Relations between two operator inequalities via operator means

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Relations between two operator inequalities via operator means

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Abstract

Let $A$ and $B$ be (not necessarily invertible) positive operators. Recently, the author and Yamazaki discussed relations between

$$(B^{\frac{r}{2}}AB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B$$

and

$$A^{p} \geq (A^{\frac{r}{2}}B^{r}A^{\frac{r}{2}})^{\frac{r}{p+r}}$$

for $p \geq 0$ and $r \geq 0,$ and also Yamazaki and Yanagida discussed relations between

$$\frac{p}{p+r}I + \frac{r}{p+r}B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}} \geq B$$

and

$$A^{p} \geq \frac{A^{\frac{r}{2}}B^{r}A^{\frac{r}{2}}}{\frac{r}{p+r}A^{\frac{p}{2}}B^{r}A^{\frac{p}{2}} + \frac{r}{p+r}I}$$

for $p \geq 0$ and $r \geq 0.$

In this report, as a generalization of their results via the representing functions of operator means, we shall show relations between two operator inequalities

$$f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B$$

and

$$A \geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}}),$$

where $f$ and $g$ are non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t.$

1 Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space $\mathcal{H}.$ An operator $T$ is said to be positive (in symbol: $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in \mathcal{H}.$ We denote the set of positive operators by $B(\mathcal{H})_{+}.$

Kubo-Ando [8] investigated an axiomatic approach for operator means (see also [5]). A binary operation $\sigma : B(\mathcal{H})_{+} \times B(\mathcal{H})_{+} \rightarrow B(\mathcal{H})_{+}$ is called an operator connection if it satisfies the following conditions (i), (ii) and (iii) for $A, B, C, D \in B(\mathcal{H})_{+}:

(i) $A \leq C$ and $B \leq D$ imply $A\sigma B \leq C\sigma D,$

(ii) $C(A\sigma B)C \leq (CAC)\sigma(CBC),$

(iii) $A_{n}, B_{n} \in B(\mathcal{H})_{+}, A_{n} \downarrow A$ and $B_{n} \downarrow B$ imply $A_{n}\sigma B_{n} \downarrow A\sigma B,$

where $A_{n} \downarrow A$ means that $A_{1} \geq A_{2} \geq \cdots$ and $A_{n}$ converges strongly to $A.$
An operator connection $\sigma$ is called an operator mean if

(iv) $I\sigma I = I$.

There exists a one-to-one correspondence between an operator connection $\sigma$ and an operator monotone function $f \geq 0$ on $[0, \infty)$. The operator connection $\sigma$ can be defined via the corresponding function $f$, which is called the representing function of $\sigma$, by

$$A\sigma B = A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$$

if $A$ is invertible, and $\sigma$ is an operator mean if and only if $f(1) = 1$.

The following are typical examples of operator means. For positive invertible operators $A$ and $B$, and for $\alpha \in [0,1]$,

(i) Arithmetic mean: $A\nabla_\alpha B = (1 - \alpha)A + \alpha B$,

(ii) Geometric mean ($\alpha$-power mean): $A\&_\alpha B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$,

(iii) Harmonic mean: $A!_\alpha B = \{(1 - \alpha)A^{-1} + \alpha B^{-1}\}^{-1}$.

The representing functions of $\nabla_\alpha$, $\&_\alpha$ and $!_\alpha$ are $(1 - \alpha) + \alpha t$, $t^{\alpha}$ and $(1 - \alpha) + \alpha t^{-1} = \frac{t}{(1 - \alpha)t + \alpha}$, respectively. On these operator means, the following relations are known. We remark that (1.1) was shown in [4], and (1.1) and (1.2) can be proved without using properties of operator means. Let $A$ and $B$ be positive invertible operators. For each $p \geq 0$ and $r \geq 0$,

$$B^{-r}\&_{\frac{p+r}{p}} A^p \geq I \iff I \geq A^{-p}\&_{\frac{p+r}{p}} B^r \tag{1.1}$$

and

$$B^{-r}\nabla_{\frac{p+r}{p}} A^p \geq I \iff I \geq A^{-p}!_{\frac{p+r}{p}} B^r \tag{1.2}$$

