On spectra of $q$-deformed operators

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1.

The formal algebraic relation $xx^* = qx^*x$ $(q > 0, q \neq 1)$ appears in several different situations related to the theory of quantum groups. This leads us to the study of an operator obeying this relation in a Hilbert space. Let $q$ be a positive real number with $q \neq 1$. Let $T$ be a closed densely defined operator in $\mathcal{H}$. If $T$ satisfies

$$TT^* = qT^*T,$$

then $T$ is called a deformed normal operator with deformation parameter $q$. Let $T$ be a closed densely defined operator in $\mathcal{H}$ with polar decomposition $T = U|T|$. If $T$ satisfies the relation

$$U|T| = \sqrt{q}|T|U,$$

then $T$ is called a deformed quasinormal operator with deformation parameter $q$. For a deformed normal (resp. deformed quasinormal) operator $T$ with deformation parameter $q$, we will simply say $T$ is $q$-normal. (resp. $q$-quasinormal)

If $T$ is $q$-normal then $T$ is $q$-quasinormal. A closed densely defined operator $T$ is $q$-normal if and only if

$$\mathcal{D}(T) = \mathcal{D}(T^*) \quad \text{and} \quad \|T^*\eta\| = \sqrt{q}\|T\eta\| \quad (\eta \in \mathcal{D}(T)).$$

A densely defined operator $T$ is called a $q$-hyponormal operator (or a deformed hyponormal operator with deformation parameter $q$) if it satisfies

$$\mathcal{D}(T) \subseteq \mathcal{D}(T^*) \quad \text{and} \quad \|T^*\eta\| \leq \sqrt{q}\|T\eta\|$$

for all $\eta \in \mathcal{D}(T)$. If $T$ is $q$-quasinormal, then $T$ is $q$-deformed hyponormal.

Let $T$ be a $q$-deformed hyponormal operator in $\mathcal{H}$. Then there exists uniquely a contraction $K_T$ such that

$$T^* \supseteq \sqrt{q}K_TW \quad \text{and} \quad \ker T^* \supseteq \ker T.$$
$K_T$ is called the attached contraction to $T$. If, in addition, $T$ is closed and $T = U|T|$ is the polar decomposition, then $T$ is $q$-quasinormal if and only if $K_T = (U^*)^2$.

2. Unbounded weighted shifts

Let $S_b$ be a closed densely defined operator in a separable Hilbert space $\mathcal{H}$. If there are an orthonormal basis $\{e_n\}$ $(n \in \mathbb{Z})$ and a sequence $\{w_n\}(w_n \neq 0, n \in \mathbb{Z})$ of complex numbers such that

$$D(S_b) = \left\{ \sum_{-\infty}^{\infty} \alpha_n e_n \in \mathcal{H} : \sum_{-\infty}^{\infty} |\alpha_n|^2 |w_n|^2 < \infty \right\}$$

and

$$S_b e_n = w_n e_{n+1}$$

for all $n \in \mathbb{Z}$, then $S_b$ is called a bilateral (injective) weighted shift with weight sequence $\{w_n\}$ (with respect to $\{e_n\}$). A unilateral weighted shift $S_u$ is defined analogously.

**Proposition.** The following statements hold:

1. A unilateral weighted shift $S_u$ in $\mathcal{H}$ with weights $\{w_n\}$ is $q$-quasinormal if and only if

$$|w_n| = \left( \frac{1}{\sqrt{q}} \right)^n |w_0|$$

for all $n \geq 0$. In particular, a unilateral weighted shift cannot be $q$-normal.

2. A bilateral weighted shift $S_b$ in $\mathcal{H}$ with weights $\{w_n\}$ is $q$-normal if and only if the above equation is valid for all $n \in \mathbb{Z}$

3. A weighted shift $S_u$ (resp. $S_b$) is $q$-hyponormal if and only if

$$|w_{n+1}| \geq \frac{1}{\sqrt{q}} |w_n|$$

for all $n \geq 0$ (resp. $n \in \mathbb{Z}$).

The spectrum of a $q$-normal weighted shift $S_b$:

<table>
<thead>
<tr>
<th>$S_b$ $(0 &lt; q &lt; 1)$</th>
<th>$\sigma_p$</th>
<th>$\sigma_c$</th>
<th>$\sigma_r$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_b$ $(0 &lt; q &lt; 1)$</td>
<td>$\emptyset$</td>
<td>${0}$</td>
<td>$\mathbb{C} \setminus {0}$</td>
<td>$\mathbb{C}$</td>
</tr>
<tr>
<td>$S_b$ $(q &gt; 1)$</td>
<td>$\mathbb{C} \setminus {0}$</td>
<td>${0}$</td>
<td>$\emptyset$</td>
<td>$\mathbb{C}$</td>
</tr>
</tbody>
</table>
The spectrum of a $q$-quasinormal weighted shift $S_u$:

<table>
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<tr>
<th></th>
<th>$\sigma_p$</th>
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</tr>
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<tbody>
<tr>
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<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\mathbb{C}$</td>
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<td>${0}$</td>
<td>${0}$</td>
</tr>
</tbody>
</table>

3. Spectra of a $q$-hyponormal operator

**Theorem.** Let $T_1$ and $T_2$ be $q$-hyponormal operators in a Hilbert space $\mathcal{H}$. Then $T_1 \oplus T_2$ is also $q$-hyponormal in $\mathcal{H} \oplus \mathcal{H}$ and

$$K_{T_1 \oplus T_2} = K_{T_1} \oplus K_{T_2}.$$ 

Moreover, $T_1 \oplus T_2$ is $q$-normal (resp. $q$-quasinormal) if and only if both $T_1$ and $T_2$ are $q$-normal (resp. $q$-quasinormal).

