An application of proper forcings with models as side conditions

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Abstract
The method in the title has been introduced by Stevo Todorcević. In this note, we give one application of this method, i.e. we show that it is consistent that Martin's Axiom holds and there exist $(\mathfrak{c}, \mathfrak{c})$-gaps but no $(\omega_1, \mathfrak{c})$-gaps.

1 Introduction

Proper forcing notions have been introduced by Saharon Shelah. These are very useful forcing notions to lead consistency results. Forcing notions in the title is one of types of proper forcing notions which has been introduced by Stevo Todorcević ([19]). A condition of a forcing notion of this type consists of two parts: a working part $D$ and a side part $\mathcal{N}$ which is a finite $\in$-chain of countable elementary submodels of some large enough structure $H(\theta)$. To define such a forcing notion, we always require that $\mathcal{N}$ separates $D$, i.e.

$$\forall x \neq y \in D \exists N \in \mathcal{N}(\{x, y\} \cap N \text{ has exactly one element}).$$

(See also [10].) Todorcević used the method to show that the conjecture (S) is true under the Proper Forcing Axiom. Zapletal also applied it to study a strongly almost disjoint family ([24]).

The topic of this note is gaps in $\mathcal{P}(\omega)/\text{fin}$, in particular specific types of gaps in $\mathcal{P}(\omega)/\text{fin}$, namely $(\omega_1, \mathfrak{c})$-gaps and $(\mathfrak{c}, \mathfrak{c})$-gaps (where $\mathfrak{c}$ is the size of the continuum). The subject of gaps in $\mathcal{P}(\omega)/\text{fin}$ has been investigated by many

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mathematicians for a long time. We already know many ZFC results and many consistency results about gaps. It is one of the classical results that there exist some types of gaps: $(\omega_1, \omega_1)$-gaps and $(\omega, b)$-gaps (where $b$ is the (un)bounding number). (These are due to Hausdorff [7] and Rothberger [14], see also [15].)

Concerning the existence of $(\kappa, \lambda)$-gaps, we know that it is consistent with ZFC that if there exists a $(\kappa, \lambda)$-gap where $\kappa$ and $\lambda$ are regular cardinals with $\kappa \leq \lambda$, then either $(\kappa = \omega$ and $\lambda = \epsilon)$ or $\kappa = \lambda = \omega_1$. In this paper, we give one result of the existence of $(\kappa, \lambda)$-gaps under Martin’s Axiom (MA) for regular cardinals $\kappa$ and $\lambda$. This subject has also been studied in the past.

It is one of the classical results that any $(\omega, \omega)$-pregap is separated. And if $\kappa$ and $\lambda$ are regular cardinals so that $\kappa$ or $\lambda$ is not $\omega_1$, then for any $(\kappa, \lambda)$-gap $(A, B)$ there is a ccc forcing notion which forces that $(A, B)$ is separated (see [13], [15]). (So any such gap is not indestructible.) Therefore under MA if $(A, B)$ is an $(\kappa, \lambda)$-gap, then $\kappa = \lambda = \omega_1$ or $(\kappa = \epsilon \vee \lambda = \epsilon)$ holds. In fact, we know that there is a $(\omega, b)$-gap (see also [15], [20]), so under MA there always exists an $(\omega, c)$-gap. In [7], Hausdorff has proved that there always exists an $(\omega_1, \omega_1)$-gap. In particular, his proof gives that there exists an indestructible $(\omega_1, \omega_1)$-gap, hence under MA, there exists an $(\omega_1, \omega_1)$-gap. (In fact under MA, every $(\omega_1, \omega_1)$-gap is indestructible.) So the remaining problems are about the existence of $(\omega_1, c)$-gaps and $(c, c)$-gaps under MA.

In [11], Kunen has proved that the following statements are consistent with ZFC:

1. MA + $\exists(c, c)$-gaps + $\exists(\omega_1, c)$-gaps, and
2. MA + $\neg\exists(c, c)$-gaps + $\neg\exists(\omega_1, c)$-gaps.

In this note, we see the outline of a proof of the following theorem, which answers a problem addressed in [15], using forcing notions with models as side conditions. For the detail proof, see [23].

**Theorem 1.1** ([23], Theorem 1.2). If PFA is consistent, then it is also consistent that MA holds and there exist $(c, c)$-gaps but no $(\omega_1, c)$-gaps.

Following definitions are needed to explain the above theorem and its proof.

**Definition and Notation 1.2.** Let $a$ and $b$ be elements of $P(\omega)$, and $A$ and $B$ subsets of $P(\omega)$.

1. $a \perp b$ denotes that $a \cap b$ is finite.
2. $a \subseteq^* b$ denotes that $a \setminus b$ is finite.
3. $A^\perp := \{c \subseteq \omega; \forall a \in A(a \perp c)\}$.
4. $(A, B)$ is called a pregap if for any $a \in A$ and $b \in B$, $a \perp b$, i.e. $B \subseteq A^\perp$.
5. A pregap $(A, B)$ is separated if there is $c \in A^\perp$ such that $b \subseteq^* c$ for all $b \in B$. 
6. A pregap \((A, B)\) is countably separated if there is a sequence \((c_n; n \in \omega)\) of elements of \(P(\omega)\) such that for all \((a, b) \in A \times B\) there is an \(n \in \omega\) with \(a \perp c_n\) and \(b \subseteq^* c_n\).

7. \((A, B)\) is called a gap if it is a pregap and not separated.

