数理解析研究所講究録 1359 巻 2004 年 38-45

Relations among operator orders and operator inequalities

東京理科大・理　柳田 昌宏 (Masahiro Yanagida)
Department of Mathematical Information Science,
Tokyo University of Science

東京理科大・理　伊藤 公智 (Masatoshi Ito)
Department of Mathematical Information Science,
Tokyo University of Science

神奈川大・工　山崎 丈明 (Takeaki Yamazaki)
Department of Mathematics,
Kanagawa University

1 The Furuta inequality and the chaotic order

In what follows, an operator means a bounded linear operator on a Hilbert space H and is denoted by a capital letter. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and also T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

We start this report with introduction of the following order preserving operator inequalities.

Theorem F (Furuta inequality [5]).

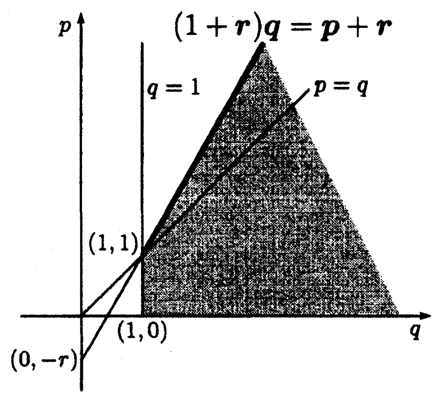
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $\quad (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $\quad (A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.



Theorem F yields the famous Löwner-Heinz theorem "*$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$*" by putting $r = 0$ in (i) or (ii) of Theorem F. An elementary one-page proof of Theorem F was given in [6]. It was shown in [15] that the domain of the parameters is the best possible in Theorem F.

The order defined by $\log A \geq \log B$ for $A, B > 0$ is called the chaotic order. The chaotic order is weaker than the usual order since $\log\cdot$ is an operator monotone function. The following characterization of the chaotic order is an application of Theorem F and an extension of a result in [1].

Theorem 1.A **([3][7]).** *Let $A, B > 0$. Then the following are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ *for all $p > 0$ and $r > 0$.*

(iii) $A^r \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}$ *for all $p > 0$ and $r > 0$.*

We remark the correspondence of Theorem 1.A to the essential part of Theorem F: *$A \geq B \geq 0$ ensures*

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geq B^{1+r} \quad \text{and} \quad A^{1+r} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}$$

for all $p > 1$ and $r > 0$. Another simple proof of Theorem 1.A was given in [17]. It was shown in [18] that the domain of the parameters is the best possible in Theorem 1.A. It can be proved by the following Lemma F that

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \iff A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}} \qquad (*)$$

holds for $A, B > 0$ and $p, r > 0$.

Lemma F **([9]).** *Let $A > 0$ and B be an invertible operator. Then*

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number λ.

It was shown in [14] that similar relations to $(*)$ hold even if A and B are not invertible.

Theorem 1.B **([14]).** *Let $A, B \geq 0$. Then for each $p > 0$ and $r > 0$, the following hold:*

(i) *If $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$, then $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$.*

(ii) *If $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ and $N(A) \subseteq N(B)$, then $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$.*

2 Operator inequalities related to the relative operator entropy

The relative operator entropy was defined in [2] as $S(A \mid B) = A^{\frac{1}{2}}\log(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}$ for $A, B > 0$. We remark that $S(A \mid I) = -A\log A$ is the operator entropy. In case $p, r > 0$,

$$A^r \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}} \Longrightarrow \log A^{p+r} \geq \log(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})$$

holds for $A, B > 0$, so that (iii) $\Longrightarrow$ (i) of the following Theorem 2.A is an extension of (iii) $\Longrightarrow$ (i) of Theorem 1.A.

Theorem 2.A ([8]). *Let $A, B > 0$. Then the following are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ *for all* $p > 0$ *and* $r > 0$.

(iii) $\log A^{p+r} \geq \log(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})$ *for all* $p > 0$ *and* $r > 0$.

(iv) $S(A^{-r} \mid A^p) \geq S(A^{-r} \mid B^p)$ *for all* $p > 0$ *and* $r > 0$.

Here we consider the case $p > 0 > r$. We obtain the following result by applying Theorem 1.A.

Proposition 2.1. *Let $A, B > 0$ and $p > 0$.*

(i) *In case* $s > -p$, $\log A^{p+s} \geq \log(A^{\frac{s}{2}} B^p A^{\frac{s}{2}}) \Longleftrightarrow A^{-s+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{-s+r}{p+r}}$ *for all* $r > s$.

