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An operator transform from class A to the class of hyponormal operators and its application

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ABSTRACT

In this report, we shall give an operator transform $\hat{T}$ from class A to the class of hyponormal operators. Then we shall show that $\sigma(\hat{T})=\sigma(T)$ and $\sigma_a(\hat{T})=\sigma_a(T)$ in case T belongs to class A. Next, as an application of $\hat{T}$, we will show that every class A operator has SVEP and property (β).

1. INTRODUCTION

As a research on non-normal operators on a Hilbert space, many authors studied properties of hyponormal operators. Recently, in the development of operator inequality, many operator classes which include the class of hyponormal operators were defined, and many authors studied these new classes. In the study of these new classes, the Aluthge transform is a very useful tool. It is an operator transform from the class of w-hyponormal and semi-hyponormal operators to the class of semi-hyponormal and hyponormal operators, respectively. By using Aluthge transform, we can treat spectrum properties of these new operator classes like hyponormal operators. But until now, we have not obtained any property of Aluthge transform of a class A operator which is a weaker class than the class of w-hyponormal operators, so it was difficult to discuss on properties of class A operators. In this report, we shall give a new operator transform $\hat{T}$ of T from class A to the class of hyponormal operators with modulus $|\hat{T}|=|T^2|^{\frac{1}{2}}$. Then we will show that the spectrum of $\hat{T}$ coincides with one of T in case T belongs to class A, and can obtain some properties of class A operators by using hyponormality of $\hat{T}$.

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space $\mathcal{H}$. An operator T is said to be positive (denoted by $T\geq 0$) if $(Tx,x)\geq 0$ for all $x\in\mathcal{H}$. For a positive number p, an operator T is said to be p-hyponormal if

$(T^*T)^p \geq (TT^*)^p$ holds. Especially, a p-hyponormal operator T is said to be hyponormal and semi-hyponormal if $p = 1$ and $p = \frac{1}{2}$, respectively. For positive numbers s and t, an operator T belongs to class A(s,t) if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$. Especially, we denote class A$(1,1)$ by class A, simply. We remark that class A was first defined by the inequality $|T^2| \geq |T|^2$, and it is known that inequalities $|T^2| \geq |T|^2$ and $(|T^*||T|^2|T^*|)^{\frac{1}{2}} \geq |T^*|^2$ are equivalent. Class A operator has been defined in [9] as a nice application of Furuta inequality [8]. Then as a generalization of class A, class A(s,t) was defined in [7]. Inclusion relations among these classes are known as follows:

$$
\begin{aligned}
\{\text{hyponormal}\} &\subset \{p\text{-hyponormal}, 0 < p < 1\} \\
&\subset \{\text{class A}(s,t),\ s,t \in (0,1]\} \\
&\subset \{\text{class A}\} \\
&\subset \{\text{paranormal, i.e., } \|T^2x\| \geq \|Tx\|^2 \text{ for } \|x\| = 1\}.
\end{aligned} \tag{1.1}
$$

The first relation was shown by using Löwner-Heinz inequality, the second one was shown in [7], the third one was shown in [12] (if T is invertible, it was shown in [7], see also [11]), and the last one was shown in [9].

An operator T has the single valued extension property (simply denoted by SVEP) at $\lambda \in \mathbb{C}$ if the following assertion holds:

> If $\mathcal{D} \subset \mathbb{C}$ is an open neighborhood of λ and if $f : \mathcal{D} \longrightarrow \mathcal{H}$ is a vector-valued analytic function such that $(T - \mu)f(\mu) = 0$ for all $\mu \in \mathcal{D}$, then f is identically zero on $\mathcal{D}$.

When T has SVEP for every $\lambda \in \mathbb{C}$, we simply say that T has SVEP.

SVEP has been much studied by many authors. This is a good property for operators and there are plenty of applications in operator theory. For example, if T has SVEP, then for any $\lambda \in \mathbb{C}$, $T - \lambda$ is invertible if and only if it is surjective. This result was suggested in Finch [6].

As a generalization of SVEP, an operator T has property (β) at $\lambda \in \mathbb{C}$ if the following assertion holds:

> If $\mathcal{D} \subset \mathbb{C}$ is an open neighborhood of λ and if $f_n : \mathcal{D} \longrightarrow \mathcal{H}$ $(n = 1, 2, \ldots)$ are vector-valued analytic functions such that $(T - \mu)f_n(\mu) \longrightarrow 0$ uniformly on every compact subset of $\mathcal{D}$, then $f_n(\mu) \longrightarrow 0$, again uniformly on every compact subset of $\mathcal{D}$.