8. If \(\text{ot}(A, \subseteq^*) = \kappa\) and \(\text{ot}(B, \subseteq^*) = \lambda\), then \((A, B)\) is called a \((\kappa, \lambda)\)-pregap (or a \((\kappa, \lambda)\)-gap) if it is a pregap (or gap).

9. An \((\omega_1, \omega_1)\)-gap \((A, B)\) is called indestructible if for every forcing extension in which cardinalities are preserved, \((A, B)\) is still a gap.

10. For a collection \(P\) of forcing notions and a cardinal \(\kappa \leq \omega\), \(\text{MA}_\kappa(P)\) means that for every \(P \in P\) and \(\kappa\) many dense sets \(\{D_\alpha; \alpha < \kappa\}\), there exists a filter \(G\) in \(P\) which meets all \(D_\alpha\).

11. Martin's Axiom (MA) is the statement \(\text{MA}_\omega(\text{ccc})\).

The Proper Forcing Axiom (PFA) is the statement \(\text{MA}_{\aleph_1}(\text{proper})\).

12. For a collection \(P\) of forcing notions, we denote by \(m(P)\) the least cardinal \(\kappa\) so that \(\text{MA}_\kappa(P)\) fails. If \(P\) is a singleton \(\{P\}\), then \(m(P)\) is denoted by \(m(\{P\})\).

1.1 Preparation

Definition 1.3. Let \(A\) and \(B\) be subsets of \(P(\omega)\).

1. \(B^+ := \{c \subseteq \omega; \exists b \in B(c \subseteq^* b)\}\).

2. \(A \otimes B := \{(a, b) \in A \times B; a \cap b = \emptyset\}\).

3. (Todorčević [19]) Coloring: \([A \otimes B]^2 = K_0 \cup K_1\), where
   \[
   \{(a, b), (a', b')\} \in K_0 : \iff (a \cap b') \cup (a' \cap b) \neq \emptyset.
   \]

   For \(X, Y \subseteq A \otimes B\), we write \(X \ast Y := \{\{x, y\} \in [A \otimes B]^2; x \in X \land y \in Y \land x \neq y\}\).

   For \(X \subseteq A \otimes B\) and \(i = 0\) or \(1\), \(X\) is called \(K_i\)-homogeneous if \(X \ast X \subseteq K_i\).

\(P(\omega)\) is identified with the Cantor space. Now we fix a linear order \(\prec_{P(\omega)}\) in \(P(\omega)\) and then we identify \([A \otimes B]^2\) with the topological space \(\{(a, b) \in P(\omega) \times P(\omega); a \prec_{P(\omega)} b\}\). Then we notice that \([A \otimes B]^2 = (A \times B \cup B \times A) \cap \{(a, b) \in P(\omega) \times P(\omega); a \prec_{P(\omega)} b\}\) and \(K_0\) is open in this topology. For \(A \subseteq P(\omega)\), \(A\) is called \(\sigma\)-directed if for every countable subset \(X\) of \(A\), there is \(a \in A\) so that for all \(x \in X, x \subseteq^* a\). The following propositions are well-known.

Proposition 1.4. (Folklore, [5]) If both \(A\) and \(B\) are \(\sigma\)-directed, \((A, B)\) is separated iff \((A, B)\) is countably separated.
Proposition 1.5. (Folklore, [5]) Let \((A, B)\) be a pregap. Then \(A \otimes B^+\) is a union of countably many \(K_1\)-homogeneous subsets if \((A, B)\) is countably separated.

Proposition 1.6. (Kunen [11], see also [15], [19] or [20]) Let \((A, B)\) is an \((\omega_1, \omega_1)\)-pregap. Suppose that \(\{a_\alpha; \alpha < \omega_1\} \subseteq A, \{b_\alpha; \alpha < \omega_1\} \subseteq B, a_\alpha \cap b_\alpha = \emptyset\) for all \(\alpha < \omega_1\) and \(\{\langle a_\alpha, b_\alpha \rangle; \alpha < \omega_1\}\) is \(K_0\)-homogeneous. Then \(\langle\{a_\alpha; \alpha < \omega_1\}, \{b_\alpha; \alpha < \omega_1\}\rangle\) forms a gap and is indestructible, i.e. still forms a gap in any extension with a forcing doesn't collapse \(\aleph_1\).

2 A proof of the theorem

Suppose that PFA holds in the ground model \(V\). Then \(c = \aleph_2\) and there is a decreasing sequence \(\langle X_\alpha; \alpha < \omega_2\rangle\) of elements of \(\mathcal{P}(\omega)\) which is a generator of an ultrafilter, i.e.

1. \(\forall \alpha < \beta < \omega_2 (X_\beta \subseteq^* X_\alpha)\)
2. \(\forall Y \subseteq \omega \exists \alpha < \omega_1 (X_\alpha \subseteq^* Y \lor X_\alpha \subseteq^* \omega \setminus Y)\)

Let \(\mathcal{U}\) be the ultrafilter generated by \(\langle X_\alpha; \alpha < \omega_2\rangle\), and \(\mathcal{U}^*\) the dual ideal of \(\mathcal{U}\). We define a forcing notion \(\mathbb{P}(\mathcal{U}) = \mathbb{P}\) := \(\bigcup_{X \in \mathcal{U}} 2^X\), for conditions \(f, g\) in \(\mathbb{P}\) \(f \leq g\) iff \(g \subseteq f\). And we let \(\mathbb{P}'(\mathcal{U}) = \mathbb{P}'\) := \(\bigcup_{X \in \mathcal{U}} 2^X\) be Grigoroff forcing ([6]), i.e. for conditions \(f, g\) in \(\mathbb{P}', f \leq g\) iff \(g \subseteq f\). (\(\mathbb{P}\) and \(\mathbb{P}'\) have the same underlying set. The only difference is the ordering, but \(1(=\emptyset)\) is the strongest condition in both \(\mathbb{P}\) and \(\mathbb{P}'\). ) We must note that \(\mathbb{P}'(\mathcal{U})\) is proper if \(\mathcal{U}\) is a fat p-filter (by Shelah, see [16]). Now, since \(\mathcal{U}\) satisfies the properties of fat-ness and p-filter, \(\mathbb{P}'(\mathcal{U})\) is a proper forcing notion.