(ii) *In case* $s < -p$, $\log A^{p+s} \geq \log(A^{\frac{s}{2}} B^p A^{\frac{s}{2}}) \Longleftrightarrow A^{-s+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{-s+r}{p+r}}$ *for all* $r < s$.

The following is an immediate corollary of Proposition 2.1.

Corollary 2.2. *Let $A, B > 0$ and $p > t > 0$.*

$$A^p \geq B^p \Longrightarrow \log A^{p-t} \geq \log(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}}) \Longrightarrow A^t \geq B^t.$$

Corollary 2.2 corresponds to the case $\beta \nearrow t$ of the following result.

Proposition 2.B ([12]). *Let $A, B > 0$ and $p > t > \beta \geq 0$.*

$$A^\gamma \geq B^\gamma \Longrightarrow A^{t-\beta} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{t-\beta}{p-t}} \Longrightarrow A^\delta \geq B^\delta,$$

where $\gamma = \max\{2t - \beta, p\}$ *and* $\delta = \min\{2t - \beta, p\}$.

Proof of Proposition 2.1. $\log A^{p+s} \geq \log(A^{\frac{s}{2}} B^p A^{\frac{s}{2}})$ implies

$$A^{(p+s)r_1} \geq \left\{ A^{\frac{(p+s)r_1}{2}} (A^{\frac{s}{2}} B^p A^{\frac{s}{2}}) A^{\frac{(p+s)r_1}{2}} \right\}^{\frac{r_1}{1+r_1}}$$

for $r_1 = \frac{-s+r}{p+s} > 0$ by Theorem 1.A, then we have ($\Rightarrow$). ($\Leftarrow$) is obtained by taking the logarithms of both sides of $A^{-s+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{-s+r}{p+r}}$ and letting $r \to s$. □

Proof of Corollary 2.2. The first implication is obvious since $\log \cdot$ is operator monotone, and the second is obtained by putting $s = -t < 0$ and $r = 0$ in (i) of Proposition 2.1. □

We can summarize relations among orders and the inequality $\log A^{p+r} \geq \log(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})$ as follows.

(i) In case $p, r > 0$,

$$\begin{matrix} A^p \geq B^p \searrow & & \\ & \log A \geq \log B \Longrightarrow \log A^{p+r} \geq \log(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}). \\ A^r \geq B^r \nearrow & & \end{matrix}$$

(ii) In case $p > t > 0$,

$$A^p \geq B^p \Longrightarrow \log A^{p-t} \geq \log(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}}) \Longrightarrow A^t \geq B^t \Longrightarrow \log A \geq \log B.$$

(iii) In case $t > p > 0$,

$$A^t \geq B^t \Longrightarrow A^p \geq B^p \begin{matrix} \nearrow \log A \geq \log B \\ \searrow \log A^{p-t} \geq \log(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}}). \end{matrix}$$

We obtain the following result on the best possibility of Corollary 2.2.

Proposition 2.3.

(i) *Let $p > q > 0$ and $t > 0$. Then there exist $A, B > 0$ such that*

$$A^q \geq B^q \quad \textit{and} \quad \log A^{p-t} \not\geq \log(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}}).$$

(ii) *Let $p > t > 0$ and $q > t$. Then there exist $A, B > 0$ such that*

$$\log A^{p-t} \geq \log(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}}) \quad \textit{and} \quad A^q \not\geq B^q.$$

Proposition 2.3 can be proved by applying the following results.

Theorem 2.C ([16]). *Let $p > 1$ and $t > 0$. If $\alpha > 0$, then there exist $A, B > 0$ such that*

$$A \geq B \quad \textit{and} \quad A^{(p-t)\alpha} \not\geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^\alpha.$$

Theorem 2.D ([18]). *Let $p > 0$ and $r > 0$. If $\alpha > 1$, then there exist $A, B > 0$ such that*

$$\log A \geq \log B \quad \textit{and} \quad A^{r\alpha} \not\geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r\alpha}{p+r}}.$$

Proof of Proposition 2.3.

