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Relations between two operator inequalities via operator means

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Abstract

Let $A$ and $B$ be (not necessarily invertible) positive operators. Recently, the author and Yamazaki discussed relations between

$$(B^{rac{r}{2}}A^{rac{1}{2}}B^{rac{r}{2}})^{rac{r}{p+r}} \geq B^r \quad \text{and} \quad A^p \geq (A^{rac{r}{2}}B^rA^{rac{r}{2}})^{rac{p}{p+r}}$$

for $p \geq 0$ and $r \geq 0$, and also Yamazaki and Yanagida discussed relations between

$$\frac{\frac{p}{p+r}I + \frac{r}{p+r}B^{rac{r}{2}}A^pB^{rac{r}{2}}}{A^p} \geq B^r \quad \text{and} \quad A^p \geq \frac{A^{rac{r}{2}}B^rA^{rac{r}{2}}}{\frac{r}{p+r}A^{rac{p}{2}}B^rA^{rac{p}{2}} + \frac{p}{p+r}I}$$

for $p \geq 0$ and $r \geq 0$.

In this report, as a generalization of their results via the representing functions of operator means, we shall show relations between two operator inequalities

$$f(B^{rac{1}{2}}AB^{rac{1}{2}}) \geq B \quad \text{and} \quad A \geq g(A^{rac{1}{2}}BA^{rac{1}{2}}),$$

where $f$ and $g$ are non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$.

1 Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space $\mathcal{H}$. An operator $T$ is said to be positive (in symbol: $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$. We denote the set of positive operators by $B(\mathcal{H})_+$.  

Kubo-Ando [8] investigated an axiomatic approach for operator means (see also [5]). A binary operation $\sigma : B(\mathcal{H})_+ \times B(\mathcal{H})_+ \to B(\mathcal{H})_+$ is called an operator connection if it satisfies the following conditions (i), (ii) and (iii) for $A, B, C, D \in B(\mathcal{H})_+$:

(i) $A \leq C$ and $B \leq D$ imply $A\sigma B \leq C\sigma D$,

(ii) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$,

(iii) $A_n, B_n \in B(\mathcal{H})_+, A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$,

where $A_n \downarrow A$ means that $A_1 \geq A_2 \geq \cdots$ and $A_n$ converges strongly to $A$. 

An operator connection $\sigma$ is called an *operator mean* if

(iv) $I\sigma I = I$.

There exists a one-to-one correspondence between an operator connection $\sigma$ and an operator monotone function $f \geq 0$ on $[0, \infty)$. The operator connection $\sigma$ can be defined via the corresponding function $f$, which is called the *representing function* of $\sigma$, by

$$A\sigma B = A^{\frac{1}{2}} f \left( A^{\frac{1}{2}} BA^{\frac{1}{2}} \right) A^{\frac{1}{2}}$$

if $A$ is invertible, and $\sigma$ is an operator mean if and only if $f(1) = 1$.

The following are typical examples of operator means. For positive invertible operators $A$ and $B$, and for $\alpha \in [0,1]$,

(i) Arithmetic mean: $A\nabla_\alpha B = (1-\alpha)A + \alpha B$,

(ii) Geometric mean (\alpha-power mean): $A^\#_\alpha B = A^{\frac{1}{2}} \left( A^{\frac{1}{2}} BA^{\frac{1}{2}} \right)^\alpha A^{\frac{1}{2}}$,

(iii) Harmonic mean: $A!_\alpha B = \{(1-\alpha)A^{-1} + \alpha B^{-1}\}^{-1}$.