The following proposition is very similar to [21] and [22].

Proposition 2.1. \(\mathbb{P}\) is \(\sigma\)-closed, \(\omega_2\)-Baire and adds an \((\omega_2, \omega_2)\)-gap (under PFA).

Proof. For the first two statements, see [21]. (Since the length of the generating sequence of \(\mathcal{U}\) is \(\omega_2\), if \(m(\mathbb{P}') = \aleph_2\), then it follows that \(\mathbb{P}\) is \(\omega_2\)-Baire.)

For the last statement, let \(G\) be a \(\mathbb{P}\)-generic filter over \(V\). Then we may take a condition \(f_\alpha \in G \cap 2^{X_\alpha}\) for every \(\alpha < \omega_2\), and let \(a_\alpha := \{n \in X_\alpha; f_\alpha(n) = 0\}\) and \(b_\alpha := \{n \in X_\alpha; f_\alpha(n) = 1\}\). Then it is trivial that \(\langle a_\alpha; \alpha < \omega_2\rangle, \langle b_\alpha; \alpha < \omega_2\rangle\) is an \((\omega_2, \omega_2)\)-gap by the genericity.

Therefore in the extension with \(\mathbb{P}\) over \(V\), there are no new reals and MA holds. So to finish the proof, we have only to show that \(\mathbb{P}\) adds no \((\omega_1, \omega_2)\)-gaps (under PFA). To prove this, we will use the method in the title.

2.1 A proof of non-existence of \((\omega_1, \omega_2)\)-gaps in the extension with \(\mathbb{P}\) over \(V\)

Assume that in the extension with \(\mathbb{P}\), there exists an \((\omega_1, \omega_2)\)-gap, whose P-name is \((\dot{A}, \dot{B})\), i.e.

\(|\mathbb{P}|^\langle (\dot{A}, \dot{B}) \rangle \text{ is an } (\omega_1, \omega_2)\text{-gap}\).
Since $\mathbb{P}$ is $\omega_2$-Baire in $\mathbb{V}$, there are $A \in \mathbb{V}$ and $f \in \mathbb{P}$ such that $f \Vdash_{\mathbb{P}} \check{A} = \bar{A}$. So by the homogeneity of $\mathbb{P}$, without loss of generality, we may assume that $\Vdash_{\mathbb{P}} \check{A} = \bar{A}$.

We recall that

$$\Vdash_{\mathbb{P}} \bar{B}^+ = \{c \subseteq \omega; \exists b \in \bar{B} (c \subseteq^* b)\}.$$

Then

$$\Vdash_{\mathbb{P}} (\bar{A}, \bar{B}^+) \text{ also forms a gap }.$$

In $\mathbb{V}$, for all $f \in \mathbb{P}$, let $B^+(f) := \{b \subseteq \omega; \exists g \leq_{\mathbb{P}} f (g \Vdash_{\mathbb{P}} \check{b} \in \bar{B}^{+})\}$. It is trivial that for conditions $f, g$ in $\mathbb{P}$, $f \Vdash_{\mathbb{P}} B^+(f) \supseteq B^+(g)$ and if $f \leq_{\mathbb{P}} g$, then $B^+(f) \subseteq B^+(g)$. The next proposition is used to show Lemma 2.6.

**Proposition 2.2 ([23], Proposition 2.7).** For every $f \in \mathbb{P}$, $A \otimes B^+(f)$ is not a union of countably many $K_0$-homogeneous subsets.

We will find $\bar{f} \in \mathbb{P}$ and $\mathcal{X} \subseteq A \otimes B^+(1)$ such that

1. $\mathcal{X}$ is uncountable and $K_0$-homogeneous, and
2. for all $(a, b) \in \mathcal{X}$, $\bar{f} \Vdash_{\mathbb{P}} \check{b} \in \bar{B}^{+}$,

which completes the proof, because then

$$\bar{f} \Vdash_{\mathbb{P}} \mathcal{X} \text{ forms an } (\omega_1, \omega_1) \text{-indestructible gap in } (\bar{A}, \bar{B}^+),$$

which is a contradiction.

In fact we can get an uncountable $K_0$-homogeneous subset of $A \otimes B^+(1)$ applying OCA. But now we need the condition $\bar{f}$ as above to get a contradiction. To get the desired objects, we consider the extension by the following forcing notion $Q(A, \mathcal{B}, \mathcal{U})$. This is an example of a forcing notion with models as side conditions.

