SOME EXTENSIONS

OF

KANTOROVICH TYPE INEQUALITIES

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Abstract

We consider Kantorovich type inequalities for bounded strictly positive operators on a Hilbelt space. Mićić-Pečarić-Seo recently obtained Kantorovich type inequalities between A^q and B^p for the case $p > 1, q > 1$ under the assumption $A \geq B$. We extend it to more generalized Kantorovich type inequalities between $(Tx, x)^q$ and $(T^p x, x)$ for the case (a) $p > 1, q > 1$, (b) $p < 0, q < 0$, (c) $0 < p < 1, 0 < q < 1$. We further prove that these results are applied to the case chaotic order.

1 Introduction

This report is based on the following papers:

- T.Furuta and M.Giga, A complementary result of Kantorovich type order preserving inequalities by Mićić-Pečarić-Seo, Linear Alg. and Its Appl., **369**(2003), 27-40.

- M.Giga, Kantorovich type inequalities for $1 > p > 0$, J.Ineq. Pure and Applied Math., (to appear).

In this report, capital letters T, A, B are bounded linear operators on a Hilbert space. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

Theorem A. (Hölder-McCarthy inequality)

Let A be a positive operator on a Hilbert space H. For any unit vector x,

$$\text{(a)} \quad (A^\lambda x, x) \geq (Ax, x)^\lambda \quad \textit{for any } \lambda > 1,$$
$$\text{(b)} \quad (A^\lambda x, x) \geq (Ax, x)^\lambda \quad \textit{for any } \lambda < 0 \quad \textit{if } A \textit{ is invertible,}$$
$$\text{(c)} \quad (A^\lambda x, x) \leq (Ax, x)^\lambda \quad \textit{for any } \lambda \in [0, 1].$$

In Theorem A, (a),(b) and (c) are mutually equivalent.

Theorem B. (Kantorovich inequality)

Let A be a positive operator on a Hilbert space H such that $MI \geq A \geq mI > 0$. For any unit vector x ,

$$(Ax, x)^{-1} \leq (A^{-1}x, x) \leq \frac{(m+M)^2}{4mM}(Ax, x)^{-1}$$

$$(Ax, x)^2 \leq (A^2x, x) \leq \frac{(m+M)^2}{4mM}(Ax, x)^2.$$

The left hand side is Hölder-McCarthy inequality. The right hand side is a reverse inequality of Hölder-McCarthy inequality. The constant is interesting. It is a square of an arithemetic mean over a geometric mean.

Many mathematicians investigated Kantorovich inequality. Among others, there is a long research series of Mond-Pecaric, some of them are [10] and [11]. In [9], Mićić-Pečarić-Seo showed the two variable version of [Theorem 2.1, 3] where the variables are $p > 1, q > 1$. That is a Kantorovich type inequality concerning ordered operators A and B

In this report we extend the Kantorovich type inequality of two variable version to the inequality on (a) $p > 1, q > 1$, (b) $p < 0, q < 0$ and (c) $0 < p < 1, 0 < q < 1$ by simple proof [Theorem 2.1, Theorem 2.2].

Secondly we apply the Kantorovich type inequality to the inequality concerning ordered operators A and B, whose variables are (a) $p > 1, q > 1$, (b) $p < 0, q < 0$, (c) $0 < p < 1, 0 < q < 1$ [Corollary 3.1].

Furthermore we apply these results to the Kantorovich type inequality of chaotic order version[Theorem 4.1, Corollary 4.2].

2 Extended Kantorovich type inequalities

We state Theorem 2.1 and Theorem 2.2 that are our main results. Theorem 2.2 is more general. We obtain Theorem 2.2 by using that t^p ($p > 1, p < 0$) is a convex function and t^p ($0 < p < 1$) is a concave function.