When T has property (β) for every $\lambda \in \mathbb{C}$, we simply say that T has property (β). This was first introduced by Bishop [4], in an attempt to develop a general spectral theory for operators on Banach spaces. According to Putinar [17], "*every hyponormal operator has property* (β)."

An operator $T = U|T|$ is said to be w-hyponormal if $|\widetilde{T}| \geq |T| \geq |\widetilde{T}^*|$ hold, where $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is the Aluthge transform of T (see [2] and [3]). It is known that the class of w-hyponormal operators coincides with class $\mathrm{A}(\frac{1}{2}, \frac{1}{2})$ (see [11] and [12].) Recently, Kimura [16] showed that every w-hyponormal operator has SVEP and property (β).

To study some properties of semi-hyponormal operators, we often consider the following transforms.

$$\text{(i) } S = U|T|^{\frac{1}{2}}, \qquad \text{(ii) } \widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \text{ (Aluthge transform).}$$

If T is semi-hyponormal then S and $\widetilde{T}$ are both hyponormal. Therefore we can expect to obtain some properties of semi-hyponormal operators by using above transforms and properties of hyponormal operators. But it is well known that $\sigma(S) \neq \sigma(T)$ and $\sigma(\widetilde{T}) = \sigma(T)$, so that, to study some spectral properties of semi-hyponormal operators, (ii) is a better transform than (i). Aluthge obtained more general result as follows: "*If T is p-hyponormal, then* (i) *$\widetilde{T}$ is $p+\frac{1}{2}$-hyponormal in case $0 < p \leq \frac{1}{2}$, and* (ii) *$\widetilde{T}$ is hyponormal in case $p \geq \frac{1}{2}$*" in [2]. Aluthge transform has more interesting properties itself. For example, $\|\widetilde{T}\| \leq \|T\|$ and $\overline{W(\widetilde{T})} \subseteq \overline{W(T)}$ in [14, 15, 19, 21], where $W(T)$ means the numerical range of an operator T. Moreover, by considering n-th iterated of Aluthge transform $\widetilde{T_n}$ of T, we obtained the following parallel results $\lim_{n\to\infty} \|\widetilde{T_n}\| = r(T)$ in [22] and $\bigcap_n \overline{W(\widetilde{T_n})} = \mathrm{conv}\sigma(T)$ in [1].

But until now, we do not know that for a class A operator T, whether $\tilde{T}$ belongs to the class of w-hyponormal operators or not. We obtained a transform from class A to the class of w-hyponormal operators is only T^2 in [12], but obviously $\sigma(T) \neq \sigma(T^2)$. In this report, first we shall give an operator transform $\hat{T}$ from class A to the class of hyponormal operators as an analogue of Aluthge transform satisfying $\sigma(\hat{T}) = \sigma(T)$, and obtain some spectral properties of class A operators. Next as an application of this transform, we shall show that every class A operator has SVEP and property (β) which is an extension of Kimura's result.

2. An operator transform from class A to the class of hyponormal operators

Let us start this section to prove the following result:

Theorem 2.1. *Let $T = U|T|$ be the polar decomposition of a class A operator. Then*

$$\hat{T} = WU|T^2|^{\frac{1}{2}}$$

is hyponormal, where $|T||T^*| = W\big||T||T^*|\big|$ is the polar decomposition.

To prove this result, we need the following theorems:

Theorem A ([12]). *Let A and B be positive operators. Then for each $p \geq 0$ and $r \geq 0$, the following assertions hold:*

(i) *If $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$, then $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$.*

(ii) *If $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ and $N(A) \subset N(B)$, then $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$.*

Theorem B ([13]). *Let $T = U|T|$, $S = V|S|$ and*

$$|T||S^*| = W\big||T||S^*|\big|$$

be the polar decompositions. Then $TS = UWV|TS|$ is also the polar decomposition.

Proof of Theorem 2.1. Since T is a class A operator, the following inequalities hold:

$$(|T|U^*|T|^2U|T|)^{\frac{1}{2}} = |T^2| \geq |T|^2 \Longleftrightarrow (|T^*||T|^2|T^*|)^{\frac{1}{2}} \geq |T^*|^2. \tag{2.1}$$

By (i) of Theorem A, we have

$$|T|^2 \geq (|T||T^*|^2|T|)^{\frac{1}{2}} = (|T|U|T|^2U^*|T|)^{\frac{1}{2}}. \tag{2.2}$$

Then by (2.1) and (2.2), $|T|U|T|$ is semi-hyponormal.