### 2.2 An example of proper forcings with models as side conditions

**Definition 2.3.** A condition of $Q(A, \mathcal{B}, \mathcal{U})$ is a triple $p = (f_p, X_p, \mathfrak{N}_p)$ satisfying the following statements:

(a) $f_p$ is a member of $\bigcup_{X \in \mathcal{U}} 2^X$,

(b) $X_p$ is a finite $K_0$-homogeneous subset of $A \otimes B^+(1)$,

(c) $\mathfrak{N}_p$ is a finite $\in$-chain of countable elementary submodels of $H(c^+) (= H(\aleph_3))$ containing everything we need for our discussion, e.g. $A, \mathcal{B}, \mathcal{U},$ etc ... (i.e. $\mathfrak{N}_p$ can be enumerated by $\{N_i; i < n\}$ such that for all $i < n - 1$, $N_i \in N_{i+1}$ and $N_i$ is an elementary submodel of $N_{i+1}$ (say $N_i < N_{i+1}$).
(d) for any \( x = (a_x, b_x) \in X_p \), \( f_p \Vdash \bar{b}_x \in \dot{B}^+\),

(e) for any \( x, y \in X_p \) with \( x \neq y \) there exists \( N \in \mathcal{M}_p \) so that \( |N \cap \{x, y\}| = 1 \),

(d) define \( x < y : \iff \exists N \in \mathcal{M}_p(x \in N \& y \notin N)\),

(f) for all \( N \in \mathcal{M}_p \), \( f_p \) is \((N, \mathbb{P}^')\)-generic, and

(g) for every \( x \in X_p \) and \( N \in \mathcal{M}_p \) with \( x \notin N \),

\[
 f_p \Vdash \forall Y \in \dot{N}[\dot{G}](Y \subseteq A \otimes B^+(1) \& Y \star Y \subseteq K_1 \Rightarrow \bar{x} \notin Y).
\]

For conditions \( p, q \in \mathbb{Q}(A, \dot{B}, U) \),

\[
 p \leq_{\mathbb{Q}(A, \dot{B}, U)} q : \iff f_p \supseteq f_q \ (i.e. \ f_p \leq_{\mathbb{P}^'} f_q) \& X_p \supseteq X_q \& \mathcal{M}_p \supseteq \mathcal{M}_q.
\]

We note that \( \mathcal{M}_p \) is an element of \( H(\aleph_3) \) because every element of \( \mathcal{M}_p \) is a countable subset of \( H(\aleph_3) \) and \( \mathcal{M}_p \) is finite. We must show that \( \mathbb{Q}(A, \dot{B}, U) \) is proper and adds desired objects. To apply the PFA, we need to show the following lemma.

**Lemma 2.4.** For \( A, \dot{B} \) and \( U \), \( \mathbb{Q}(A, \dot{B}, U) \) is proper.

**Proof.** Let \( \theta \) be a large enough regular cardinal, \( M \prec H(\theta) \) a countable elementary submodel containing everything needed for our discussion, e.g. \( A, \dot{B}, U, H(\aleph_3) \) etc, and \( p = (f_p, X_p, \mathcal{M}_p) \in M \) a condition of \( \mathbb{Q}(A, \dot{B}, U) (= \mathbb{Q}) \).

Since \( \mathbb{P}^' \) is proper, we can choose an extension \( f_q \leq_{\mathbb{P}^'} f_p \) such that \( f_q \) is \((M \cap H(\aleph_3), \mathbb{P}^')\)-generic. (We note that \( M \cap H(\aleph_3) \) is an elementary submodel of \( H(\aleph_3) \).) Then let \( q := (f_q, X_p, \mathcal{M}_p \cup \{M \cap H(\aleph_3)\}) \), which is a condition of \( \mathbb{Q} \). (We note that \( \mathcal{M}_p \subseteq M \cap H(\aleph_3) \), in fact \( \mathcal{M}_p \in M \cap H(\aleph_3) \) holds since \( p \in M \) and \( \mathcal{M}_p \in H(\aleph_3) \).) Show that \( q \) is \((M, \mathbb{Q})\)-generic, i.e. for every dense open subset \( D \in M \) in \( \mathbb{Q} \) and an extension \( r \in \mathbb{Q} \) of \( q \) there exists a condition \( s \in D \cap M \) such that \( r \) and \( s \) are compatible in \( \mathbb{Q} \) (i.e. \( D \cap M \) is predense in \( \mathbb{Q} \) below \( q \)).

Taking such \( D \in M \) and \( r \leq q \), without loss of generality, we may assume that \( r \) is in \( D \). Let \( X_r \setminus M = \{x_i: i \leq n\} \) where \( x_i \prec x_{i+1} \) for \( i < n \) and \( N_0 := M \cap H(\aleph_3) \), and pick \( N_i \in \mathcal{M}_r \) such that \( x_{i+1-1} \in N_i \) but \( x_i \notin N_i \) for \( 1 \leq i \leq n \). We choose rational open intervals \( U_i \subseteq A \otimes B^+(1) \) such that

- \( x_i \in U_i \) for \( i \leq n \),
- \( U_i \cap U_j = \emptyset \) and \( U_i \star U_j \subseteq K_0 \) for every \( i, j \leq n \) with \( i \neq j \).

(We recall that \( K_0 \) is open, so this can be done.) We note that all rational open intervals are in any model of ZFC because those codes consist of finite elements. Let \( G \) be \( \mathbb{P}^'\)-generic over \( H(\theta) \) with \( f_r \in G \).

**Claim 2.5.** In \( H(\theta)[G] \), there are rational open intervals \( V_i^0, V_i^1 \subseteq U_i \) and \( y_i \in V_i^1 \cap V \) for \( i \leq n \) and \( s \in D \cap V \) such that
1. \( x_i \in V^0_i \) for all \( i \leq n \),
2. \( V^0_i \cap V^1_i = \emptyset \) and \( V^0_i \ast V^1_i \subseteq K_0 \) for all \( i \leq n \),
3. \( f_s \in G \),
4. \( X_s = (X_r \cap M) \cup \{y_i; i \leq n\} \) and for any \( x \in X_r \cap M \) and \( i < j \leq n \), and \( x < y_i < y_j \),
5. \( \mathcal{N}_s \) is an end extension of \( \mathcal{N}_r \cap M \).