The representing functions of $\nabla_\alpha$, $^\#_\alpha$ and $!_\alpha$ are $(1-\alpha)+\alpha t$, $t^\alpha$ and $(1-\alpha)+\alpha t^{-1}$, respectively. On these operator means, the following relations are known. We remark that (1.1) was shown in [4], and (1.1) and (1.2) can be proved without using properties of operator means. Let $A$ and $B$ be positive invertible operators. For each $p \geq 0$ and $r \geq 0$,

$$B^{-r} \nabla^{\frac{r}{p+r}} A^p \geq I \iff I \geq A^{-p} \nabla^{\frac{r}{p+r}} B^r \quad (1.1)$$

and

$$B^{-r} \nabla^{\frac{r}{p+r}} A^p \geq I \iff I \geq A^{-p} \nabla^{\frac{r}{p+r}} B^r \quad (1.2)$$

(1.1) is closely related to Furuta inequality [3], and a mean theoretic approach to Furuta inequality was discussed in [1], [7] and others. We remark the following relations on inequalities in (1.1) and (1.2): Let $A$ and $B$ be positive invertible operators. For each $p \geq 0$ and $r \geq 0$,

$$A \geq B \implies \log A \geq \log B \implies \begin{cases} B^{-r} \nabla^{\frac{r}{p+r}} A^p \geq I, \\ I \geq A^{-p} \nabla^{\frac{r}{p+r}} B^r \end{cases} \implies \begin{cases} B^{-r} \nabla^{\frac{r}{p+r}} A^p \geq I, \\ I \geq A^{-p} \nabla^{\frac{r}{p+r}} B^r \end{cases}.$$

The first relation holds since $\log t$ is operator monotone, the second was shown in [2], [4], and the third holds since $(1-\alpha) + \alpha t \geq t^\alpha \geq \frac{t}{(1-\alpha)+t+\alpha}$ for $t \geq 0$ and $\alpha \in [0,1]$. We remark that it was shown in [2], [4] that

$$\log A \geq \log B \iff B^{-r} \nabla^{\frac{r}{p+r}} A^p \geq I \quad \text{for all } p \geq 0 \text{ and } r \geq 0,$$

$$\iff I \geq A^{-p} \nabla^{\frac{r}{p+r}} B^r \quad \text{for all } p \geq 0 \text{ and } r \geq 0.$$
In this report, firstly we attempt a mean theoretic approach to (1.1) and (1.2). In other words, we shall state a result corresponding to (1.1) and (1.2) on a general operator mean for invertible operators. Secondly we shall show relations between

\[ f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B \quad \text{and} \quad A \geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}}) \]

for (not necessarily invertible) positive operators \( A \) and \( B \), where \( f \) and \( g \) are non-negative continuous functions on \([0, \infty)\) satisfying \( f(t)g(t) = t \). This result is a further generalization of the former argument via the representing functions of operator means. Moreover this result includes the ones by the author and Yamazaki [6] and by Yamazaki and Yanagida [11].

2 A result on a general operator mean

In this section, we shall state a result corresponding to (1.1) and (1.2) on a general operator mean for invertible operators. At first we state definitions and properties of some operator means via an operator mean \( \sigma \).

**Definition ([8]).** Let \( \sigma \) be the operator mean with a representing function \( f \).

(i) \( \sigma' \) is said to be the transpose of \( \sigma \) if \( \sigma' \) is the operator mean with a representing function \( tf(t^{-1}) \).

(ii) \( \sigma^* \) is said to be the adjoint of \( \sigma \) if \( \sigma^* \) is the operator mean with a representing function \( \{f(t^{-1})\}^{-1} \).

(iii) \( \sigma^\perp \) is said to be the dual of \( \sigma \) if \( \sigma^\perp \) is the operator mean with a representing function \( \frac{t}{f(t)} \).

We remark that these representing functions can be defined on \([0, \infty)\) by setting the value on 0 by the limit to +0 since \( f \) is operator monotone.

**Proposition 2.A ([8]).** Let \( \sigma \) be an operator mean and \( A, B \in \mathcal{B}(\mathcal{H})_+ \).

(i) \( A\sigma' B = B\sigma A \).

(ii) \( A\sigma^* B = (A^{-1}\sigma B^{-1})^{-1} \) if \( A \) and \( B \) are invertible.

