Title	作用素に対する任意区間上のJensen不等式 (作用素不等式に関わる最近の話題)
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Citation	数理解析研究所講究録 (2004), 1359: 93-102
Issue Date	2004-02
URL	http://hdl.handle.net/2433/25238
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録 1359 巻 2004 年 93-102

作用素に対する任意区間上の Jensen 不等式

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Jensen の不等式は、多くの応用もあり作用素版にも拡張されてきた。われわれの興味は、Hilbert 空間上のエルミット作用素に関する Jensen 不等式であるが、最近、 F.Hansen と G.K.Pedersen [12] が、彼ら自身の結果 [11, 10] を見直した。

Hansen-Pedersen's theorem. Let n be an integer with $n \geq 2$. Then,

(A) *A continuous function f on $\mathcal{I}$ is convex if and only if*

$$\mathrm{Tr}\left(f\left(\sum_{k=1}^{n} C_k^* A_k C_k\right)\right) \leq \mathrm{Tr}\left(\sum_{k=1}^{n} C_k^* f(A_k) C_k\right)$$

hold for all selfadjoint matrices A_k with $\sigma(A_k) \subset \mathcal{I}$ and matrices C_k with $\sum_{k=1}^{n} C_k^* C_k = 1$.

(B) *The following conditions are all equivalent to that f is operator convex on $\mathcal{I}$:*

1. $f\left(\sum_{k=1}^{n} C_k^* A_k C_k\right) \leq \sum_{k=1}^{n} C_k^* f(A_k) C_k$ *hold for all selfadjoint A_k with $\sigma(A_k) \subset \mathcal{I}$ and C_k with $\sum_{k=1}^{n} C_k^* C_k = 1$.*
2. $f(C^*AC) \leq C^* f(A) C$ *hold for all selfadjoint A with $\sigma(A) \subset \mathcal{I}$ and isometries C.*
3. $Pf\big(PAP + s(1-P)\big) \leq Pf(A)P$ *for all selfadjoint operators A with $\sigma(A) \subset \mathcal{I}$, scalars $s \in \mathcal{I}$ and projections P.*

実は、われわれもほぼ同時期に同様の結果を得ており、この研究集会で指摘されたように、以前にも内山充先生の報告 [15, 16] や奇しくもこの講究録にもある古田孝之先生の講演 [9] でも同様の結果が報告されている。ここでは、Hansen-Pederesen の結果を見直す形で、結果としては知られていることかもしれないが、おそらくはもっとも統一的であろうと思われる見方をしてみたので、それを提示する。

まず、トレース方不等式では、古典的な von Neumann の「任意の凸関数 f に対して、$\mathrm{Tr} \circ f$ が作用素凸」[17] という結果がある（証明は付けられていないようである。）：

von Neumann's trace inequality. If f is convex on $\mathcal{I}$, then

$$\mathrm{Tr}\ \big(f\big((1-t)A+tB\big)\big) \le \mathrm{Tr}\ \big((1-t)f(A)+tf(B)\big)$$

for all trace class selfadjoint operators A and B with $\sigma(A)$, $\sigma(B) \subset \mathcal{I}$.

この方向で述べられた Jensen 不等式の結果には、たとえば Brown-幸崎 の結果 [4] がある:

Brown-Kosaki inequality. If f is convex and continuous on $[0,\infty)$ with $f(0)=0$, then

$$\tau(f(C^*AC)) \le \tau(C^*f(A)C)$$

for any positive A, contractive C and trace τ on a semi-finite von Neumann algebra.

もともとの Hansen-Pesersen [11] は:

Hansen-Pedersen-Jensen inequality. *Let f be a (continuous) real function on $[0,r)$. Then the following conditions are equivalent:*

(1) *f is operator convex and $f(0) \le 0$.*
(2) *$f(C^*AC) \le C^*f(A)C$ for all positive operators $A \le r$ and contractions C.*
(3) *$f(C^*AC + D^*BD) \le C^*f(A)C + D^*f(B)D$ for all positive operators A, $B \le r$ and operators C and D with $C^*C + D^*D \le 1$.*
(4) *$f(PAP) \le Pf(A)P$ for all positive operators $A \le r$ and projections P*

さらに、$r=\infty$、$f \le 0$ のときは、次の結果とも同等である:

(5) *$-f$ is operator monotone.*

一方、作用素不等式では、 Davis [6] と Choi [5] (see also Ando [1]) の結果がある:

Davis-Choi-Jensen inequality. *Let Φ be a unital positive linear map between C^*-algebras $\mathcal{A},\mathcal{B}$. If f is an operator convex function on an interval $\mathcal{I}$, then*

$$f\left(\Phi(A)\right) \le \Phi\left(f(A)\right)$$

for all selfadjoint operators $A \in \mathcal{A}$ with $\sigma(A) \subset \mathcal{I}$.