On the other hand, since $|T| = U^*U|T|$ and $U|T|$ are the polar decompositions, by Theorem B we have the polar decomposition of $|T|U|T|$ as follows:

$$|T| \cdot U|T| = U^*UWU\big||T|U|T|\big|, \tag{2.3}$$

where $|T||T^*| = W\big||T||T^*|\big|$ is the polar decomposition. Here by the definition of W, we have $N(U) \subset N(|T^*||T|) = N(W^*)$ and $W^*U^*U = W^*$ on $\mathcal{H} = N(U) \oplus R(U^*)$. Then we can arrangement (2.3) as follows:

$$|T|U|T| = U^*UWU\big||T|U|T|\big| = WU|T^2|. \tag{2.4}$$

Since $|T|U|T| = WU|T^2|$ is the polar decomposition of a semi-hyponormal operator, $\hat{T} = WU|T^2|^{\frac{1}{2}}$ is hyponormal. Hence the proof is complete. □

We remark that by (2.4) we can obtain the following relation for any $T \in B(\mathcal{H})$:

$$\hat{T}|T^2|^{\frac{1}{2}} = |T|T. \tag{2.5}$$

For an operator T, we denote the spectrum, the point spectrum, the approximate point spectrum and the residual spectrum by $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$ and $\sigma_r(T)$, respectively. A complex number μ is in the normal approximate point spectrum $\sigma_{na}(T)$ if there exists a sequence $\{x_n\}$ of unit vectors such that $(T-\mu)x_n \longrightarrow 0$ and $(T-\mu)^*x_n \longrightarrow 0$ as $n \to \infty$. It is easy to see that if T is hyponormal, then $\sigma_a(T) = \sigma_{na}(T)$ because the inequality $\|(T-\mu)^*x\| \leq \|(T-\mu)x\|$ always holds for all $\mu \in \mathbb{C}$ and all $x \in \mathcal{H}$, Next, we have the following spectral relation between $\hat{T}$ and T in case T belongs to class A.

Theorem 2.2. *Let T be a class A operator, Then $\sigma(\hat{T}) = \sigma(T)$.*

To prove Theorem 2.2, we shall prepare the following results.

Lemma 2.3. *If T belongs to the class A and μ is a non-zero complex number, then for a sequence $\{x_n\}$ of unit vectors, $(T-\mu)x_n \longrightarrow 0$ implies $(T-\mu)^* x_n \longrightarrow 0$.*

Lemma 2.3 is an extension of [18, Lemma 4] which discussed on a similar property for a fixed vector x.

Proof. By the assumption, we have

$$(T-\mu)x_n \longrightarrow 0 \quad \text{and} \quad (T^2-\mu^2)x_n \longrightarrow 0.$$

Since

$$\big|\|Tx_n\| - |\mu|\big| \leq \|(T-\mu)x_n\| \quad \text{and} \quad \big|\|T^2x_n\| - |\mu|^2\big| \leq \|(T^2-\mu^2)x_n\|,$$

we have

$$\|Tx_n\| \longrightarrow |\mu| \quad \text{and} \quad \|T^2x_n\| \longrightarrow |\mu|^2. \tag{2.6}$$

Since T belongs to class A, we obtain

$$\begin{aligned} \|Tx_n\|^2 &= (|T|^2x_n, x_n) \\ &\leq (|T^2|x_n, x_n) \\ &\leq \||T^2|x_n\| \quad \text{by Cauchy-Schwarz inequality} \\ &= \|T^2x_n\|, \end{aligned}$$

and by (2.6) we have

$$(|T^2|x_n, x_n) \longrightarrow |\mu|^2. \tag{2.7}$$

Therefore by (2.6) and (2.7),

$$\begin{aligned} \|(|T^2| - |\mu|^2)x_n\|^2 &= \|T^2x_n\|^2 - 2|\mu|^2(|T^2|x_n, x_n) + |\mu|^4 \\ &\longrightarrow |\mu|^4 - 2|\mu|^4 + |\mu|^4 = 0, \end{aligned}$$

that is,

$$(|T^2| - |\mu|^2)x_n \longrightarrow 0. \tag{2.8}$$

On the other hand, by (2.6) and (2.8), we have

$$\|(|T^2| - |T|^2)^{\frac{1}{2}}x_n\|^2 = (|T^2|x_n, x_n) - (|T|^2x_n, x_n) \longrightarrow 0,$$

that is,

$$(2.9) \qquad (|T^2| - |T|^2)x_n \longrightarrow 0.$$

Then by (2.8) and (2.9),

$$(|T|^2 - |\mu|^2)x_n = (|T|^2 - |T^2|)x_n + (|T^2| - |\mu|^2)x_n \longrightarrow 0.$$