(iii) \( (\sigma')' = (\sigma^*)^* = (\sigma^\perp)^\perp = \sigma \).

(iv) \( \sigma^\perp = (\sigma')^* = (\sigma^*)' = (\sigma^\perp)^* = (\sigma^\perp)^* \) and \( \sigma^* = (\sigma^\perp)' = (\sigma')^\perp \).

By using Proposition 2.A, we shall show a generalization of (1.1) and (1.2).
Proposition 2.1. Let $A$ and $B$ be positive invertible operators. For every operator mean $\sigma$,\[ B^{-1}\sigma A \geq I \iff I \geq A^{-1}\sigma B. \quad (2.1) \]

Proof. By (i) of Proposition 2.1,\[ B^{-1}\sigma A = A\sigma'B^{-1} \geq I. \quad (2.2) \]

By (ii) and (iv) of Proposition 2.1, (2.2) is equivalent to\[ I \geq (A\sigma'B^{-1})^{-1} = A^{-1}(\sigma')^*B = A^{-1}\sigma B. \]

Hence the proof is complete. \[\square\]

Since $(\#_\alpha)^{\perp} = \beta_{1-\alpha}$ and $(\nabla_\alpha)^{\perp} = \!\!_1-\alpha$, Proposition 2.1 leads (1.1) (resp. (1.2)) by replacing $A$ and $B$ with $A^p$ and $B^r$ and by putting $\sigma = \#_{\frac{r}{p+r}}$ (resp. $\sigma = \nabla_{\frac{r}{p+r}}$). We remark that (2.1) can be rewritten by\[ f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B \iff A \geq \frac{A^{\frac{1}{2}}BA^{\frac{1}{2}}}{f(A^{\frac{1}{2}}BA^{\frac{1}{2}})} \quad (2.3) \]

with the representing function $f$ of $\sigma$.

3 Main results

In this section, we shall show a further generalization of Proposition 2.1 via the representing functions of operator means.

When we rewrite (1.1) and (1.2) for positive invertible operators $A$ and $B$ by\[ (B^{\frac{1}{2}}A^pB^{\frac{1}{2}})^{\frac{r}{p+r}} \geq B^r \iff A^p \geq (A^{\frac{r}{p+r}}B^rA^{\frac{1}{2}})^{\frac{p}{p+r}} \quad (3.1) \]

and\[ \frac{p}{p+r}I + \frac{r}{p+r}B^{\frac{1}{2}}A^pB^{\frac{1}{2}} \geq B^r \iff A^p \geq \frac{A^{\frac{r}{p+r}}B^rA^{\frac{1}{2}}}{\frac{p}{p+r}A^{\frac{1}{2}}B^rA^{\frac{1}{2}} + \frac{p}{p+r}I} \quad (3.2) \]

with the representing functions, we can consider non-invertible operators on this argument. On relations between two inequalities in (3.1) and (3.2) for (not necessarily invertible) positive operators $A$ and $B$, the following results were obtained in [6] and [11].
Theorem 3.A ([6]). Let $A$ and $B$ be positive operators. Then for each $p \geq 0$ and $r \geq 0$, the following assertions hold:

(i) If $(B^{\frac{p}{2}} A^p B^{\frac{r}{2}})^{\frac{p}{p+r}} \geq B^r$, then $A^p \geq (A^\frac{p}{2} B^{\frac{r}{2}} A^\frac{p}{2})^{\frac{p}{p+r}}$.

(ii) If $A^p \geq (A^\frac{p}{2} B^{\frac{r}{2}} A^\frac{p}{2})^{\frac{p}{p+r}}$ and $N(A) \subseteq N(B)$, then $(B^\frac{p}{2} A^p B^\frac{r}{2})^{\frac{p}{p+r}} \geq B^r$.