われわれの視点は2つある。本来の Jensen 不等式は、確率分布に対してのそれであり、作用素的には等距離性、もしくは写像にすれば、単位元を保存する unital 性が、やはり本質的であるということ。さらに、等距離作用素や写像は当然のことながら、別の空間への写像であるべきだということである。それが証明の流れを自然にし、定理の statement も自然な形にするのである。

まず、トレース型のほうから述べるが、簡単のために行列に話を限る（作用素への拡張は容易だと思われるが、末梢的な条件が付随するだろう）。次の**固有汎関数** なるものを導入したほうが見やすい。エルミット行列 A に対する **eigenfunctional** τ は、

$$\tau(X) = \sum_{k=1}^{n} \alpha_k \langle X e_k, e_k \rangle$$

（$\alpha_k \geq 0$ で、$\{e_k\}$ は、A の固有ベクトルから作られる固定されたCONS。選び方は任意。）という形の正値線型汎関数である。係数 α_k がすべて1ならば、通常のトレースである。一見非負な関数 f について、$f(A)$ のトレースに過ぎないように見えるかもしれないが、重複する固有値で別の係数も取れるので、少し広い概念になっている。このとき、

Theorem 1. *Let f be a real continuous function on an interval $\mathcal{I}$, A and A_k $n \times n$ selfadjoint matrices with $\sigma(A), \sigma(A_k) \subset \mathcal{I}$ and $n \geq m$ (n and m are arbitrary). Then the following conditions are mutually equivalent:*

(i) *f is convex.*

(ii) *$\tau f(C^*AC) \leq \tau C^* f(A) C$ for all $n \times m$ isometries C and eigenfunctionals τ for C^*AC.*

(iii) *$\tau f(\sum_{k=1}^{N} C_k^* A_k C_k) \leq \tau \sum_{k=1}^{n} C_k^* f(A_k) C_k$ for all $n \times m$ matrices C_k with $\sum_k C_k^* C_k = 1_m$ and eigenfunctionals τ for $\sum_{k=1}^{N} C_k^* A_k C_k$.*

(iv) *$\tau f(\Phi(A)) \leq \tau \Phi(f(A))$ for all unital positive linear maps Φ between matrix-algebras and eigenfunctionals τ for $\Phi(A)$.*

(v) *$\tau f(\sum_{k=1}^{N} P_k A_k P_k) \leq \tau \sum_{k=1}^{n} P_k f(A_k)$ for all projections P_k with $\sum_k P_k = 1_n$ and eigenfunctionals τ for $\sum_{k=1}^{N} P_k A_k P_k$.*

(vi) $\mathrm{Tr} \circ f$ *is operator convex on $\mathcal{I}$.*

Proof. **(i)⇒(ii)** : It suffices to show the case that τ is a vector state for an eigenvector x of C^*AC. For a spectral decomposition $A = \sum_j t_j E_j$, we have

$$\sum_j \langle E_j Cx, Cx \rangle = \langle C^*Cx, x \rangle = \langle x, x \rangle = 1$$

and thereby the numerical Jensen's inequality implies

$$\langle f(C^*AC)x, x \rangle = f(\langle C^*ACx, x \rangle) = f\left(\left\langle \sum_j t_j E_j Cx, Cx \right\rangle \right) = f\left(\sum_j t_j \langle E_j Cx, Cx \rangle \right)$$
$$\leq \sum_j f(t_j) \langle E_j Cx, Cx \rangle = \left\langle \sum_j f(t_j) E_j Cx, Cx \right\rangle = \langle C^* f(A) Cx, x \rangle .$$

(ii)⇒(iii) : We have (iii) applying (ii) for

$$C = \begin{pmatrix} C_1 \\ \vdots \\ C_N \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_N \end{pmatrix}.$$