**Proof.** By induction on \( i \leq n \), we construct rational open intervals \( V^0_{n-i}, V^1_{n-i} \subseteq U_{n-i}, y_{n-i}^{n-j} \in V^1_{n-j} \cap V \) for \( j \leq i \) and \( s_{n-j} \in D \cap V \) such that

1'. \( x_{n-i} \in V^0_{n-i} \),
2'. \( V^0_{n-i} \cap V^1_{n-i} = \emptyset \) and \( V^0_{n-i} \ast V^1_{n-i} \subseteq K_0 \),
3'. \( f_{s_{n-i}} \in G \),
4'. \( X_{s_{n-i}} = (X_{s_{n-i}} \cap N_{n-i}) \cup \{y_{n-i}^{n-j}; j \leq i\} \) and for any \( x \in X_{s_{n-i}} \cap N_{n-i} \) and \( j < k \leq n, x < y_{n-i}^{n-k} < y_{n-i}^{n-j} \), and
5'. \( \mathcal{N}_{s_{n-i}} \) is an end extension of \( \mathcal{N}_{s_{n-i+1}} \cap N_{n-i} \).

**Construction** Assume that we have already constructed \( V^0_{n-i}, V^1_{n-i}, y_{n-i}^{n-j}, s_{n-j} \) for all \( j < i \). Let

\[
Y_{n-i} := \{x \in U_{n-i} \cap V; \exists z_n \in V^1_n \cap V \cdot \exists \exists z_{n-i+1} \in V^1_{n-i+1} \cap V \exists s \in D \cap V \text{ s.t.} \]

- \( f_s \in G \)
- \( X_s = (X_{s_{n-i+1}} \cap N_{n-i}) \cup \{x\} \cup \{z_{n-j}; j < i\} \)
- \( \forall z \in X_{s_{n-i+1}} \cap N_{n-i} \forall j < k \leq n (z < x < z_{n-k} < z_{n-j}) \)
- \( \mathcal{N}_s \) is an end extension of \( \mathcal{N}_{s_{n-i+1}} \cap N_{n-i} \).

Then \( Y_{n-i} \in N_{n-i}^G \) and \( x_{n-i} \in Y_{n-i} \) by 1', 4', and 5'. Since \( x_{n-i} \in Y_{n-i} \), by (g), \( Y_{n-i} \) is not \( K_1 \)-homogeneous. Let \( Y_{n-i} := \{x \in Y_{n-i}; \exists y \in Y_{n-i} \backslash \{x\}\} \{x, y \in K_0\} \). Then \( Y_{n-i} \backslash \overline{Y_{n-i}} \) is in \( N_{n-i}^G \) and \( K_1 \)-homogeneous, hence \( x_{n-i} \) belongs to \( Y_{n-i} \) by (g) again. Therefore there exists \( y_{n-i}^{n-i} \in Y_{n-i} \backslash \{x_{n-i}, y_{n-i}^{n-i}\} \) such that \( \{x_{n-i}, y_{n-i}^{n-i}\} \) is in \( K_0 \). Then we take rational open intervals \( V^0_{n-i}, V^1_{n-i} \subseteq U_{n-i} \) such that \( x_{n-i} \in V^0_{n-i}, V^0_{n-i} \nabla V^1_{n-i} = \emptyset \) and \( V^0_{n-i} \ast V^1_{n-i} \subseteq K_0 \). By \( y_{n-i}^{n-i} \in Y_{n-i} \), there are \( y_{n-i}^{n-i} \in V^1_n \cap V, \cdot \cdot \cdot, y_{n-i}^{n-i+1} \in V^1_{n-i+1} \cap V \) and \( s_{n-i} \in D \cap V \) satisfying 3', 4', and 5', which completes a construction. Put \( y_i := y_0^i \) for \( i \leq n \) and \( s := s_0 \), then these are as desired.

Since \( M \) is an elementary submodel of \( H(\theta) \), \( M[G] \) is an elementary submodel of \( H(\theta)[G] \). So by the previous claim, there are \( y_i \in V^1_i \cap M[G] \cap V \) for \( i \leq n \) and \( s \in D \cap M[G] \cap V \) satisfying 3, 4 and 5 of the claim. Then we take a condition \( g \in G \) which decides all values of \( V^0_i, V^1_i, y_i \) for all \( i \leq n \) and \( s \). By the
2.3 The end of the proof of the theorem

To get $\mathcal{C}$ and $\mathcal{X}$, we take any countable elementary submodel $M$ of $H(\theta)$ containing $A, B, U, H(\aleph_3)$, etc. Let $M_0 := M \cap H(\aleph_3)$ and pick a $(M_0, \mathbb{P}')$-generic condition $f \in \mathbb{P}'$. We notice that $\mathcal{P}(\omega) \cap M_0 = \mathcal{P}(\omega) \cap M$. Now we have the following lemma:

Lemma 2.6 ([23], Lemma 2.9). Under $m(\mathbb{P}') = c = \aleph_2$ (in particular under PFA),
\[ \vdash_{\mathbb{P}'} \text{"} A \otimes B^+(1) \text{ is not a union of countably many } K_1 \text{-homogeneous subsets"}. \]

More explicitly, for any $\mathbb{P}'$-names $\check{x}_n$ for $K_1$-homogeneous subsets, $n \in \omega$ and $f \in \mathbb{P}'$, there exist $f' \leq_{\mathbb{P}'} f$ and $(a, b) \in A \otimes B^+(1)$ such that
\[ f' \vdash \check{b} \in B^+ \text{ and } \langle a, \check{b} \rangle \notin \check{x}_n \text{ for every } n \in \omega. \]