Theorem 3.B ([11]). Let $A$ and $B$ be positive operators. Then for each $p > 0$ and $r \geq 0$, the following assertions hold:

(i) If $\frac{p}{p+r} I + \frac{r}{p+r} B^{\frac{r}{2}} A^p B^{\frac{r}{2}} \geq B^r$, then $A^p \geq \frac{A^\frac{p}{2} B^{\frac{r}{2}} A^\frac{p}{2}}{\frac{p}{p+r} B^{\frac{r}{2}} A^p B^{\frac{r}{2}} + \frac{p}{p+r} I}$.

(ii) If $A^p \geq \frac{A^\frac{p}{2} B^{\frac{r}{2}} A^\frac{p}{2}}{\frac{p}{p+r} B^{\frac{r}{2}} A^p B^{\frac{r}{2}} + \frac{p}{p+r} I}$ and $N(A) \subseteq N(B)$, then $\frac{p}{p+r} I + \frac{r}{p+r} B^{\frac{r}{2}} A^p B^{\frac{r}{2}} \geq B^r$.

Here we shall obtain a generalization of Proposition 2.1 via the form of (2.3). This result is also an extension of Theorems 3.A and 3.B.

Theorem 3.1. Let $A$ and $B$ be positive operators, and let $f$ and $g$ be non-negative continuous functions on $[0, \infty)$ satisfying

$$f(t)g(t) = t. \quad (3.3)$$

(i) If $g(0) = 0$ or $N(A^{\frac{1}{2}} B A^{\frac{1}{2}}) = \{0\}$, then $f(B^{\frac{1}{2}} A B^{\frac{1}{2}}) \geq B$ ensures $A \geq g(A^{\frac{1}{2}} B A^{\frac{1}{2}})$.

(ii) If $N(A) \subseteq N(B)$, then $A \geq g(A^{\frac{1}{2}} B A^{\frac{1}{2}})$ ensures $f(B^{\frac{1}{2}} A B^{\frac{1}{2}}) \geq B$.

In Theorem 3.1, $f$ and $g$ are not necessarily operator monotone functions. We also remark that if $f(0) > 0$, then automatically $g(0) = 0$ by (3.3).

If $A$ and $B$ are positive invertible operators and $\sigma$ is the operator mean with a representing function $f$, Theorem 3.1 ensures Proposition 2.1 since (2.1) is equivalent to (2.3). Theorem 3.1 also leads Theorem 3.A (resp. Theorem 3.B) by replacing $A$ and $B$ with $A^p$ and $B^r$ and by putting $f(t) = t^{\frac{r}{p+r}}$ and $g(t) = t^{\frac{r}{p+r}}$ (resp. $f(t) = \frac{p}{p+r} + \frac{r}{p+r} t$ and $g(t) = \frac{p}{p+r} + \frac{r}{p+r} t$). We remark that $g(0) = 0$ in these cases.

We need some lemmas in order to prove Theorem 3.1.

**Lemma 3.C.** Let $T$ be a positive operator. Then

$$\lim_{\epsilon \to 0^+} T^{\frac{1}{2}}(T + \epsilon I)^{-1}T^{\frac{1}{2}} = \lim_{\epsilon \to 0^+} (T + \epsilon I)^{-1}T = P_M,$$

where $P_M$ is a projection onto a closed subspace $M$. 

Lemma 3.2. Let $f$ be a non-negative continuous function on $[0, \infty)$ such that $f(0) = 0$ and $f(t) > 0$ for $t > 0$. Then $N(f(T)) = N(T)$ for every positive operator $T$.

Proof. Let $T = \int_0^{||T||} \! tdE_t$ be the spectral decomposition of a positive operator $T$. Then

\[
(f(T)x, y) = \int_0^{||T||} \! f(t)d(E_t x, y) \quad \text{for } x, y \in \mathcal{H}.
\]  

(3.4)

We remark that $E_{-0} = 0$.

Assume that $x \in N(T)$. Then $E_0 x = (E_0 - E_{-0}) x = P_{N(T)} x = x$, and $(f(T)x, y) = f(0)(x, y) = 0$ for any $y \in \mathcal{H}$ by (3.4). Therefore $f(T)x = 0$, so that $x \in N(f(T))$.