(iii)⇒(iv) : For a spectral decomposition $A = \sum_k t_k E_k$ and $C_k = \sqrt{\Phi(E_k)}$, it follows from (iii) that

$$\tau f(\Phi(A)) = \tau f\left(\sum_k t_k \Phi(E_k)\right) \leq \tau \sum_k f(t_k)\Phi(E_k) = \tau\Phi\left(\sum_k f(t_k)E_k\right) = \tau\Phi(f(A)).$$

(iv)⇒(v) : Clear.
(v)⇒(vi) : Let $\sigma(A), \sigma(B) \subset \mathcal{I}$. For

$$X = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, P_2 = P_1^{\perp}, U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

the operator U is unitaries, so that

$$\begin{aligned} 2\mathrm{Tr}\ f\left(\frac{A+B}{2}\right) &= \mathrm{Tr}\ \begin{pmatrix} f\left(\frac{A+B}{2}\right) & 0 \\ 0 & f\left(\frac{A+B}{2}\right) \end{pmatrix} = \mathrm{Tr}\ f(P_1U^*XUP_1 + P_2U^*XUP_2) \\ &\leq \mathrm{Tr}\ (P_1 f(U^*XU)P_1 + P_2 f(U^*XU)P_2) = \mathrm{Tr}\ (P_1U^*f(X)UP_1 + P_2U^*f(X)UP_2) \\ &= \mathrm{Tr}\ \begin{pmatrix} \frac{f(A)+f(B)}{2} & 0 \\ 0 & \frac{f(A)+f(B)}{2} \end{pmatrix} = 2\mathrm{Tr}\ \frac{f(A)+f(B)}{2}. \end{aligned}$$

Thus the continuity of f implies (vi), cf.[11].
(vi)⇒(i) : (i) is the 1-dimensional case of (vi). □

Remark 1. As in Hansen-Pedersen's theorem (A), the above theorem holds even if τ is Tr alone. Actually, suppose (ii) holds for $\tau = \mathrm{Tr}$ only:

(ii') $\mathrm{Tr}\ f(C^*AC) \leq \mathrm{Tr}\ C^*f(A)C$ for all $n \times m$ isometries C.

We may assume that C^*AC is diagonal. Let C_k be a unit eigenvector of A whose nonzero entries are the k-th one only and $\tau_{(k)}$ be the eigenfunctional for C_k. Then

$$\tau_{(k)} f(C^*AC) = \mathrm{Tr}\ C_k^* f(C^*AC)C_k = \mathrm{Tr}\ f(C_k^*C^*ACC_k) \leq \mathrm{Tr}\ C_k^*C^*f(A)CC_k = \tau_{(k)}C^*f(A)C.$$

For all eigenfunctionals τ, we can choose nonnegative numbers α_k with $\tau = \sum_{j=1}^n \alpha_j \tau_{(j)}$, so that (ii) holds for all τ.

固有汎関数は、行列の majorization に使えることはすぐにわかるだろう。エルミット行列 A の大きい順に並べられた固有値 t_j について、$N_k(A)=\sum_{j=1}^k t_j$ とおくとき、A **weakly majorizes** B ($A \prec_w B$) とは、$N_k(A) \le N_k(B)$ ($\forall k = 1,2,...,n$) が成り立つことで、さらに $k=n$ のとき等号が成立すれば、A **majorizes** B ($A \prec B$) と呼ばれる。f が **isotone** であるとは、$A \prec B \implies f(A) \prec_w f(B)$ が常に成り立つことで、**strongly isotone** とは、$A \prec_w B \implies f(A) \prec_w f(B)$ な成り立つことである。このとき、$A \prec B$ は、$A=\sum_j a_j U_j^* B U_j$ となるユニタリ U_j と $\sum_j a_j = 1$ となる $a_j > 0$ が存在することと同値であることが知られている cf. [2]。すると、本質的には知られていることだがより Jensen との関連をはっきりさせることができた：

Corollary 2. *A continuous function f is convex if and only if f is isotone. Moreover, suppose that f is increasing. Then f is convex if and only if f is strongly isotone.*