Theorem 2.1 *Let T be a strictly positive operator on a Hilbert space H such that $MI \geq T \geq mI > 0$. Then for any unit vector x,*

(a) *If $p > 1$ and $q > 1$,* (b) *If $p < 0$ and $q < 0$,*

$$K(m, M, p, q)(Tx, x)^q \geq (T^px, x) \geq (Tx, x)^p.$$

(c)*If $0 < p < 1$ and $0 < q < 1$,*

$$K(m, M, p, q)(Tx, x)^q \leq (T^px, x) \leq (Tx, x)^p.$$

$$K(m, M, p, q) = \begin{cases} K^{(1)}(m, M, p, q) & \text{if case 1 holds} \\ m^{p-q} & \text{if case 2 holds} \\ M^{p-q} & \text{if case 3 holds,} \end{cases}$$

where Kantorovich constant $K^{(1)}(m, M, p, q)$ *is*

$$K^{(1)}(m, M, p, q) = \frac{(mM^p - Mm^p)}{(q-1)(M-m)} \left\{ \frac{(q-1)(M^p - m^p)}{q(mM^p - Mm^p)} \right\}^q.$$

$$\begin{cases} case\ 1.\ m^{p-1}q \begin{cases} \leq \dfrac{M^p - m^p}{M - m} \leq M^{p-1}q & \text{(a),(b)} \\[2mm] \geq \dfrac{M^p - m^p}{M - m} \geq M^{p-1}q & \text{(c)} \end{cases} \\[6mm] case\ 2.\ m^{p-1}q \begin{cases} > \dfrac{M^p - m^p}{M - m} & \text{(a),(b)} \\[2mm] < \dfrac{M^p - m^p}{M - m} & \text{(c)} \end{cases} \\[6mm] case\ 3.\ M^{p-1}q \begin{cases} < \dfrac{M^p - m^p}{M - m} & \text{(a),(b)} \\[2mm] > \dfrac{M^p - m^p}{M - m} & \text{(c).} \end{cases} \end{cases}$$

Theorem 2.2 *Let T be a strictly positive operator on a Hilbert space H such that $MI \geq T \geq mI > 0$. Also let $f(t)$ be a real valued continuous function on $[m, M]$. Then for any unit vector x,*

(a) *If $q > 1$ and f is convex,* (b)*If $q < 0$ and f is convex,*

$$K(m, M, f, q)(Tx, x)^q \geq (f(T)x, x) \geq f((Tx, x)).$$

(c) *If $0 < q < 1$ and f is concave,*

$$K(m, M, f, q)(Tx, x)^q \leq (f(T)x, x) \leq f((Tx, x)).$$

$$K(m, M, f, q) = \begin{cases} K^{(1)}(m, M, f, q) & \text{if case 1 holds} \\[2mm] \dfrac{f(m)}{m^q} & \text{if case 2 holds} \\[2mm] \dfrac{f(M)}{M^q} & \text{if case 3 holds,} \end{cases}$$

where Kantorovich constant $K^{(1)}(m, M, p, q)$ *is*

$$K^{(1)}(m, M, f, q) = \frac{(mf(M) - Mf(m))}{(q-1)(M-m)} \left(\frac{(q-1)(f(M) - f(m))}{q(mf(M) - Mf(m))} \right)^q.$$

(a) $\begin{cases} case\ 1.\ f(M) > f(m),\ \dfrac{f(M)}{M} > \dfrac{f(m)}{m},\ \dfrac{f(m)}{m}q \leq \dfrac{f(M)-f(m)}{M-m} \leq \dfrac{f(M)}{M}q \\[3mm] case\ 2.\ f(M) > f(m),\ \dfrac{f(M)}{M} > \dfrac{f(m)}{m},\ \dfrac{f(m)}{m}q > \dfrac{f(M)-f(m)}{M-m} \\[3mm] case\ 3.\ f(M) > f(m),\ \dfrac{f(M)}{M} > \dfrac{f(m)}{m},\ \dfrac{f(M)}{M}q < \dfrac{f(M)-f(m)}{M-m}, \end{cases}$