In case that $0 < q < 1$, a non-trivial $q$-hyponormal operator is always unbounded and the planar Lebesgue measure of its spectrum is positive.

Let $q > 1$. Then, there are various kinds of $q$-deformed operators, bounded or unbounded:

- A $q$-quasinormal unilateral weighted shift is always bounded.
- There exist $q$-quasinormal operators which are unbounded; they are $q$-normal ones.
- Using Theorem, one can construct an unbounded $q$-quasinormal operator which is not $q$-normal. (For this take $T_1$ to be any $q$-normal operator (which must be unbounded) and $T_2$ to be a bounded $q$-quasinormal unilateral weighted shift.)
- There exists a $q$-hyponormal operator which has empty spectrum, which is given in the following section; this is in contrast to the fact that every closed densely defined hyponormal operator ($q = 1$) has to have non-empty spectrum.
4. A $q$-deformed operator with empty spectrum

Let $T$ be a closed densely defined operator in a Hilbert space $\mathcal{H}$. Recall that the resolvent set $\rho(T)$ of $T$ is defined as the set of all $\lambda \in \mathbb{C}$ for which $\ker(\lambda - T) = \{0\}$, $\mathcal{R}(\lambda - T) = \mathcal{H}$ and the inverse $(\lambda - T)^{-1}$ is bounded on $\mathcal{H}$. Especially,

$$0 \in \rho(T)$$

if and only if there is a bounded operator $S$ on $\mathcal{H}$ such that

$$ST \subseteq 1 \text{ and } TS = 1.$$

**Lemma.** Let $T$ be a closed densely defined operator in $\mathcal{H}$. Suppose that

$$\rho(T) \ni 0.$$ If $\sigma(T^{-1}) = \{0\}$, then

$$\sigma(T) = \phi.$$ Let $q > 1$. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n\}_{n \in \mathbb{Z}}$. Take numbers $r$ and $\ell$ such that

$$\ell > 1 > r \geq \frac{1}{\sqrt{q}}.$$ Put

$$w_n = \begin{cases} r^n & \text{if } n \geq 0, \\ \ell^n & \text{if } n \leq -1. \end{cases}$$

Let us consider the weighted shift $S_0$ with the weight sequence $\{w_n\}$. Then, clearly $S_0$ is bounded with $\mathcal{D}(S_0) = \mathcal{H}$. Since the sequence $\{w_n\}$ tends to zero as $|n| \to \infty$, $S_0$ is compact and so $\sigma(S_0)$ is countable. On the other hand,

$$\sigma(S_0) = c \sigma(S_0)$$

for all $c \in \mathbb{C}$ with $|c| = 1$. It follows that $\sigma(S_0) = \{0\}$. Since $\ker(S_0) = \ker(S_0^+) = \{0\}$, $S_0$ is injective and has dense range. This means that the inverse $S_0^{-1}$ is closed and densely defined. Hence, it follows from Lemma that $S_0^{-1}$ has empty spectrum. On the other hand, we have

$$\frac{w_{n+1}}{w_n} = r \geq \frac{1}{\sqrt{q}} \quad \text{for} \quad n \geq 0,$$
and
\[ \frac{w_{n+1}}{w_n} = \ell > 1 > \frac{1}{\sqrt{q}} \quad \text{for} \quad n \leq -1. \]

These inequalities imply that \( S_0 \) is \( q \)-hyponormal. Therefore, \( S_0^{-1} \) is also \( q \)-hyponormal. Thus we have:

**Theorem.** Let \( q > 1 \). Then, there exists a \( q \)-hyponormal operator with empty spectrum.

5. Order relations for \( q \)-deformed operators

Let us recall some inequalities by Kato and Rellich ([1] and [5]):

\[ S \ll T \text{ means } D(T) \subseteq D(S), \text{ and } ||S\eta|| \leq ||T\eta|| \quad \text{for } \eta \in D(T) \]

and

\[ S \preceq T \text{ means } D(T^{\frac{1}{2}}) \subseteq D(S^{\frac{1}{2}}) \text{ and } ||S^{\frac{1}{2}}\eta|| \leq ||T^{\frac{1}{2}}\eta|| \quad \text{for } \eta \in D(T^{\frac{1}{2}}) \]

provided \( S \) and \( T \) are selfadjoint and nonegative.

**Definition.** Let \( S \) and \( T \) be symmetric (densely defined) operators in \( \mathcal{H} \). If

\[ D(T) \subseteq D(S) \quad \text{and} \quad \langle S\eta, \eta \rangle \leq \langle T\eta, \eta \rangle \]

for all \( \eta \in D(T) \), then we write

\[ S \preceq T. \]

**Theorem.** Let \( T \) be a closed densely defined operator in \( \mathcal{H} \). We consider the following statements:

(1) \( T \) is \( q \)-hyponormal.

(2) \( T \) satisfies the condition \( |T^*| \ll \sqrt{q} |T| \).

(3) \( T \) satisfies the condition \( |T^*| \leq \sqrt{q} |T| \).

(4) \( T \) satisfies the condition \( |T^*| \leq \sqrt{q} |T| \).
Then, \((1) \iff (2) \implies (3) \implies (4)\).

Especially, if \(T\) is a weighted shift, unilateral or bilateral, then all these statements are equivalent.

**Theorem.** If a closed densely defined operator \(T\) in \(\mathcal{H}\) satisfies condition

\[ TT^* \leq q T^* T, \]

then \(T\) is \(q\)-hyponormal.

\[ \textbf{参考文献} \]