Proof. Suppose f is convex and $A \prec B$. We may assume $A = \mathrm{diag}(t_1,...,t_n)$ such that $f(t_i)$ is decreasing. Then there exist unitaries U_j and positive weights a_j with $A = \sum a_j U_j^* B U_j$. Putting $\tau_k = \sum_{j=1}^k \tau_{(j)}$ in the above remark, then τ_k is an eigenfunctional for A and $\tau_k(f(A)) = N_k(f(A))$. For selfadjoint X, $\tau_k(X) \le N_k(X)$ in general (e.g. by the Courant-Fisher minimax theorem). It follows from Theorem 1 that

$$N_k(f(A)) = \tau_k(f(A)) = \tau_k f\left(\sum a_j U_j^* B U_j\right) \le \tau_k \sum a_j U_j^* f(B) U_j \le N_k\left(\sum a_j U_j^* f(B) U_j\right),$$

and hence $f(A) \prec_w \sum a_j U_j^* f(B) U_j \prec f(B)$.

Next suppose f is an increasing convex function and $A = \mathrm{diag}(t_1,...,t_n) \prec_w B$. Take $B' \equiv \mathrm{diag}(s_1,...,s_n) = U^*BU$ for some unitary U where s_j themselves are decreasing. Then putting

$$B_0 = B' - \mathrm{diag}(0,...,0,\sum_j (s_j - t_j)) \le B',$$

we have $\mathrm{Tr}\ A = \mathrm{Tr}\ B_0$ and $A \prec B_0$. Since $f(A) \prec_w f(B_0) \le f(B')$ by monotonicity of f for diagonal matrices, we have $f(A) \prec_w f(B') = f(U^*BU)$ and hence $f(A) \prec_w f(B)$.

Conversely suppose f is isotone. Considering

$$A = \begin{pmatrix} (x+y)/2 & \\ & (x+y)/2 \end{pmatrix} \prec B = \begin{pmatrix} x & \\ & y \end{pmatrix}$$

for $x, y \in \mathcal{I}$, we have

$$2f\left(\frac{x+y}{2}\right) = \mathrm{Tr}\ f(A) \le \mathrm{Tr}\ f(B) = f(x) + f(y),$$

and hence f is convex by the continuity if f. Suppose f is strongly isotone. Then the above argument also shows f is convex. Considering 1-dimensional case, we have f is increasing. □

つぎに、作用素不等式の場合は（結果自体は、(iv) を除いて内山 [15, Theo3.3] で言及されている）：

Theorem 3. *Let f be a real function on an interval $\mathcal{I}$, A or A_k a selfadjoint operator with $\sigma(A), \sigma(A_k) \subset \mathcal{I}$, and H or K a Hilbert space. Then the following conditions are mutually equivalent:*

(i) *f is operator convex on $\mathcal{I}$.*

(ii) *$f(C^*AC) \leq C^*f(A)C$ for all $A \in B(H)$ and isometries $C \in B(K,H)$.*

(ii') *$f(C^*AC) \leq C^*f(A)C$ for all A and isometries C in $B(H)$.*

(iii) *$f(\sum_{k=1}^n C_k^*A_kC_k) \leq \sum_{k=1}^n C_k^*f(A_k)C_k$ for all $A_k \in B(H)$ and $C_k \in B(K,H)$ with $\sum_k C_k^*C_k = 1_K$.*

(iii') *$f(\sum_{k=1}^n C_k^*A_kC_k) \leq \sum_{k=1}^n C_k^*f(A_k)C_k$ for all A_k, $C_k \in B(H)$ with $\sum_k C_k^*C_k = 1_H$.*

(iv) *$f(\Phi(A)) \leq \Phi(f(A))$ for all unital positive linear map Φ between C^*-algebras $\mathcal{A}, \mathcal{B}$ and all $A \in \mathcal{A}$.*

(v) *$f(\sum_{k=1}^n P_kA_kP_k) \leq \sum_{k=1}^n P_kf(A_k)P_k$ for all A_k, and projections $P_k \in B(H)$ with $\sum_k P_k = 1_H$.*