Proof of (i). The case $p = t$ can be proved easily since $0 \geq \log(A^{\frac{-p}{2}} B^p A^{\frac{-p}{2}})$ is equivalent to $A^p \geq B^p$. In case $p > t$, there exist $A_1, B_1 > 0$ such that

$$A_1 \geq B_1 \quad \text{and} \quad A_1^{(p_1 - t_1)\alpha} \not\geq (A_1^{\frac{-t_1}{2}} B_1^{p_1} A_1^{\frac{-t_1}{2}})^\alpha$$

for $p_1 = \frac{p}{q} > 1$, $t_1 = \frac{t}{2q} > 0$ and $\alpha = \frac{t}{2p-t} > 0$ by Theorem 2.C. Put $A = A_1^{\frac{1}{q}}$, $B = B_1^{\frac{1}{q}}$ and $r_1 = \frac{t}{2(p-t)} > 0$, then we have

$$A^q \geq B^q \quad \text{and} \quad A^{(p-t)r_1} \not\geq \left\{ A^{\frac{(p-t)r_1}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}}) A^{\frac{(p-t)r_1}{2}} \right\}^{\frac{r_1}{1+r_1}},$$

so that $\log A^{p-t} \not\geq \log(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})$ by Theorem 1.A.

In case $p < t$, there exist $A_1, B_1 > 0$ such that

$$A_1 \geq B_1 \quad \text{and} \quad A_1^{(p_1-t_1)\alpha} \not\geq (A_1^{\frac{-t_1}{2}} B_1^{p_1} A_1^{\frac{-t_1}{2}})^{\alpha}$$

for $p_1 = \frac{p}{q} > 1$, $t_1 = \frac{2t}{q} > 0$ and $\alpha = \frac{-t}{p-2t} > 0$ by Theorem 2.C. Put $A = A_1^{\frac{1}{q}}$, $B = B_1^{\frac{1}{q}}$ and $r_1 = \frac{-t}{p-t} > 0$, then we have

$$A^q \geq B^q \quad \text{and} \quad A^{(p-t)r_1} \not\geq \left\{ A^{\frac{(p-t)r_1}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}}) A^{\frac{(p-t)r_1}{2}} \right\}^{\frac{r_1}{1+r_1}},$$

so that $\log A^{p-t} \not\geq \log(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})$ by Theorem 1.A.

Proof of (ii). There exist $A_1, B_1 > 0$ such that

$$\log A_1 \geq \log B_1 \quad \text{and} \quad A_1^{r_1\alpha} \not\geq (A_1^{\frac{r_1}{2}} B_1 A_1^{\frac{r_1}{2}})^{\frac{r_1\alpha}{1+r_1}}$$

for $r_1 = \frac{t}{p-t} > 0$ and $\alpha = \frac{q}{t} > 1$ by Theorem 2.D, then we have the desired conclusion by putting $A = A_1^{\frac{1}{p-t}}$ and $B = (A_1^{\frac{t}{2(p-t)}} B_1 A^{\frac{t}{2(p-t)}})^{\frac{1}{p}}$, that is, $A_1 = A^{p-t}$ and $B_1 = A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}}$. □

We obtain the following result by applying (i) of Proposition 2.3.

Theorem 2.4. *Let $p > t$, $s > 1$ and $r < 0$. Then there exist $A, B > 0$ such that*

$$A^p \geq B^p \quad \textit{and} \quad \log A^{(p-t)s+r} \not\geq \log\{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}.$$

Proof. There exist $A_1, B_1 > 0$ such that

$$A_1 \geq B_1 \quad \text{and} \quad \log A_1^{s-t_1} \not\geq \log(A_1^{\frac{-t_1}{2}} B_1^s A_1^{\frac{-t_1}{2}}).$$

for $t_1 = \frac{-r}{p-t} > 0$ by (i) of Proposition 2.3, then we have the desired conclusion by putting $A = A_1^{\frac{1}{p-t}}$ and $B = (A_1^{\frac{t}{2(p-t)}} B_1 A^{\frac{t}{2(p-t)}})^{\frac{1}{p}}$, that is, $A_1 = A^{p-t}$ and $B_1 = A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}}$. □

It turns out by Theorem 2.4 that the generalized Furuta inequality ([9])

"$A \geq B \geq 0$ *with* $A > 0 \Longrightarrow A^{1-t+r} \geq \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$

for $p \geq 1$, $t \in [0,1]$, $s \geq 1$ *and* $r \geq t$"

is not valid for $p \geq 1$, $p > t$, $s > 1$ and $r < 0$.

3 Operator inequalities in a characterization of the chaotic order

The following relation holds between the inequalities in Theorem 1.A for $0 < p_1 \leq p_2$ and $0 < r_1 \leq r_2$. In fact, this relation can be proved by Theorem F and Lemma F in case A and B are invertible, and by Theorem 1.B in case they are not invertible.