Therefore

$$(T-\mu)^*x_n = \frac{1}{\mu}\{(|T|^2 - |\mu|^2)x_n - T^*(T-\mu)x_n\} \longrightarrow 0.$$

Hence the proof is complete. □

Theorem C ([10]).

(i) *If A is normal, then for any $B \in B(\mathcal{H})$, $\sigma(AB) = \sigma(BA)$.*

(ii) *Let $T = U|T|$ be the polar decomposition of a p-hyponormal operator ($p > 0$). Then for any $t > 0$,*

$$\sigma(U|T|^t) = \{e^{i\theta}r^t \ : \ e^{i\theta}r \in \sigma(T)\}.$$

Theorem D ([20]). *Let $\mathcal{R}$ be a set of the complex plane $\mathbb{C}$, $T(t)$ be an operator-valued function of $t \in [0,1]$ which is continuous in the norm topology, $\tau_t, t \in [0,1]$, be a family of bijective mapping from $\mathcal{R}$ onto $\tau_t(\mathcal{R}) \subset \mathbb{C}$, and for any fixed $z \in \mathcal{R}$, $\tau_t(z)$ be a continuous function of $t \in [0,1]$ such that τ_0 is the identity function. Suppose*

$$\sigma_a(T(t)) \cap \tau_t(\mathcal{R}) = \tau_t(\sigma_a(T(0)) \cap \mathcal{R})$$

for all $t \in [0,1]$. Then for all $t \in [0,1]$,

$$\sigma_r(T(t)) \cap \tau_t(\mathcal{R}) = \tau_t(\sigma_r(T(0)) \cap \mathcal{R}),$$
$$\sigma(T(t)) \cap \tau_t(\mathcal{R}) = \tau_t(\sigma(T(0)) \cap \mathcal{R}).$$

Let $\mathcal{F}$ be the set of all strictly monotone increasing continuous nonnegative functions on $\mathbb{R}^+ = [0,\infty)$. Let $\mathcal{F}_0 = \{\Psi \in \mathcal{F} : \Psi(0) = 0\}$ and $T = U|T|$. For $\Psi \in \mathcal{F}_0$, the mapping $\tilde{\Psi}$ is defined by $\tilde{\Psi}(\rho e^{i\theta}) = e^{i\theta}\Psi(\rho)$ and $\tilde{\Psi} = U\Psi(|T|)$.

Theorem E ([5]). *Let $T = U|T|$ and $\Psi \in \mathcal{F}_0$. Then*

$$\sigma_{na}(\tilde{\Psi}(T)) = \tilde{\Psi}(\sigma_{na}(T)).$$

Proof of Theorem 2.2. Let $T = U|T|$ be the polar decomposition. First, we shall prove that if T is a class A operator then

$$(2.10) \qquad \sigma(U|T|^2) = \{r^2e^{i\theta} : re^{i\theta} \in \sigma(T)\}.$$

Let $T(t) = U|T|^{1+t}$ and $\tau_t(re^{i\theta}) = e^{i\theta}r^{1+t}$. Since

$$|T(t)| = |T|^{1+t} \quad \text{and} \quad |T(t)^*| = |T^*|^{1+t},$$

we obtain

$$\begin{aligned} T \text{ belongs to class A} &\Longleftrightarrow (|T^*||T|^2|T^*|)^{\frac{1}{2}} \geq |T^*|^2 \\ &\Longleftrightarrow (|T(t)^*|^{\frac{1}{1+t}}|T(t)|^{\frac{2}{1+t}}|T(t)^*|^{\frac{1}{1+t}})^{\frac{1}{2}} \geq |T(t)^*|^{\frac{2}{1+t}} \\ &\Longleftrightarrow T(t) \text{ belongs to class } \mathrm{A}(\tfrac{1}{1+t}, \tfrac{1}{1+t}) \\ &\Longrightarrow T(t) \text{ belongs to class A by (1.1).} \end{aligned}$$