Conversely, assume that $x \in N(f(T))$. Then for $\epsilon > 0$,

\[
0 = (f(T)x, x) = \int_0^{\epsilon} \! f(t)d(E_t x, x) + \int_{\epsilon}^{||T||} \! f(t)d(E_t x, x)
\]

by (3.4). Since $f(t) > 0$ for $t > 0$, $E_\epsilon x = x$ for $\epsilon > 0$. By tending $\epsilon \to +0$, we have $P_{N(T)} x = E_0 x = x$, so that $x \in N(T)$.

Lemma 3.3. Let $T = U|T|$ be the polar decomposition of an operator $T$, and let $f$ be a continuous function on $[0, \infty)$. Then

\[
Uf(|T|)U^* = f(|T^*|) - f(0)(I - UU^*).
\]

Proof. First we shall show the case $f(0) = 0$ by the same way to [10, Lemma]. Since $U|T|^nU^* = |T^*|^n$ for each positive integer $n$, $Uf(|T|)U^* = p(|T^*|)$ holds for any polynomials $p$ such that $p(0) = 0$. By taking a sequence $\{p_n\}$ of polynomials with $p_n(0) = 0$ which converges uniformly to $f$ on $[0, ||T||]$, we obtain $Uf(|T|)U^* = f(|T^*|)$ for general $f$ with $f(0) = 0$.

Next, let $g(t) = f(t) - f(0)$. Then $g(0) = 0$, so that

\[
Uf(|T|)U^* = U\{g(|T|) + f(0)I\}U^* = Ug(|T|)U^* + f(0)UU^*
\]

\[
= g(|T^*|) + f(0)I - f(0)(I - UU^*) = f(|T^*|) - f(0)(I - UU^*).
\]

Hence the proof is complete.

Proof of Theorem 3.1. Let $\epsilon > 0$.

Proof of (i). Since $f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B$, we obtain

\[
(B + \epsilon I)^{-1} \geq \{f(A^{\frac{1}{2}}B^{\frac{1}{2}})^2\} + \epsilon I\}^{-1}.
\]
Let $A^{\frac{1}{2}}B^{\frac{1}{2}} = U|A^{\frac{1}{2}}B^{\frac{1}{2}}|$ be the polar decomposition of $A^{\frac{1}{2}}B^{\frac{1}{2}}$. Then we have

$$A^{\frac{1}{2}}B^{\frac{1}{2}}(A + \epsilon I)^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}} \geq U\{f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2}) + \epsilon I\}^{-1}f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2})$$

by (3.3).

In (3.5), by tending $\epsilon \rightarrow +0$ and Lemma 3.C, we obtain

$$A^{\frac{1}{2}}P_{N(B)^{[\perp]}}A^{\frac{1}{2}} \geq UP_{N(f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2}))^{[\perp]}}g(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2})U^{*}$$

by the following: If $f(0) > 0$, then $f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2})$ is invertible and $P_{N(f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2}))^{[\perp]}} = I$. If $f(0) = 0$, then $U_{N(f(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2}))^{[\perp]}} = U_{N(A^{\frac{1}{2}})} = U$ by Lemma 3.2.

Therefore, noting that $UU^{*} = P_{N(B)^{[\perp]}} = P_{N(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{[\perp]}} = U$ if $N(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = \{0\}$, we have

$$A \geq A^{\frac{1}{2}}P_{N(B)^{[\perp]}}A^{\frac{1}{2}}$$

$$\geq U_{g(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2})}U^{*}$$

by (3.6)

$$= g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^{2}) - g(0)(I - UU^{*})$$

by Lemma 3.3

$$= g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^{2})$$

since $g(0) = 0$ or $N(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = \{0\}$

$$= g(A^{\frac{1}{2}}BA^{\frac{1}{2}}).$$

Proof of (ii). Since $A \geq g(A^{\frac{1}{2}}BA^{\frac{1}{2}})$, we obtain

$$(A + \epsilon I)^{-1} \leq \{g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^{2}) + \epsilon I\}^{-1}.$$

