a finite sequence of shell moves if and only if

(i) $J_n(K_1; L) = J_n(K'_1; L')$ for any $n \neq 0, 1, -1$,

(ii) $J_n(K_2; L) = J_n(K'_2; L')$ for any $n \neq 0, 1, 2$,

(iii) $F(L) = F(L')$.

$\square$

**Theorem 4.6.** Let $L = K_1 \cup K_2$ and $L' = K'_1 \cup K'_2$ be oriented 2-component virtual links with $\lambda = \lambda' \geq 2$. Then $L$ and $L'$ are related by a finite sequence of shell moves if and only if
(i) \( J_n(K_1; L) = J_n(K_1'; L') \) for any \( n \neq 0, 1, -\lambda, -\lambda + 1 \),

(ii) \( J_n(K_2; L) = J_n(K_2'; L') \) for any \( n \neq 0, 1, \lambda, \lambda + 1 \),

(iii) \( F(L) = F(L') \), and

(iv) \( J_1(K_1; L) + J_{-\lambda + 1}(K_1; L) + J_1(K_2; L) + J_{\lambda + 1}(K_2; L) \)

\[ = J_1(K_1'; L') + J_{-\lambda + 1}(K_1'; L') + J_1(K_2'; L') + J_{\lambda + 1}(K_2'; L') \].

\[ \square \]

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