By Lemma 2.3 and Theorem E, we have

$$\begin{aligned} \sigma_a(T(t)) - \{0\} &= \sigma_{na}(T(t)) - \{0\} \\ &= \tau_t(\sigma_{na}(T) - \{0\}) \\ &= \tau_t(\sigma_a(T) - \{0\}) \\ &= \tau_t(\sigma_a(T)) - \{0\}. \end{aligned}$$

On the other hand, if $0 \in \sigma_a(T(t))$, then there exists a sequence $\{x_n\}$ of unit vectors such that $U|T|^{1+t}x_n \longrightarrow 0$. Hence by

$$\|Tx_n\|^2 = (U|T|^{1+t}x_n, U|T|^{1-t}x_n) \longrightarrow 0,$$

we have $0 \in \sigma_a(T)$. Conversely, if $0 \in \sigma_a(T)$, then we have $0 \in \sigma_a(T(t))$ by

$$\|U|T|^{1+t}x_n\| \leq \||T|^t\| \cdot \|Tx_n\| \longrightarrow 0.$$

Hence we obtain $\sigma_a(T(t)) = \tau_t(\sigma_a(T))$ for all $t \in [0,1]$, and by Theorem D we have $\sigma(T(t)) = \tau_t(\sigma(T))$ for all $t \in [0,1]$. Especially, put $t = 1$ we have (2.10).

Next, by (i) of Theorem C and (2.10) we obtain

$$\sigma(WU|T^2|) = \sigma(|T|U|T|) = \sigma(U|T|^2) = \{e^{i\theta}r^2 \ : \ e^{i\theta}r \in \sigma(T)\}.$$

By Theorem 2.1, $\hat{T}$ is hyponormal. Hence by (ii) of Theorem C, we have

$$\sigma(\hat{T}) = \sigma(WU|T^2|^{\frac{1}{2}}) = \{e^{i\theta}r \ : \ e^{i\theta}r^2 \in \sigma(U|T|^2)\} = \sigma(T).$$

Therefore the proof is complete. □

In general, Theorem 2.2 does not hold for an arbitrary operator. In fact let

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $\sigma(T) = \{0, 1\}$. Let $T = U|T|$ be the polar decomposition of T, then we obtain $|T|U|T| = |T| \geq 0$ because $T^2 = T$ holds. Hence by (2.4) and the definition of $\hat{T}$, we have

$$\hat{T} = |T|^{\frac{1}{2}}.$$

On the other hand, by the simple calculation, we have

$$|T|^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad |T|^{\frac{1}{2}} = \frac{1}{2^{3/4}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Hence $\sigma(\hat{T}) = \{0, \sqrt[4]{2}\} \neq \sigma(T)$.

But in case T belongs to class A, we can precise Theorem 2.2 as follows:

Theorem 2.4. *Let T be a class A operator. For a complex number μ and a sequence $\{x_n\}$ of unit vectors,*

$$(T-\mu)x_n \longrightarrow 0 \textit{ if and only if } (\hat{T}-\mu)x_n \longrightarrow 0.$$

Proof. Let $T = U|T|$ be the polar decomposition. (a) We shall prove that $(T-\mu)x_n \longrightarrow 0$ implies $(\hat{T}-\mu)x_n \longrightarrow 0$. In case $\mu = 0$, it is obvious by

$$\|\hat{T}x_n\| = \||T^2|^{\frac{1}{2}}x_n\| = (|T^2|x_n, x_n)^{\frac{1}{2}} \longrightarrow 0.$$

So we shall prove the case $\mu \neq 0$. By Lemma 2.3, we have $(T-\mu)^* x_n \longrightarrow 0$. Then we obtain

$$(|T| - |\mu|)x_n \longrightarrow 0, \qquad (|T^*| - |\mu|)x_n \longrightarrow 0, \tag{2.11}$$

and

$$\left(\big||T||T^*|\big|^2 - |\mu|^4\right)x_n \longrightarrow 0.$$

Hence we have

$$\left(\big||T||T^*|\big|^{\frac{1}{2}} - |\mu|\right)x_n \longrightarrow 0.$$

On the other hand, if $\mu = e^{i\theta}|\mu|$, then by (2.11) we have

$$(U - e^{i\theta})x_n = \frac{1}{|\mu|}\{U(|\mu| - |T|)x_n + (T-\mu)x_n\} \longrightarrow 0.$$