Proposition 3.A ([11][14]). *Let $A, B \geq 0$, $0 < p_1 \leq p_2$ and $0 < r_1 \leq r_2$.*

$$(B^{\frac{r_1}{2}} A^{p_1} B^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}} \geq B^{r_1} \Longrightarrow (B^{\frac{r_2}{2}} A^{p_2} B^{\frac{r_2}{2}})^{\frac{r_2}{p_2+r_2}} \geq B^{r_2}.$$

Here we consider the case $p_1 > p_2$ or $r_1 > r_2$ in Proposition 3.A. In case A and B are not invertible, the following was shown in the proof of [13, Theorems 5, 6].

Theorem 3.B ([13]). *Let $p_1 > 0$ and $r_1 > 0$. Then there exist $A, B \geq 0$ such that*

$$(B^{\frac{r_1}{2}} A^{p_1} B^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}} \geq B^{r_1} \quad \text{and} \quad (B^{\frac{r_2}{2}} A^{p_2} B^{\frac{r_2}{2}})^{\frac{r_2}{p_2+r_2}} \not\geq B^{r_2}$$

for all $p_2 > 0$ and $r_2 > 0$ such that $p_1 > p_2$.

In case A and B are invertible, the following was given as a concrete example for $p_1 = r_1 = 2$ and $p_2 = r_2 = 1$.

Example 3.C ([4][10]).

Let $A = \begin{pmatrix} 17 & 7 \\ 7 & 5 \end{pmatrix}^2$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}^2$. Then $A, B > 0$, $(BA^2B)^{\frac{1}{2}} \geq B^2$ and $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \not\geq B$.

We obtain the following result by applying Proposition 3.A and Example 3.C.

Theorem 3.1. *Let $p_1 > p_2 > 0$ and $r_1 > r_2 > 0$. Then there exist $A, B > 0$ such that*

$$(B^{\frac{r_1}{2}} A^{p_1} B^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}} \geq B^{r_1} \quad \text{and} \quad (B^{\frac{r_2}{2}} A^{p_2} B^{\frac{r_2}{2}})^{\frac{r_2}{p_2+r_2}} \not\geq B^{r_2}.$$

It turns out by Lemma F that A and B in Theorem 3.1 also satisfy

$$A^{p_1} \geq (A^{\frac{p_1}{2}} B^{r_1} A^{\frac{p_1}{2}})^{\frac{p_1}{p_1+r_1}} \quad \text{and} \quad A^{p_2} \not\geq (A^{\frac{p_2}{2}} B^{r_2} A^{\frac{p_2}{2}})^{\frac{p_2}{p_2+r_2}}.$$

Proof. Assume that the following holds for $A, B > 0$:

$$(B^{\frac{r_1}{2}} A^{p_1} B^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}} \geq B^{r_1} \Longrightarrow (B^{\frac{r_2}{2}} A^{p_2} B^{\frac{r_2}{2}})^{\frac{r_2}{p_2+r_2}} \geq B^{r_2}. \tag{3.1}$$

By Proposition 3.A and (3.1), we have

$$(B^{\frac{r_1}{2}} A^{p_1} B^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}} \geq B^{r_1} \Longrightarrow (B^{\frac{\theta r_1}{2}} A^{\theta p_1} B^{\frac{\theta r_1}{2}})^{\frac{r_1}{p_1+r_1}} \geq B^{\theta r_1}, \tag{3.2}$$

where $\theta = \max\{\frac{p_2}{p_1}, \frac{r_2}{r_1}\} < 1$. Let n be an integer such that $\theta^n \leq \min\{\frac{p_1}{2r_1}, \frac{r_1}{2p_1}\}$. By applying (3.2) n times, we have

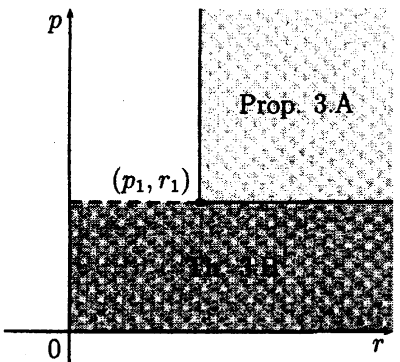
$$(B^{\frac{r_1}{2}} A^{p_1} B^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}} \geq B^{r_1} \Longrightarrow (B^{\frac{\theta^n r_1}{2}} A^{\theta^n p_1} B^{\frac{\theta^n r_1}{2}})^{\frac{r_1}{p_1+r_1}} \geq B^{\theta^n r_1}. \tag{3.3}$$