(1.1) is closely related to Furuta inequality [3], and a mean theoretic approach to Furuta inequality was discussed in [1][7] and others. We remark the following relations on inequalities in (1.1) and (1.2): Let $A$ and $B$ be positive invertible operators. For each $p \geq 0$ and $r \geq 0$,

$$A \geq B \implies \log A \geq \log B \implies \left\{ \begin{array}{l} B^{-r}\&_{\frac{p+r}{p}} A^p \geq I, \\ I \geq A^{-p}\&_{\frac{p+r}{p}} B^r \end{array} \right\} \implies \left\{ \begin{array}{l} B^{-r}\nabla_{\frac{p+r}{p}} A^p \geq I, \\ I \geq A^{-p}!_{\frac{p+r}{p}} B^r \end{array} \right\}.$$

The first relation holds since $\log t$ is operator monotone, the second was shown in [2][4], and the third holds since $(1 - \alpha) + \alpha t \geq t^\alpha \geq \frac{t}{(1 - \alpha)t + \alpha}$ for $t \geq 0$ and $\alpha \in [0,1]$. We remark that it was shown in [2][4] that

$$\log A \geq \log B \iff B^{-r}\&_{\frac{p+r}{p}} A^p \geq I \quad \text{for all } p \geq 0 \text{ and } r \geq 0$$

$$\iff I \geq A^{-p}\&_{\frac{p+r}{p}} B^r \quad \text{for all } p \geq 0 \text{ and } r \geq 0.$$
In this report, firstly we attempt a mean theoretic approach to (1.1) and (1.2). In other words, we shall state a result corresponding to (1.1) and (1.2) on a general operator mean for invertible operators. Secondly we shall show relations between

$$f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B \quad \text{and} \quad A \geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}})$$

for (not necessarily invertible) positive operators $A$ and $B$, where $f$ and $g$ are non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$. This result is a further generalization of the former argument via the representing functions of operator means. Moreover this result includes the ones by the author and Yamazaki [6] and by Yamazaki and Yanagiida [11].

2 A result on a general operator mean

In this section, we shall state a result corresponding to (1.1) and (1.2) on a general operator mean for invertible operators. At first we state definitions and properties of some operator means via a operator mean $\sigma$.

**Definition ([8]).** Let $\sigma$ be the operator mean with a representing function $f$.

(i) $\sigma'$ is said to be the transpose of $\sigma$ if $\sigma'$ is the operator mean with a representing function $tf(t^{-1})$.

(ii) $\sigma^*$ is said to be the adjoint of $\sigma$ if $\sigma^*$ is the operator mean with a representing function $\{f(t^{-1})\}^{-1}$.

(iii) $\sigma^\perp$ is said to be the dual of $\sigma$ if $\sigma^\perp$ is the operator mean with a representing function $\frac{t}{f(t)}$.

We remark that these representing functions can be defined on $[0, \infty)$ by setting the value on 0 by the limit to $+0$ since $f$ is operator monotone.

**Proposition 2.A ([8]).** Let $\sigma$ be an operator mean and $A,B \in \mathcal{B}(\mathcal{H})_+$. 

(i) $A\sigma' B = B\sigma A$.

(ii) $A\sigma^* B = (A^{-1}\sigma B^{-1})^{-1}$ if $A$ and $B$ are invertible.

(iii) $(\sigma')' = (\sigma^*)^* = (\sigma^\perp)^\perp = \sigma$.

(iv) $\sigma^\perp = (\sigma')^* = (\sigma^*)'$, $\sigma^\prime = (\sigma^*)^\perp = (\sigma^\perp)^* = (\sigma^\prime)^* = (\sigma^\perp)^\perp$. 

By using Proposition 2.A, we shall show a generalization of (1.1) and (1.2).
Proposition 2.1. Let $A$ and $B$ be positive invertible operators. For every operator mean $\sigma$,

$$B^{-1}\sigma A \geq I \iff I \geq A^{-1}\sigma^\perp B.$$ \hfill (2.1)

Proof. By (i) of Proposition 2.A,

$$B^{-1}\sigma A = A\sigma'B^{-1} \geq I.$$ \hfill (2.2)