Hence

$$\begin{aligned}
(\hat{T}-\mu)x_n &= (WU|T^2|^{\frac{1}{2}} - |\mu|e^{i\theta})x_n \\
&= \{WU(|T|U^*|T|^2U|T|)^{\frac{1}{4}} - |\mu|e^{i\theta}\}x_n \\
&= \{W(|T^*||T|^2|T^*|)^{\frac{1}{4}}U - |\mu|e^{i\theta}\}x_n \\
&= W\big||T||T^*|\big|^{\frac{1}{2}}(U - e^{i\theta})x_n + e^{i\theta}(W\big||T||T^*|\big|^{\frac{1}{2}} - |\mu|)x_n,
\end{aligned}$$

and we only prove $(W\big||T||T^*|\big|^{\frac{1}{2}} - |\mu|)x_n \longrightarrow 0$. By the fact $|T||T^*| = W\big||T||T^*|\big|$ and

$$(W\big||T||T^*|\big|^{\frac{1}{2}} - |\mu|)x_n = \frac{1}{|\mu|}\{-W\big||T||T^*|\big|^{\frac{1}{2}}(\big||T||T^*|\big|^{\frac{1}{2}} - |\mu|)x_n + (|T||T^*| - |\mu|^2)x_n\},$$

we obtain $(W\big||T||T^*|\big|^{\frac{1}{2}} - |\mu|)x_n \longrightarrow 0$. Hence $(\hat{T}-\mu)x_n \longrightarrow 0$.

(b) We shall show that $(\hat{T}-\mu)x_n \longrightarrow 0$ implies $(T-\mu)x_n \longrightarrow 0$. In case $\mu = 0$, it is easy since $|T^2| \geq |T|^2$ holds. So we shall prove the case $\mu(= |\mu|e^{i\theta}) \neq 0$. By Theorem 2.1, $\hat{T}$ is hyponormal. Then it is known that $(\hat{T}-\mu)x_n \longrightarrow 0$ implies $(\hat{T}-\mu)^*x_n \longrightarrow 0$, and also we have

$$(|\hat{T}| - |\mu|)x_n \longrightarrow 0 \quad \text{and} \quad (|\hat{T}^*| - |\mu|)x_n \longrightarrow 0.$$

Then by

$$|\hat{T}| = |T^2|^{\frac{1}{2}} = (|T|U^*|T|^2U|T|)^{\frac{1}{4}} \quad \text{and} \quad |\hat{T}^*| = (|T|U|T|^2U^*|T|)^{\frac{1}{4}},$$

we obtain

$$\left((|T|U^*|T|^2U|T|)^{\frac{1}{2}} - |\mu|^2\right)x_n \longrightarrow 0 \quad \text{and} \quad \left((|T|U|T|^2U^*|T|)^{\frac{1}{2}} - |\mu|^2\right)x_n \longrightarrow 0.$$

On the other hand, since T belongs to class A, by (2.1) and (2.2) we have

$$(|T|U^*|T|^2U|T|)^{\frac{1}{2}} \geq |T|^2 \geq (|T|U|T|^2U^*|T|)^{\frac{1}{2}}. \tag{2.12}$$

Hence $\left((|T|^2 - |\mu|^2)x_n, x_n\right) \longrightarrow 0$ holds. By (2.12), since

$$\begin{aligned} 0 &\leq \left\| \{(|T|U^*|T|^2U|T|)^{\frac{1}{2}} - |T|^2\}^{\frac{1}{2}}x_n \right\|^2 \\ &= (\{(|T|U^*|T|^2U|T|)^{\frac{1}{2}} - |\mu|^2\}x_n, x_n) - \left((|T|^2 - |\mu|^2)x_n, x_n\right) \longrightarrow 0, \end{aligned}$$

we have

$$(|T|^2 - |\mu|^2)x_n = \{|T|^2 - (|T|U^*|T|^2U|T|)^{\frac{1}{2}}\}x_n + \{(|T|U^*|T|^2U|T|)^{\frac{1}{2}} - |\mu|^2\}x_n \longrightarrow 0.$$

By the polar decompositions $\hat{T} = WU|T^2|^{\frac{1}{2}}$ and $|T|U|T| = WU|T^2|$, we have

$$(|T|U - \mu)x_n = \frac{1}{|\mu|}|T|U(|\mu| - |T|)x_n + \frac{1}{|\mu|}\hat{T}(|T^2|^{\frac{1}{2}} - |\mu|)x_n + (\hat{T} - \mu)x_n \longrightarrow 0.$$