Therefore, there are $x \in A \otimes B^+(1)$ and $g \leq_{\mathbb{P}'} f$ such that
\[ g \vdash_{\mathbb{P}'} \check{x} \notin \bigcup \{ Y \in \check{M}_0[\mathcal{G}]; Y \subseteq A \otimes B^+(1) \} \text{ and } b_x \in B^+. \]

Let $p := \langle g, \{ x \}, \{ M_0 \} \rangle$ which is a condition of $\mathbb{Q}$ and we can show that $p$ is $(M, \mathbb{Q})$-generic by the same argument as in the proof of Lemma 2.4. The following lemma indicates the density argument of $\mathbb{Q}$.

Lemma 2.7.

\[ p \vdash_{\mathbb{Q}} \text{"} \mathcal{X} := \bigcup \{ X_q; q \in \check{G} \} \text{ is uncountable } K_0 \text{-homogeneous"}. \]

Proof. It is trivial that $\vdash_{\mathbb{Q}} \text{"} \mathcal{X} \text{ is } K_0 \text{-homogeneous"}$. From now on we show that $\vdash_{\mathbb{Q}} \text{"} \mathcal{X} \text{ is uncountable"}.

Assume not, then there is $q \leq_{\mathbb{Q}} p$ so that $q \vdash_{\mathbb{Q}} \text{"} \mathcal{X} \text{ is countable"}$. Now $q \vdash_{\mathbb{Q}} \text{"} \check{x} \notin \mathcal{X} \text{"}$. Since $\vdash_{\mathbb{Q}} \text{"} \check{G} \in \check{M}_0[\check{G}] \text{"}$, $\vdash_{\mathbb{Q}} \text{"} \mathcal{X} \subseteq \check{M}_0[\check{G}] \text{"}$. Since $q \vdash_{\mathbb{Q}} \text{"} \check{x} \text{ is countable"}$, $q \vdash_{\mathbb{Q}} \text{"} \check{x} \notin \mathcal{X} \subseteq \check{M}_0[\check{G}] \text{"}$. Now $q$ is $(M, \mathbb{Q})$-generic, $q \vdash_{\mathbb{Q}} \text{"} M[\check{G}] \cap \mathcal{V} = M, \check{M}_0[\check{G}] \cap \mathcal{P}(\omega) \cap \mathcal{V} = \check{M}_0 \cap \mathcal{P}(\omega) \text{"}$. Thus $q \vdash_{\mathbb{Q}} \text{"} \check{x} \notin \check{M}_0[\check{G}] \text{"}$, because of $x \notin M_0$, which is a contradiction.

Applying PFA to $\mathbb{Q}$, we can get a filter $G \subseteq \mathbb{Q}$ such that $\mathcal{X}' = \bigcup \{ X_p; p \in G \}$ is uncountable and $K_0$-homogeneous. Let $\{ x_\alpha; \alpha < \omega_1 \}$ list $\mathcal{X}'$. For each $\alpha < \omega_1$, we choose $p_\alpha \in G$ with $x_\alpha \in X_{p_\alpha}$. Then we take a fusion $\tilde{f}$ of $\langle f_{p_\alpha}; \alpha < \omega_1 \rangle$, i.e. take $\check{X} \in U^*$, a natural number $n$ and an uncountable subset $A$ of $\omega_1$ such that $\text{dom}(f_{p_\alpha}) \subseteq \check{X} \cup n$ for all $\alpha \in A$ and take a condition $\check{f} \in 2^{\mathcal{X} \cup n}$ of $\mathbb{P}'$ with $\check{f} \supseteq \bigcup_{\alpha \in A} f_{p_\alpha}$. Then $\tilde{f}$ is an extension of $f_{p_\alpha}$ in $\mathbb{P}$ for all $\alpha \in A$, so $\tilde{f} \vdash_{\mathbb{P}'} \text{"} b_{x_{\alpha}} \in B^+ \text{"}$. Put $\mathcal{X} := \{ x_\alpha; \alpha \in A \}$, then these are as desired so we finish the proof of the theorem.
3 Appendix: An iteration of the method of models as side conditions

3.1 Redefinition of the freezing forcing

In this section, we prove the following theorem, i.e. we can eliminate any large cardinal property of Theorem 1.1.

Theorem 3.1. It is consistent with ZFC that Martin’s Axiom holds and there are \((\omega, \omega)\)-gaps but no \((\omega_1, \omega)\)-gaps.

The key-point of the proof of Theorem 3.1 is same as the proof of Theorem 1.1. To prove Theorem 3.1, we use a countable support iteration instead of PFA. The problem is that in general \(Q(\mathcal{A}, \mathcal{B}, \mathcal{U})\) collapses \(\aleph_2\), so we cannot force by an iteration of \(Q(\mathcal{A}, \mathcal{B}, \mathcal{U})\). To overcome this problem, we redefine the freezing forcing using the following objects:

Definition 3.2. 1. For a model \(N\) of ZFC (i.e. a model of sufficiently large fragments of ZFC), denote the transitive collapse of \(N\) by \(\overline{N}\) and denote the unique isomorphism from \(N\) onto \(\overline{N}\) by \(\pi_N\), i.e. for \(x \in N\), \(\pi_N(x) := \{\pi_N(y) : y \in N \& y \in x\}\). (This is defined by the \(\epsilon\)-recursion.)