By (ii) and (iv) of Proposition 2.A, (2.2) is equivalent to

$$I \geq (A\sigma'B^{-1})^{-1} = A^{-1}(\sigma'^\ast)B = A^{-1}\sigma^\perp B.$$

Hence the proof is complete. \hfill \square

Since $(\#_\alpha)^\perp = \beta_{1-\alpha}$ and $(\nabla_\alpha)^\perp = \!_1^{1-\alpha}$, Proposition 2.1 leads (1.1) (resp. (1.2)) by replacing $A$ and $B$ with $A^p$ and $B^r$ and by putting $\sigma = \#_{\frac{r}{p+r}}$ (resp. $\sigma = \nabla_{\frac{r}{r+p}}$). We remark that (2.1) can be rewritten by

$$f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B \iff A \geq \frac{A^{\frac{1}{2}}BA^{\frac{1}{2}}}{f(A^{\frac{1}{2}}BA^{\frac{1}{2}})}.$$ \hfill (2.3)

with the representing function $f$ of $\sigma$.

3 Main results

In this section, we shall show a further generalization of Proposition 2.1 via the representing functions of operator means.

When we rewrite (1.1) and (1.2) for positive invertible operators $A$ and $B$ by

$$(B^\frac{p}{p+r}A^pB^\frac{r}{p+r})^{\frac{p}{p+r}} \geq B^r \iff A^p \geq (A^\frac{p}{p+r}B^rA^\frac{p}{p+r})^{\frac{p}{p+r}}$$ \hfill (3.1)

and

$$\frac{p}{p+r}I + \frac{r}{p+r}B^\frac{p}{p+r}A^pB^\frac{r}{p+r} \geq B^r \iff A^p \geq \frac{A^\frac{p}{p+r}B^rA^\frac{p}{p+r}}{\frac{p}{p+r}A^\frac{p}{p+r}B^rA^\frac{p}{p+r} + \frac{p}{p+r}I}$$ \hfill (3.2)

with the representing functions, we can consider non-invertible operators on this argument. On relations between two inequalities in (3.1) and (3.2) for (not necessarily invertible) positive operators $A$ and $B$, the following results were obtained in [6] and [11].
Theorem 3.A ([6]). Let $A$ and $B$ be positive operators. Then for each $p \geq 0$ and $r \geq 0$, the following assertions hold:

(i) If $(B^{rac{r}{2}}A^{p}B^{rac{r}{2}})^{\frac{r}{p+r}} \geq B^{r}$, then $A^{p} \geq (A^{rac{p}{2}}B^{r}A^{rac{p}{2}})^{\frac{p}{p+r}}$.

(ii) If $A^{p} \geq (A^{rac{p}{2}}B^{r}A^{rac{p}{2}})^{\frac{p}{p+r}}$ and $N(A) \subseteq N(B)$, then $(B^{rac{r}{2}}A^{p}B^{rac{r}{2}})^{\frac{r}{p+r}} \geq B^{r}$.

Theorem 3.B ([11]). Let $A$ and $B$ be positive operators. Then for each $p > 0$ and $r \geq 0$, the following assertions hold:

(i) If $\frac{p}{p+r}I + \frac{r}{p+r}B^{rac{r}{2}}A^{p}B^{rac{r}{2}} \geq B^{r}$, then $A^{p} \geq \frac{A^{E}2B^{r}A^{E}2}{\frac{r}{p+r}A^{E}2B^{r}A^{E}2 + \frac{p}{p+r}I}$.

(ii) If $A^{p} \geq \frac{A^{E}2B^{r}A^{E}2}{\frac{r}{p+r}A^{E}2B^{r}A^{E}2 + \frac{r}{p+r}I}$ and $N(A) \subseteq N(B)$, then $\frac{p}{p+r}I + \frac{r}{p+r}B^{rac{r}{2}}A^{p}B^{rac{r}{2}} \geq B^{r}$.

Here we shall obtain a generalization of Proposition 2.1 via the form of (2.3). This result is also an extension of Theorems 3.A and 3.B.

Theorem 3.1. Let $A$ and $B$ be positive operators, and let $f$ and $g$ be non-negative continuous functions on $[0, \infty)$ satisfying

$$f(t)g(t) = t.$$  \hspace{1cm} (3.3)

(i) If $g(0) = 0$ or $N(A^{\frac{1}{2}}BA\frac{1}{2}) = \{0\}$, then $f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B$ ensures $A \geq g(A^{\frac{1}{2}}BA\frac{1}{2})$.