Hence we obtain $(T-\mu)Ux_n = U(|T|U - \mu)x_n \longrightarrow 0$. Then by Lemma 2.3, we obtain $(T-\mu)^*Ux_n \longrightarrow 0$ and $(e^{i\theta}|T| - |\mu|U)x_n = e^{i\theta}(T-\mu)^*Ux_n \longrightarrow 0$. Therefore we have

$$(T-\mu)x_n = U(|T| - |\mu|)x_n + (|\mu|U - e^{i\theta}|T|)x_n + e^{i\theta}(|T| - |\mu|)x_n \longrightarrow 0.$$

Hence the proof is complete. □

Corollary 2.5. *Let T be a class A operator, then $\sigma_p(\hat{T}) = \sigma_p(T)$ and $\sigma_a(\hat{T}) = \sigma_a(T)$.*

3. An application of $\hat{T}$ to SVEP and property (β)

In this section, we shall show that every class A operator has SVEP and property (β) as an application of $\hat{T}$.

Theorem 3.1. *If T belongs to class A, then T has SVEP and property (β).*

To prove Theorem 3.1, we prepare the following lemma which is a slight modification of [16, Lemma 2.5].

Lemma 3.2. *Let $\mathcal{D}$ be an open subset of $\mathbb{C}$ and $f_n : \mathcal{D} \longrightarrow \mathcal{H}$ $(n = 1, 2, \ldots)$ be vector-valued analytic functions such that $\mu^2 f_n(\mu) \longrightarrow 0$ uniformly on every compact subset of $\mathcal{D}$. Then $f_n(\mu) \longrightarrow 0$, again uniformly on every compact subset of $\mathcal{D}$.*

Proof. Let us fix an arbitrary $\lambda \in \mathcal{D}$. It suffices to show that there exists a constant $r > 0$ such that $\{|\mu - \lambda| \leq r\} \subset \mathcal{D}$ and $f_n(\mu) \longrightarrow 0$ uniformly on $\{|\mu - \lambda| \leq r\}$. If $\lambda \neq 0$, then we need merely to take r such as $0 \notin \{|\mu - \lambda| \leq r\} \subset \mathcal{D}$. So we consider the case where $\lambda = 0$. Take any constant $r > 0$ such that $\{|\mu| \leq r\} \subset \mathcal{D}$. Then for each $n = 1, 2, \ldots$, we can find an ω_n with $|\omega_n| = r$ such that $\|f_n(\mu)\| \leq \|f_n(\omega_n)\|$ on $\{|\mu| \leq r\}$ by the maximum principle. Thus

$$\|f_n(\mu)\| = \frac{1}{|\omega_n|^2}|\omega_n|^2\|f_n(\mu)\| \leq \frac{1}{r^2}\|\omega_n^2 f_n(\omega_n)\| \longrightarrow 0$$

uniformly on $\{|\mu| \leq r\}$. □

Proof of Theorem 3.1. By the definition of SVEP and property (β), we have only to prove that T has property (β).

Let $\mathcal{D}$ be an open neighborhood of $\lambda \in \mathbb{C}$ and f_n $(n = 1, 2, \ldots)$ be vector-valued analytic functions on $\mathcal{D}$ such that $(T - \mu)f_n(\mu) \longrightarrow 0$ uniformly on every compact subset of $\mathcal{D}$. We may assume that $\sup\limits_n \|f_n(\mu)\| < +\infty$ on every compact subset of $\mathcal{D}$. In fact, let M_n be a positive number such that $\|f_n(\mu)\| \leq M_n$. Then by replacing $f_n(\mu)$ with $\frac{f_n(\mu)}{M_n+1}$, we have $\sup\limits_n \|f_n(\mu)\| \leq 1$ and $(T-\mu)f_n(\mu) \longrightarrow 0$ uniformly on every compact subset of $\mathcal{D}$.