Proof. **(i)⇒(ii):** Take $B = B^* \in B(K)$ with $\sigma(B) \in \mathcal{I}$. For $P = \sqrt{1_H - CC^*}$, putting

$$X = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in B(H \oplus K),\ U = \begin{pmatrix} C & P \\ 0 & -C^* \end{pmatrix}, V = \begin{pmatrix} C & -P \\ 0 & C^* \end{pmatrix} \in B(K \oplus H, H \oplus K),$$

we have

$$C^*P = \sqrt{1_K - C^*C}C^* = 0 \in B(H,K),\ PC = C\sqrt{1_K - C^*C} = 0 \in B(K,H),$$

so that both U and V are unitaries. Since

$$U^*XU = \begin{pmatrix} C^*AC & C^*AP \\ PAC & PAP + CBC^* \end{pmatrix}, \qquad V^*XV = \begin{pmatrix} C^*AC & -C^*AP \\ -PAC & PAP + CBC^* \end{pmatrix},$$

then the operator convexity of f implies

$$\begin{pmatrix} f(C^*AC) & 0 \\ 0 & f(PAP+CBC^*) \end{pmatrix} = f\begin{pmatrix} C^*AC & 0 \\ 0 & PAP+CBC^* \end{pmatrix}$$
$$= f\left(\frac{U^*XU+V^*XV}{2}\right)$$
$$\leq \frac{f(U^*XU)+f(V^*XV)}{2} = \frac{U^*F(X)U+V^*f(X)V}{2}$$
$$= \begin{pmatrix} C^*f(A)C & 0 \\ 0 & Pf(A)P+Cf(B)C^* \end{pmatrix}.$$

Thus we have **(ii)** by seeing the $(1,1)$-components.

(ii)⇒(iii): Putting

$$\tilde{A} = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{pmatrix} \in B(H\oplus\cdots\oplus H), \qquad \tilde{C} = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} \in B(K, H\oplus\cdots\oplus H),$$

we have $\tilde{C}^*\tilde{C} = 1_K$. It follows from (ii) that

$$f\left(\sum_{k=1}^n C_k^*A_kC_k\right) = f\left(\tilde{C}^*\tilde{A}\tilde{C}\right) \leq \tilde{C}^*f(\tilde{A})\tilde{C} = \sum_{k=1}^n C_k^*f(A_k)C_k.$$

(iii)⇒(iv): Considering the universal enveloping von Neumann algebras and the uniquely extended linear map, we may assume that $\mathcal{A}$ is a von Neumann algebra. Thereby a selfadjoint operator $A \in \mathcal{A}$ can be approximated uniformly by a simple function $A' = \sum_k t_kE_k$ where $\{E_k\}$ is a decomposition of the unit $1_{\mathcal{A}}$. Since $\sum_k \Phi(E_k) = 1_{\mathcal{B}}$ by the unitality of Φ, then applying **(iii)** to $C_k = \sqrt{\Phi(E_k)}$, we have

$$f(\Phi(A')) = f\left(\sum_k t_k\Phi(E_k)\right) \leq \sum_k f(t_k)\Phi(E_k) = \Phi\left(\sum_k f(t_k)E_k\right) = \Phi(f(A')).$$

The continuity of Φ implies **(iv)**.

The implications **(iv)⇒(ii)⇒(ii')⇒(i)** , **(iii)⇒(iii')⇒(i)** and **(iii')⇒(v)** are clear. So the following implication completes the proof:

(v)⇒(i): Putting

$$X = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, U = \begin{pmatrix} \sqrt{1-t} & -\sqrt{t} \\ \sqrt{t} & \sqrt{1-t} \end{pmatrix},$$

we have

$$\begin{pmatrix} f((1-t)A+tB) & \\ & f((1-t)B+tA) \end{pmatrix} = f(PU^*XUP + (1-P)U^*XU(1-P))$$

$$\leq PU^*f(X)UP + (1-P)U^*f(X)U(1-P)) = \begin{pmatrix} (1-t)f(A)+tf(B) & \\ & (1-t)f(B)+tf(A) \end{pmatrix},$$

so that f is operator convex. □

Remark 2. Modifying the proof in [7], we can also show **(ii')⇒(iii')** directly. In fact, we show the case $n = 2$, which is essential. Putting

$$\tilde{X} = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & A_2 & \\ & & & \ddots \end{pmatrix}, \quad \tilde{V} = \begin{pmatrix} C_1 & 0 & \cdots & & \\ C_2 & 0 & \cdots & & \\ 0 & 1 & 0 & \cdots & \\ \vdots & \ddots & \ddots & \ddots & \end{pmatrix} \in B(H \oplus H \oplus \cdots),$$