(ii) If $N(A) \subseteq N(B)$, then $A \geq g(A^{\frac{1}{2}}BA\frac{1}{2})$ ensures $f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B$.

In Theorem 3.1, $f$ and $g$ are not necessarily operator monotone functions. We also remark that if $f(0) > 0$, then automatically $g(0) = 0$ by (3.3).

If $A$ and $B$ are positive invertible operators and $\sigma$ is the operator mean with a representing function $f$, Theorem 3.1 ensures Proposition 2.1 since (2.1) is equivalent to (2.3). Theorem 3.1 also leads Theorem 3.A (resp. Theorem 3.B) by replacing $A$ and $B$ with $A^{p}$ and $B^{r}$ and by putting $f(t) = t^{\frac{r}{p+r}}$ and $g(t) = t^{\frac{r}{p+r}}$ (resp. $f(t) = \frac{p}{p+r} + \frac{r}{p+r}t$ and $g(t) = \frac{t}{\frac{p}{p+r} + \frac{r}{p+r}}$). We remark that $g(0) = 0$ in these cases.

We need some lemmas in order to prove Theorem 3.1.

Lemma 3.C. Let $T$ be a positive operator. Then

$$\lim_{\varepsilon \to +0} T^{\frac{1}{2}}(T + \varepsilon I)^{-1}T^{\frac{1}{2}} = \lim_{\varepsilon \to +0} (T + \varepsilon I)^{-1}T = P_{N(T)^{\perp}},$$

where $P_{\mathcal{M}}$ is a projection onto a closed subspace $\mathcal{M}$. 

Lemma 3.2. Let $f$ be a non-negative continuous function on $[0, \infty)$ such that $f(0) = 0$ and $f(t) > 0$ for $t > 0$. Then $N(f(T)) = N(T)$ for every positive operator $T$.

Proof. Let $T = \int_{0}^{\|T\|} tdE_t$ be the spectral decomposition of a positive operator $T$. Then

$$ (f(T)x, y) = \int_{0}^{\|T\|} f(t)d(E_t x, y) \quad \text{for } x, y \in \mathcal{H}. \quad (3.4) $$

We remark that $E_{-0} = 0$.

Assume that $x \in N(T)$. Then $E_0 x = (E_0 - E_{-0}) x = P_{N(T)} x = x$, and $(f(T)x, y) = f(0)(x, y) = 0$ for any $y \in \mathcal{H}$ by (3.4). Therefore $f(T)x = 0$, so that $x \in N(f(T))$.

Conversely, assume that $x \in N(f(T))$. Then for $\epsilon > 0$,

$$ 0 = (f(T)x, x) = \int_{0}^{\epsilon} f(t)d(E_t x, x) + \int_{\epsilon}^{\|T\|} f(t)d(E_t x, x) $$

by (3.4). Since $f(t) > 0$ for $t > 0$, $E_t x = x$ for $\epsilon > 0$. By tending $\epsilon \to +0$, we have $P_{N(T)} x = E_0 x = x$, so that $x \in N(T)$.

Lemma 3.3. Let $T = |T| U$ be the polar decomposition of an operator $T$, and let $f$ be a continuous function on $[0, \infty)$. Then

$$ U f(|T|) U^* = f(|T^*|) - f(0)(I - U U^*). $$

Proof. First we shall show the case $f(0) = 0$ by the same way to [10, Lemma]. Since $U|T|^n U^* = |T^*|^n$ for each positive integer $n$, $U p(|T|) U^* = p(|T^*|)$ holds for any polynomials $p$ such that $p(0) = 0$. By taking a sequence $\{p_n\}$ of polynomials with $p_n(0) = 0$ which converges uniformly to $f$ on $[0, \|T\|]$, we obtain $U f(|T|) U^* = f(|T^*|)$ for general $f$ with $f(0) = 0$.

Next, let $g(t) = f(t) - f(0)$. Then $g(0) = 0$, so that

$$ U f(|T|) U^* = U \{g(|T|) + f(0)I\} U^* = U g(|T|) U^* + f(0)UU^* $$

$$ = g(|T^*|) + f(0)I - f(0)(I - U U^*) = f(|T^*|) - f(0)(I - U U^*). $$

Hence the proof is complete.

Proof of Theorem 3.1. Let $\epsilon > 0$.