By Proposition 3.A and (3.3), we have

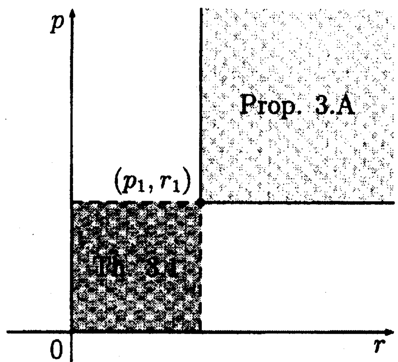
$$(B^{\frac{t}{2}} A^t B^{\frac{t}{2}})^{\frac{1}{2}} \geq B^t \Longrightarrow (B^{\frac{t}{4}} A^{\frac{t}{2}} B^{\frac{t}{4}})^{\frac{1}{2}} \geq B^{\frac{t}{2}}, \tag{3.4}$$

where $t = \min\{p_1, r_1\}$. The proof is complete since (3.4) contradict to Example 3.C. □

The domains of (p_2, r_2) in Proposition 3.A, Theorem 3.B and Theorem 3.1 are as in the following figures.



$A, B \geq 0$



$A, B > 0$

The following remains an open problem which corresponds to the case A and B are invertible in Theorem 3.B.

Conjecture 3.2. *Let $p_1 > 0$ and $r_1 > 0$. Then there exist $A, B > 0$ such that*

$$(B^{\frac{r_1}{2}} A^{p_1} B^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}} \geq B^{r_1} \quad \textit{and} \quad (B^{\frac{r_2}{2}} A^{p_2} B^{\frac{r_2}{2}})^{\frac{r_2}{p_2+r_2}} \not\geq B^{r_2}$$

for all $p_2 > 0$ and $r_2 > 0$ such that $p_1 > p_2$.

The following follows from Conjecture 3.2 by Lemma F since A and B are invertible in Conjecture 3.2.

Conjecture 3.3. *Let $p_1 > 0$ and $r_1 > 0$. Then there exist $A, B > 0$ such that*

$$(B^{\frac{r_1}{2}} A^{p_1} B^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}} \geq B^{r_1} \quad \textit{and} \quad (B^{\frac{r_2}{2}} A^{p_2} B^{\frac{r_2}{2}})^{\frac{r_2}{p_2+r_2}} \not\geq B^{r_2}$$

for all $p_2 > 0$ and $r_2 > 0$ such that $p_1 > p_2$ or $r_1 > r_2$.

References

[1] T. Ando, *On some operator inequalities*, Math. Ann. **279** (1987), 157–159.

[2] J. I. Fujii and E. Kamei, *Relative operator entropy in noncommutative information theory*, Math. Japon. **34** (1989), 341–348.

[3] M. Fujii, T. Furuta and E. Kamei, *Furuta's inequality and its application to Ando's theorem*, Linear Algebra Appl. **179** (1993), 161–169.

[4] M. Fujii, T. Furuta, D. Wang, *An application of the Furuta inequality to operator inequalities on chaotic orders*, Math. Japon. **40** (1994), 317–321.

[5] T. Furuta, *$A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$*, Proc. Amer. Math. Soc. **101** (1987), 85–88.

[6] T. Furuta, *An elementary proof of an order preserving inequality*, Proc. Japan Acad. Ser. A Math. Sci. **65** (1989), 126.

[7] T. Furuta, *Applications of order preserving operator inequalities*, Oper. Theory Adv. Appl. **59** (1992), 180–190.

[8] T. Furuta, *Furuta's inequality and its application to the relative operator entropy*, J. Operator Theory **30** (1993), 21–30.

[9] T. Furuta, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Algebra Appl. **219** (1995), 139–155.

[10] T. Furuta, M. Ito and T. Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and several related classes*, Sci. Math. **1** (1998), 389–403.

[11] T. Furuta, T. Yamazaki and M. Yanagida, *Order preserving operator inequalities via Furuta inequality*, Math. Japon. **48** (1998), 471–476.

[12] M. Ito, *Some order preserving operator inequality via Furuta inequality*, Sci. Math. Jpn. **55** (2002), 25–32.

[13] M. Ito, *On classes of operators generalizing class A and paranormality*, Sci. Math. Jpn. **57** (2003), 287–297.

[14] M. Ito and T. Yamazaki, *Relations between two inequalities $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ and $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$ and their applications*, Integral Equations Operator Theory **44** (2002), 442–450.

[15] K. Tanahashi, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc. **124** (1996), 141–146.

[16] K. Tanahashi, *The Furuta inequality with negative powers*, Proc. Amer. Math. Soc. **127** (1999), 1683–1692.

[17] M. Uchiyama, *Some exponential operator inequalities*, Math. Inequal. Appl. **2** (1999), 469–471.

[18] M. Yanagida, *Some applications of Tanahashi's result on the best possibility of Furuta inequality*, Math. Inequal. Appl. **2** (1999), 297–305.