By the assumption $(T - \mu)f_n(\mu) \longrightarrow 0$ uniformly, we have $(T^2 - \mu^2)f_n(\mu) \longrightarrow 0$ also uniformly. Since

$$\big|\|Tf_n(\mu)\| - \|\mu f_n(\mu)\|\big| \leq \|(T-\mu)f_n(\mu)\| \text{ and } \big|\|T^2 f_n(\mu)\| - \|\mu^2 f_n(\mu)\|\big| \leq \|(T^2-\mu^2)f_n(\mu)\|,$$

we have

$$\|Tf_n(\mu)\| - \|\mu f_n(\mu)\| \longrightarrow 0 \quad \text{and} \quad \|T^2 f_n(\mu)\| - \|\mu^2 f_n(\mu)\| \longrightarrow 0 \quad \text{uniformly.} \tag{3.1}$$

Since T belongs to class A, we obtain

$$\begin{aligned}
\|Tf_n(\mu)\|^2 - \|\mu f_n(\mu)\|^2 &= \big(|T|^2 f_n(\mu), f_n(\mu)\big) - \big(|\mu|^2 f_n(\mu), f_n(\mu)\big) \\
&\leq \big(|T^2| f_n(\mu), f_n(\mu)\big) - \big(|\mu|^2 f_n(\mu), f_n(\mu)\big) \\
&\leq \||T^2| f_n(\mu)\| \cdot \|f_n(\mu)\| - \|\mu^2 f_n(\mu)\| \cdot \|f_n(\mu)\| \\
&= \big(\|T^2 f_n(\mu)\| - \|\mu^2 f_n(\mu)\|\big)\, \|f_n(\mu)\|
\end{aligned}$$

by Cauchy-Schwarz inequality, and by (3.1) we have

$$\big(|T^2| f_n(\mu), f_n(\mu)\big) - \big(|\mu|^2 f_n(\mu), f_n(\mu)\big) \longrightarrow 0 \quad \text{uniformly.} \tag{3.2}$$

Therefore by (3.1) and (3.2), we have

$$\begin{aligned}\left\|(|T^2|-|\mu|^2)f_n(\mu)\right\|^2 &= \|T^2 f_n(\mu)\|^2 - 2|\mu|^2(|T^2|f_n(\mu), f_n(\mu)) + |\mu|^4\|f_n(\mu)\|^2 \\ &= \|T^2 f_n(\mu)\|^2 - \|\mu^2 f_n(\mu)\|^2 - 2|\mu|^2\left((|T^2|-|\mu|^2)f_n(\mu), f_n(\mu)\right) \\ &\longrightarrow 0 \quad \text{uniformly,}\end{aligned}$$

that is,

$$(3.3) \qquad (|T^2|-|\mu|^2)f_n(\mu) \longrightarrow 0 \quad \text{and} \quad (|T^2|^{\frac{1}{2}}-|\mu|)f_n(\mu) \longrightarrow 0 \quad \text{uniformly.}$$

On the other hand, by (3.1) and (3.2),

$$\begin{aligned}0 &\le \|(|T^2|-|T|^2)^{\frac{1}{2}} f_n(\mu)\|^2 \\ &= (|T^2|f_n(\mu), f_n(\mu)) - (|T|^2 f_n(\mu), f_n(\mu)) \longrightarrow 0 \quad \text{uniformly,}\end{aligned}$$

that is,

$$(3.4) \qquad (|T^2|-|T|^2)f_n(\mu) \longrightarrow 0 \quad \text{uniformly.}$$

Hence by (3.3) and (3.4), we have

$$(3.5) \qquad (|T|^2-|\mu|^2)f_n(\mu) \longrightarrow 0 \quad \text{and} \quad (|T|-|\mu|)f_n(\mu) \longrightarrow 0 \quad \text{uniformly.}$$

Therefore we obtain

$$\begin{aligned}(\hat{T}-\mu)|T^2|^{\frac{1}{2}} f_n(\mu) &= (|T|T - \mu|T^2|^{\frac{1}{2}})f_n(\mu) \quad \text{by (2.5)} \\ &= |T|(T-\mu)f_n(\mu) + \mu(|T|-|\mu|)f_n(\mu) + \mu(|\mu|-|T^2|^{\frac{1}{2}})f_n(\mu) \\ &\longrightarrow 0 \quad \text{uniformly by (3.3) and (3.5).}\end{aligned}$$

By Theorem 2.1, $\hat{T}$ is hyponormal, so $\hat{T}$ has property (β), that is,

$$|T^2|^{\frac{1}{2}} f_n(\mu) \longrightarrow 0 \quad \text{uniformly,}$$

that is,

$$T^2 f_n(\mu) \longrightarrow 0 \quad \text{uniformly.}$$

Hence we have $\mu^2 f_n(\mu) \longrightarrow 0$ uniformly, and also $f_n(\mu) \longrightarrow 0$ uniformly by Lemma 3.2. This completes the proof. □

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