Let $B^{\frac{1}{2}}A^{\frac{1}{2}} = V|B^{\frac{1}{2}}A^{\frac{1}{2}}|$ be the polar decomposition of $B^{\frac{1}{2}}A^{\frac{1}{2}}$. Then we have

$$B^{\frac{1}{2}}A^{\frac{1}{2}}(A + \epsilon I)^{-1}A^{\frac{1}{2}}B^{\frac{1}{2}}$$

$$\leq B^{\frac{1}{2}}A^{\frac{1}{2}}\{g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^{2}) + \epsilon I\}^{-1}A^{\frac{1}{2}}B^{\frac{1}{2}}$$

$$= V|B^{\frac{1}{2}}A^{\frac{1}{2}}|\{g(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^{2}) + \epsilon I\}^{-1}|B^{\frac{1}{2}}A^{\frac{1}{2}}|V^{*}$$

by (3.3).

In (3.7), by tending $\epsilon \rightarrow +0$ and Lemma 3.C, we obtain

$$B^{\frac{1}{2}}P_{N(A)^{[\perp]}}B^{\frac{1}{2}} \leq VP_{N(g(B^{\frac{1}{2}}A^{\frac{1}{2}}|^{2}))^{[\perp]}}f(|B^{\frac{1}{2}}A^{\frac{1}{2}}|^{2})V^{*}$$

by (3.8)
by the following: If \( g(0) > 0 \), then \( g(|B^\frac{1}{2}A^\frac{1}{2}|^2) \) is invertible and \( P_{N(g(|B^\frac{1}{2}A^\frac{1}{2}|^2))} = I \). If 
\( g(0) = 0 \), then \( VP_{N(g(|B^\frac{1}{2}A^\frac{1}{2}|^2)^\perp} = VP_{N(B^\frac{1}{2}A^\frac{1}{2})^\perp} = V \) by Lemma 3.2.

Therefore, noting that \( N(A) \subseteq N(B) \) is equivalent to \( P_{N(A)^\perp} \geq P_{N(B)^\perp} \), we have

\[
B = B^\frac{1}{2}P_{N(B)^\perp}B^\frac{1}{2} \\
\leq B^\frac{1}{2}P_{N(A)^\perp}B^\frac{1}{2} \\
\leq Vf(|B^\frac{1}{2}A^\frac{1}{2}|^2)V^* \\
= f(|A^\frac{1}{2}B^\frac{1}{2}|^2) - f(0)(I - VV^*) \text{ by Lemma 3.3} \\
\leq f(|A^\frac{1}{2}B^\frac{1}{2}|^2) \\
= f(B^\frac{1}{2}AB^\frac{1}{2}).
\]

Hence the proof is complete. \( \square \)

**Corollary 3.4.** Let \( A \) and \( B \) be positive operators, and let \( f \) and \( g \) be positive continuous functions on \([0, \infty)\) satisfying \( f(t)g(t) = t \). If \( N(A^\frac{1}{2}BA^\frac{1}{2}) = \{0\} \), then \( f(B^\frac{1}{2}AB^\frac{1}{2}) \geq B \) is equivalent to \( A \geq g(A^\frac{1}{2}BA^\frac{1}{2}) \).

**Proof.** Since \( N(A^\frac{1}{2}BA^\frac{1}{2}) = \{0\} \) ensures \( \{0\} = N(A) \subseteq N(B), f(B^\frac{1}{2}AB^\frac{1}{2}) \geq B \) is equivalent to \( A \geq g(A^\frac{1}{2}BA^\frac{1}{2}) \) by Theorem 3.1. \( \square \)

Of course \( N(A^\frac{1}{2}BA^\frac{1}{2}) = \{0\} \) if \( A \) and \( B \) are invertible.

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