$$
\text{(b)} \begin{cases}
\textit{case 1.} \ \ f(M) < f(m), \ \dfrac{f(M)}{M} < \dfrac{f(m)}{m}, \ \dfrac{f(m)}{m}q \le \dfrac{f(M)-f(m)}{M-m} \le \dfrac{f(M)}{M}q \\[2mm]
\textit{case 2.} \ \ f(M) < f(m), \ \dfrac{f(M)}{M} < \dfrac{f(m)}{m}, \ \dfrac{f(m)}{m}q > \dfrac{f(M)-f(m)}{M-m} \\[2mm]
\textit{case 3.} \ \ f(M) < f(m), \ \dfrac{f(M)}{M} < \dfrac{f(m)}{m}, \ \dfrac{f(M)}{M}q < \dfrac{f(M)-f(m)}{M-m},
\end{cases}
$$

$$
\text{(c)} \begin{cases}
\textit{case 1.} \ \ f(M) > f(m), \ \dfrac{f(M)}{M} < \dfrac{f(m)}{m}, \ \dfrac{f(m)}{m}q \ge \dfrac{f(M)-f(m)}{M-m} \ge \dfrac{f(M)}{M}q \\[2mm]
\textit{case 2.} \ \ f(M) > f(m), \ \dfrac{f(M)}{M} < \dfrac{f(m)}{m}, \ \dfrac{f(m)}{m}q < \dfrac{f(M)-f(m)}{M-m} \\[2mm]
\textit{case 3.} \ \ f(M) > f(m), \ \dfrac{f(M)}{M} < \dfrac{f(m)}{m}, \ \dfrac{f(M)}{M}q > \dfrac{f(M)-f(m)}{M-m}.
\end{cases}
$$

Proof of Theorem 2.2 We show the proof of Theorem 2.2 for (a) and (b). We can prove (c) by the parallel argument.

Let $h(t, k, K)$ be defined on $(0, \infty)$ for $q \ (\ne 0, 1)$ and $M > m > 0$.

$$
h(t, k, K) = \frac{1}{t^q}\left(k + \frac{K - k}{M - m}(t - m)\right).
$$

It has the following upper bound on $[m, M]$:

$$
BD_+(m, M, k, K, q) = \\
\begin{cases}
\dfrac{(mK - Mk)}{(q - 1)(M - m)}\left(\dfrac{(q - 1)(K - k)}{q(mK - Mk)}\right)^q & \text{if case 1 holds} \\[3mm]
\dfrac{k}{m^q} & \text{if case 2 holds} \\[3mm]
\dfrac{K}{M^q} & \text{if case 3 holds.}
\end{cases}
\tag{2.1}
$$

These are derived by an easy differential calculus, where $h(t, k, K)$ is a function for t.

As $f(t)$ is a real valued continuous convex function on $[m, M]$, we have

$$
f(t) \le f(m) + \frac{f(M) - f(m)}{M - m}(t - m).
$$

We apply the standard operational calculus of positive operator T to it. Since $M \ge (Tx, x) \ge m$, we get that for every unit vector x

$$
(f(T)x, x) \le f(m) + \frac{f(M) - f(m)}{M - m}((Tx, x) - m).
$$

Multiplying $(Tx, x)^{-q}$ on both sides, we have

$$
(Tx, x)^{-q}(f(T)x, x) \le h((Tx, x), f(M), f(m)).
$$

Then we obtain

$$(f(T)x,x) \leq \left[\max_{m \leq (Tx,x) \leq M} h((Tx,x),f(M),f(m)) \right] (Tx,x)^q.$$

Thus Theorem 2.2 is proved.