Proof of (i). Since $f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B$, we obtain

$$ (B + \epsilon I)^{-1} \geq \{f(A^{\frac{1}{2}}B^{\frac{1}{2}})^2 + \epsilon I\}^{-1}. $$
Let $A^{\frac{1}{2}}B^{\frac{1}{2}} = U|A^{\frac{1}{2}}B^{\frac{1}{2}}|$ be the polar decomposition of $A^{\frac{1}{2}}B^{\frac{1}{2}}$. Then we have

$$A^{\frac{1}{2}}B^{\frac{1}{2}}(A + \epsilon I)^{-1} = U|A^{\frac{1}{2}}B^{\frac{1}{2}}|U^*$$

and

$$A^{\frac{1}{2}}B^{\frac{1}{2}}(B + \epsilon I)^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}}$$

are non-negative functions of $|A^{\frac{1}{2}}B^{\frac{1}{2}}|$. Therefore, noting that $UU^* = P_{N(B)^\perp}P_{N(A^{\frac{1}{2}}BA^{\frac{1}{2}})^\perp} = I$ if $N(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = \{0\}$, we have

$$A \geq A^{\frac{1}{2}}P_{N(B)^\perp}A^{\frac{1}{2}}$$

by (3.5), by tending $\epsilon \to +0$ and Lemma 3.3, we obtain

$$A^{\frac{1}{2}}P_{N(B)^\perp}A^{\frac{1}{2}} \geq UP_{N(f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2})^{\perp}}f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2})U^* = Uf(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2})U^*$$

by the following: If $f(0) > 0$, then $f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2})$ is invertible and $P_{N(f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2})^{\perp}} = I$. If $f(0) = 0$, then $UP_{N(f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2})^{\perp}} = UP_{N(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2})^{\perp}} = U$ by Lemma 3.2.

Therefore, noting that $UU^* = P_{N(A^{\frac{1}{2}}BA^{\frac{1}{2}})^\perp} = P_{N(A^{\frac{1}{2}}BA^{\frac{1}{2}})^\perp} = I$ if $N(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = \{0\}$, we have

$$A \geq A^{\frac{1}{2}}P_{N(B)^\perp}A^{\frac{1}{2}}$$

by (3.6)

$$= Ug(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2})U^*$$

by Lemma 3.3

$$= g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^{2})$$

since $g(0) = 0$ or $N(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = \{0\}$

$$= g(A^{\frac{1}{2}}BA^{\frac{1}{2}}).$$

Proof of (ii). Since $A \geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}})$, we obtain

$$(A + \epsilon I)^{-1} \leq \{g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^{2}) + \epsilon I\}^{-1}.$$
by the following: If \( g(0) > 0 \), then \( g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2) \) is invertible and \( P_{N(g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2))^\perp} = I \). If \( g(0) = 0 \), then \( VP_{N(g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2))^\perp} = VP_{N(B^{\frac{1}{2}}A^{\frac{1}{2}})^\perp} = V \) by Lemma 3.2.

Therefore, noting that \( N(A) \subseteq N(B) \) is equivalent to \( P_{N(A)^\perp} \geq P_{N(B)^\perp} \), we have

\[
B = B^{\frac{1}{2}}P_{N(B)^\perp}B^{\frac{1}{2}} \\
\leq B^{\frac{1}{2}}P_{N(A)^\perp}B^{\frac{1}{2}} \\
\leq Vf(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^2)V^* \\
= f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2) - f(0)\left(\mathbf{I} - VV^*\right) \quad \text{by (3.8)} \\
\leq f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^2) \\
= f(B^{\frac{1}{2}}AB^{\frac{1}{2}}).
\]

Hence the proof is complete.

\[\square\]

**Corollary 3.4.** Let \( A \) and \( B \) be positive operators, and let \( f \) and \( g \) be positive continuous functions on \([0, \infty)\) satisfying \( f(t)g(t) = t \). If \( N(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = \{0\} \), then \( f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B \) is equivalent to \( A \geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}}) \).

**Proof.** Since \( N(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = \{0\} \) ensures \( \{0\} = N(A) \subseteq N(B) \), \( f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B \) is equivalent to \( A \geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}}) \) by Theorem 3.1. \[\square\]

Of course \( N(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = \{0\} \) if \( A \) and \( B \) are invertible.

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