2. For \(\mathcal{A}, \mathcal{B}\) and \(\mathcal{U}\), let

\[
\mathfrak{T}(\mathcal{A}, \mathcal{B}, \mathcal{U}) := \{\overline{N} : N < H(\omega) \& N \text{ is countable} \& \mathcal{A}, \mathcal{B}, \mathcal{U} \in N\}.
\]

3. For \(\mathcal{A}, \mathcal{B}, \mathcal{U}\) and \(M \in \mathfrak{T}(\mathcal{A}, \mathcal{B}, \mathcal{U})\), let

\[
\mathfrak{M}_M := \{N < H(\omega) ; N \text{ is countable} \& \mathcal{A}, \mathcal{B}, \mathcal{U} \in N \& \overline{N} = M\}.
\]

We note that

- for \(x \in \mathcal{P}(\omega) \cap N\), \(\pi_N(x) = x\), so \(\mathcal{P}(\omega) \cap \overline{N} = \mathcal{P}(\omega) \cap N\),
- for \(N, N' \in \mathfrak{M}_M\), \(N\) and \(N'\) are isomorphic and \(\pi_{N'}^{-1} \circ \pi_N\) is an isomorphism from \(N\) onto \(N'\),
- for a countable elementary submodel \(N\) of \(H(\omega)\), \(N\) is an element of \(H(\omega^+)\), so \(\mathfrak{M}_M \subseteq H(\omega^+)\) for each \(M \in \mathfrak{T}(\mathcal{A}, \mathcal{B}, \mathcal{U})\).

The following partial order \(Q'((\mathcal{A}, \mathcal{B}, \mathcal{U}), f)\) is the new freezing forcing notion designed for the iteration with countable support. This is similar to the forcing notion due to Todorcevic ([18]).

Definition 3.3. For \(\mathcal{A}, \mathcal{B}, \mathcal{U}\) and \(f \in \bigcup_{X \in \mathcal{U}} 2^X\), define \(Q'((\mathcal{A}, \mathcal{B}, \mathcal{U}), f)\) whose conditions \(p\) are triples \((f, X, N)\) such that

(a') \(f_p\) is a member of \(\bigcup_{X \in \mathcal{U}} 2^X\) with \(f_p \supseteq f\),
(b) $X_p$ is a finite $K_0$-homogeneous subset of $A \otimes B^+(1)$,
(recall that for $f \in \mathbb{P}(U)$, $B^+(f) = \{ b \subseteq \omega; \exists g \leq \mathbb{P}(U)\ f(g \Vdash \mathbb{P}(U) \ " b \in B^+ " )\})$

(c') $N_p$ is a function so that

(c1) $\text{dom}(N_p)$ is a finite $\in$-chain of elements of $\mathfrak{I}(A, \dot{B}, U)$,
(c2) for each $M \in \text{dom}(N_p)$, $N_p(M)$ is a finite subset of $\mathfrak{M}_M$,
(c3) for all $M, M' \in \text{dom}(N_p)$ with $M \subset M'$ and $N \in N_p(M)$, there exists $N' \in N(M')$ with $N \subset N'$ and $N < N'$,

(d) for any $x = \langle a_x, b_x \rangle \in X_p$, $f_p \Vdash_{\mathbb{P}(U)} \ " b_x \in \dot{B}^+ "$,

(e') for any $x, y \in X_p$ with $x \neq y$ there exists $M \in \text{dom}(N_p)$ so that $|M \cap \{ x, y \}| = 1$,

(\text{define } x < y : \iff \exists M \in \text{dom}(N_p)(x \in M \& y \notin M))$,

(f') for all $N \in \bigcup \text{ran}(N_p)$, $f_p$ is $(N, \mathbb{P}')$-generic, and

(g') for every $x \in X_p, M \in \text{dom}(N_p)$ with $x \notin M$ and $N \in N_p(M)$,

$$f_p \Vdash_{\mathbb{P}(U)} \forall Y \in N[G](Y \subseteq \check{A} \otimes B^+(1) \& Y \ast Y \subseteq K_1 \Rightarrow \check{x} \notin Y)$$.

For conditions $p, q \in \mathbb{Q}(A, \dot{B}, U)$,

$$p \leq_{\mathbb{Q}'(A, \dot{B}, U)} q : \iff f_p \supseteq f_q \ (i.e. \ f_p \leq_{\mathbb{P}} f_q) \ & X_p \supseteq X_q \ & \text{dom}(N_p) \supseteq \text{dom}(N_q) \ & \forall M \in \text{dom}(N_q)(N_q(M) \subseteq N_p(M)) \&$$

By an argument similar to the one of Lemma 2.4, we show the following lemma.

**Lemma 3.4.** For $A, \dot{B}, U$ and $f \in \bigcup_{X \in \mathcal{U}^*} 2^X$, $\mathbb{Q}'(A, \dot{B}, U, f)$ is proper.

**Proof.** Let $\theta$ be a large enough regular cardinal, $H \prec H(\theta)$ a countable elementary submodel containing all relevant objects, e.g. $A, \dot{B}, U, H(c^+)$ etc., and $p = \langle f_p, X_p, N_p \rangle \in H$ a condition of $\mathbb{Q}'(A, \dot{B}, U, f)(= \mathbb{Q}')$.

Since $\mathbb{P}'(U)(= \mathbb{P}')$ is proper, there is an extension $f_q \leq_{\mathbb{P}} f_p$ such that $f_q$ is $(H \cap H(c^+), \mathbb{P}')$-generic. Let $M_0 := H \cap H(c^+)$ and

$$q := \langle f_q, X_p, N_p \cup \{ \langle M_0, \{ H \cap H(c^+) \} \} \rangle \rangle.$$ Then $q$ is a condition of $\mathbb{Q}'$. We show that $q$ is $(H, \mathbb{Q}')$-generic.