The classification (2.1) in (a) is as follows. We consider the classification in (b) and (c) similarly.

$$(a) \begin{cases} case\ 1. & K > k, \quad \dfrac{K}{M} > \dfrac{k}{m}, \quad \dfrac{k}{m}q \leq \dfrac{K-k}{M-m} \leq \dfrac{K}{M}q \\[2mm] case\ 2. & K > k, \quad \dfrac{K}{M} > \dfrac{k}{m}, \quad \dfrac{k}{m}q > \dfrac{K-k}{M-m} \\[2mm] case\ 3. & K > k, \quad \dfrac{K}{M} > \dfrac{k}{m}, \quad \dfrac{K}{M}q < \dfrac{K-k}{M-m}. \end{cases} \qquad ///$$

The expression of Kantorovich constant $K^{(1)}(m,M,p,q)$ is derived from the constant in case 1 in upper bound of $h(t,k,K)$ (2.1). When case 2 or case 3 holds, the top of the graph of $h(t,k,K)$ is outside of the interval $[m,M]$.

3 Applications

By using Theorem 2.1 we obtain the following Corollary 3.1.

Corollary 3.1 *Let A, B be strictly positive operators on a Hilbert space H such that $M_1 I \geq A \geq m_1 I > 0$, $M_2 I \geq B \geq m_2 I > 0$ and also $A \geq B$.*

    (a) *if $p > 1$, $q > 1$, $K(m_2, M_2, p, q)A^q \geq B^p$*
    (b) *if $p < 0$, $q < 0$, $K(m_1, M_1, p, q)B^q \geq A^p$*
    (c) *if $0 < p < 1$, $0 < q < 1$, $K(m_1, M_1, p, q)B^q \leq A^p$.*

$$K(m_i, M_i, p, q) = \begin{cases} K^{(1)}(m_i, M_i, p, q) & \text{if case 1 holds} \\ m_i^{p-q} & \text{if case 2 holds} \\ M_i^{p-q} & \text{if case 3 holds,} \end{cases}$$

where Kantorovich constant $K^{(1)}(m_i, M_i, p, q)$ is

$$K^{(1)}(m_i, M_i, p, q) = \frac{(m_i M_i^p - M_i m_i^p)}{(q-1)(M_i - m_i)} \left\{ \frac{(q-1)(M_i^p - m_i^p)}{q(m_i M_i^p - M_i m_i^p)} \right\}^q.$$

The classification case 1,2 and 3 is similar to it in Theorem 2.1.

Proof of Corollary 3.1 We prove (a). (b) and (c) are proved similarly.

$$\begin{aligned} (B^p x, x) &\leq K(m_2, M_2, p, q)(Bx, x)^q \\ &\leq K(m_2, M_2, p, q)(Ax, x)^q \\ &\leq K(m_2, M_2, p, q)(A^q x, x). \end{aligned}$$

The first inequality is the result of (a) in Theorem 2.1, the second inequality is from the assumption and for the third one we derive it from (a) in Theorem A.   ///

We have an alternative proof of (a) in Corollary 3.1 by Mićić-Pečarić-Seo [9].

Proposition 3.2 *For every* p, q,

$$K^{(1)}\left(m, M, \frac{1}{2}-p, \frac{1}{2}-q\right) = (mM)^{q-p} K^{(1)}\left(m, M, \frac{1}{2}+p, \frac{1}{2}+q\right),$$

where $K^{(1)}(m, M, p, q)$ *is*

$$K^{(1)}(m, M, p, q) = \frac{(mM^p - Mm^p)}{(q-1)(M-m)}\left(\frac{(q-1)(M^p - m^p)}{q(mM^p - Mm^p)}\right)^q.$$

In particular, when $p = q$, $K^{(1)}\left(m, M, \frac{1}{2}-p, \frac{1}{2}-q\right)$ *and* $K^{(1)}\left(m, M, \frac{1}{2}+p, \frac{1}{2}+q\right)$ *are symmetric with respect to* $(p, q) = \left(\frac{1}{2}, \frac{1}{2}\right)$.

4   Applications to chaotic order

We consider applications of Theorem 2.1 and Corollary 3.1 to chaotic order.