we have $\tilde{V}^*\tilde{V} = 1$ and

$$\begin{pmatrix} f(C_1^*A_1C_1 + C_2^*A_2C_2) & & \\ & f(A_2) & \\ & & \ddots \end{pmatrix} = f(\tilde{V}^*\tilde{X}\tilde{V}) \leq \tilde{V}^*f(\tilde{X})\tilde{V}$$

$$= \begin{pmatrix} C_1^*f(A_1)C_1 + C_2^*f(A_2)C_2 & & \\ & f(A_2) & \\ & & \ddots \end{pmatrix}.$$

Remark 3. Theorem 3 includes the above two Jensen's operator inequalities. An essential part of the proof for the Hansen-Pedersen-Jensen inequality is to show that **(1)** implies **(2)**. In fact, suppose **(1)** and $C^*C \leq 1$. Then, putting $D = \sqrt{1 - C^*C}$, we have by **(iii')** and $f(0) \leq 0$ that

$$f(C^*AC + D0D) \leq C^*f(A)C + D^2f(0) \leq C^*f(A)C.$$

最後に、作用素単調関数との関連を述べておこう。Hansen-Pedersen の結果からは、密接な関係があるように見えるが、有限区間上の関数、たとえば $f(x) = \tan x$ などを見れば、全く関係がなさそうにも見えるので、次の結果を挙げておく（$\alpha = 0$ の場合は、Uchiyama[16, Theo. 3.5] でのべられている。また、実数直線全体の作用素単調関数は直線（もちろん（作用素）凸かつ凹）に限ることは [1, Cor.II.2.1] に、指摘されている。）：

Theorem 4. *For a continuous function f with $\lim_{x\to\infty} f(x) > -\infty$ (resp., $\lim_{x\to-\infty} f(x) < \infty$), the following conditions are equivalent :*

(1) *f is operator concave (resp., convex) on (α,∞) (resp., $(-\infty,\alpha)$).*
(2) *f is operator monotone on (α,∞) (resp., $(-\infty,\alpha)$).*

Proof. It suffices to show the case f is concave on (α,∞) since $-f(-x)$ is convex on $(-\infty,\alpha)$. Suppose (1). Let $\alpha\le A\le B$. For $0<t<1$, we have

$$t(B-\alpha)+\alpha = tA+(1-t)\left(\frac{t}{1-t}(B-A)+\alpha\right)$$

and

$$\alpha\le t(B-\alpha)+\alpha,\ \frac{t}{1-t}(B-A)+\alpha.$$

So the operator concavity implies

$$f\left(t(B-\alpha)+\alpha\right)\ge tf(A)+(1-t)f\left(\frac{t}{1-t}(B-A)+\alpha\right)\ge tf(A)+(1-t)m$$

where m is a lower bound and hence $f(B)\ge f(A)$ by $t\longrightarrow 1$.

Conversely suppose (2). Putting $D=\sqrt{1-CC^*}$ for a fixed isometry C,

$$X=\begin{pmatrix}A & 0\\ 0 & \alpha\end{pmatrix}\quad\text{and}\quad U=\begin{pmatrix}C & D\\ 0 & -C^*\end{pmatrix},$$

we have U is unitary. For sufficiently large $M>\alpha$ and small $\varepsilon>0$, we have

$$U^*XU=\begin{pmatrix}C^*AC & C^*AD\\ DAC & DAD+\alpha CC^*\end{pmatrix}\le\begin{pmatrix}C^*AC+\varepsilon & 0\\ 0 & M\end{pmatrix}\equiv X_{M,\varepsilon}.$$

Thereby the operator monotonicity implies

$$\begin{pmatrix}C^*f(A)C & C^*f(A)D\\ Df(A)C & Df(A)D+f(\alpha)CC^*\end{pmatrix}=f(U^*XU)\le f(X_{M,\varepsilon})=\begin{pmatrix}f(C^*AC+\varepsilon) & 0\\ 0 & f(M)\end{pmatrix}.$$

Observing (1,1)-component and tending $\varepsilon\longrightarrow 0$, we have $C^*f(A)C\le f(C^*AC)$, namely f is operator concave by Theorem 3. □

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