Theorem 4.1 *Let* T *be a strictly positive operator on a Hilbert space* H *such that* $MI \geq T \geq mI > 0$ *and* $h = \frac{M}{m} > 1$. *Then for any unit vector* x,

$$S_h(m, M, p, q)\Delta_x(T^q) \geq (T^p x, x) \geq \Delta_x(T^p) \quad for\ p > 0\ and\ q > 0,$$

where $S_h(m, M, p, q)$ *and a determinant* $\Delta_x(T)$ *are*

$$S_h(m, M, p, q) = \begin{cases} m^{p-q} \dfrac{h^{\frac{q}{h^p-1}}}{e \log h^{\frac{q}{h^p-1}}} & if\ q \leq \dfrac{h^p - 1}{\log h} \leq qh^p \\ m^{p-q} & if\ \dfrac{h^p - 1}{\log h} \leq q \\ M^{p-q} & if\ qh^p \leq \dfrac{h^p - 1}{\log h}, \end{cases}$$

$$\Delta_x(T) = \exp(((\log T)x, x)).$$

Corollary 4.2 *Let* A, B *be strictly positive operators on a Hilbert space* H *such that* $MI \geq B \geq mI > 0$. *Then* $\log A \geq \log B$ *is equivalent to*

$$S_h(m, M, p, q)A^q \geq B^p \quad for\ p > 0\ and\ q > 0.$$

Theorem 4.1 and Corollary 4.2 are the chaotic order version of Theorem 2.1 and Corollary 3.1 respectively. The only if part of Corollary 4.2 is similarly proved as [14]. When $p = q$, $S_h(m, M, p, q)$ is called Specht ratio. We have an alternative proof of Corollary 4.2 by Mićić-Pečarić-Seo [9]. The following Proposition 4.3 gives the relation between Kantorovich constant $K^{(1)}(m, M, p, q)$ and the two variable version Specht ratio.

Proposition 4.3

$$\lim_{n\to\infty} K^{(1)}\left(1+\frac{\log m}{n}, 1+\frac{\log M}{n}, np, nq\right) = m^{p-q}\frac{h^{\frac{q}{h^{p}-1}}}{e\log h^{\frac{q}{h^{p}-1}}},$$

$$\textit{where } h = \frac{M}{m} > 1.$$

Proof of Theorem 4.1 We use the following formula.

$$\lim_{n\to\infty}\left(I + \frac{1}{n}\log X\right)^n = X \quad \text{for } X > 0.$$

Put $M_1 = I + \dfrac{\log M}{n}$, $T_1 = I + \dfrac{\log T}{n}$, and $m_1 = I + \dfrac{\log m}{n}$. Then $M_1 \geq T_1 \geq m_1 > 0$ holds for sufficiently large natural number n by the hypothesis $MI \geq T \geq mI > 0$.

$$\left(\left(I + \frac{\log T}{n}\right)x, x\right)^{nq} = \left(I + \frac{((\log T)x, x)}{n}\right)^{nq} \tag{4.1}$$

$$\longrightarrow \exp\{((\log T)x, x)q\} = \Delta_x(T^q) \quad \text{as } n \to \infty.$$

$$\left(\left(I + \frac{\log T}{n}\right)^{np}x, x\right) \longrightarrow (T^p x, x) \quad \text{as } n \to \infty. \tag{4.2}$$

Since $np > 1$, $nq > 1$ for sufficiently large n, by Theorem 2.1 (a),

$$K(m_1, M_1, np, nq)\left(\left(I + \frac{\log T}{n}\right)x, x\right)^{nq} \geq \left(\left(I + \frac{\log T}{n}\right)^{np}x, x\right)$$

$$\geq \left(\left(I + \frac{\log T}{n}\right)x, x\right)^{np}. \tag{4.3}$$

Using (4.1),(4.2) and Proposition 4.3 on (4.3), the proof of Theorem 4.1 is complete.   ///

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