Let $D \in H$ be dense open in $\mathbb{Q}'$ and $r \leq_{\mathbb{Q}'} q$. We may assume that $r$ is in $D$. Let $X_r \setminus H = \{ x_i; i \leq n \}$ where $x_i < x_{i+1}$ for $i < n$ and take rational open intervals $U_i \subseteq A \otimes B^+(1)$ such that $x_i \in U_i$ for $i \leq n, U_i \cap U_j = \emptyset$ and $U_i \times U_j \subseteq K_0$ for every $i, j \leq n$ with $i \neq j$. Let $M_i \in \text{dom}(N_r)$ for $1 \leq i \leq n$ be such that $x_{i-1} \in M_i$ and $x_i \notin M_i$. And let $N_0 := H \cap H(c^+)$ and recursively pick $N_i \in N_i(M_i)$ with $N_{i-1} \in N_i$ for $1 \leq i \leq n$. Let $G$ be $\mathbb{P}'$-generic over $H(\theta)$ with $f_r \in G$. By the same argument as in the proof of claim 2.5, it is proved that in $H(\theta)[G]$, there are rational open intervals $V_i^0, V_i^1 \subseteq U_i, y_i \in V_i^1 \cap V$ for $i \leq n$ and $s \in D \cap V$ such that
1. \( x_i \in V_i^0 \) for all \( i \leq n \),
2. \( V_i^0 \cap V_i^1 = \emptyset \) and \( V_i^0 \ast V_i^1 \subseteq K_0 \) for all \( i \leq n \),
3. \( f_s \in G \),
4. \( X_s = (X_r \cap H) \cup \{ y_i; i \leq n \} \) and for any \( x \in X_r \cap H \) and \( i < j \leq n \), and \( x < y_i < y_j \),
5. \( \text{dom}(N_s) \) is an end extension of \( \text{dom}(N_r) \cap H \) and for all \( M \in \text{dom}(N_r) \cap H \), \( N_r(M) \cap H \subseteq N_s(M) \).

By \( H < H(\theta) \) and the genericity of \( f_r \), we can find \( g \in G \) which decides all values of \( V_i^0, V_i^1 \subseteq U_i, y_i \in V_i^1 \cap H \) for \( i \leq n \) and \( s \in D \cap H \) and is a common extension of \( f_r \) and \( f_s \) in \( P' \). It's enough to find a common extension of \( r \) and \( s \).

To find it, let \( \{ L_i; i < l \} \) enumerate \( N_r(M_0) \) with \( L_0 := H \cap H(\mathcal{C}^+) = N_0 \) and \( \varphi_i := \pi_{L_i}^{-1} \) for \( i < l \). We notice that for each \( M \in \text{dom}(N_s) \setminus \text{dom}(N_r) \),

- \( \varphi_i(M) \in L_i \) (because \( M \in H \cap H(\mathcal{C}^+) = M_0 \)),
- \( \varphi_i(M) = M \) (because \( M \) is transitive), and
- \( \varphi_i(M) \prec H(\mathcal{C}^+) \) (because \( \varphi_i(M) \) and \( N \) are isomorphic and \( N \prec H(\mathcal{C}^+) \) for \( N \in N_s(M) \)).

We define a function \( N' \) with domain \( \text{dom}(N_r) \cup \text{dom}(N_s) \) by:

\[
N'(M) := \begin{cases} 
N_r(M) \cup N_s(M) & \text{if } M \in \text{dom}(N_r) \cap H \\
N_s(M) \cup \{ \varphi_i(M); i < l \} & \text{if } M \in \text{dom}(N_s) \setminus \text{dom}(N_r) \\
N_r(M) \setminus H & \text{if } M \in \text{dom}(N_r) \setminus H
\end{cases}
\]

for every \( M \in \text{dom}(N') \). Then it can be checked that \( \langle g, X_r : X_s, N' \rangle \) is a common extension of \( r \) and \( s \) if it is a condition of \( Q' \). To check \( \langle g, X_r : X_s, N' \rangle \in Q' \), the only non-trivial requirement is that \( N' \) satisfies (c3), in particular the case that \( M \in \text{dom}(N_r) \cap H \), \( M' \in \text{dom}(N_s) \setminus \text{dom}(N_r) \) with \( M \in M' \) and \( N \in N_r(M) \setminus H \). Then we can find \( L_i \in N_r(M_0) \) with \( N \in L_i \), because of \( r \in Q' \) and \( M \in M_0 \). Then \( N \) is in \( \varphi_i(M') \) and an elementary submodel of \( \varphi_i(M') \), since \( M \prec M' \prec M_0 \) and \( \varphi_i \upharpoonright M = \pi_{M^{-1}} \subseteq \pi_{M,1}^{-1} = \varphi_i \upharpoonright M' \).

If \( \vdash_{P}(\forall u)^{\mathbb{A} \otimes \mathcal{B}^+}1 \) is not countably separated "', (by the argument similar to Lemma 2.7) \( X_G := \bigcup \{ X_p; p \in G \} \) is uncountable. The biggest difference between \( Q(A, \mathbb{B}, \mathcal{U}) \) and \( Q'(A, \dot{B}, \mathcal{U}, f) \) is that \( Q'(A, \dot{B}, \mathcal{U}, f) \) has a good chain condition. The following lemma says that it preserves cardinalities under CH.

**Lemma 3.5.** For \( A, \dot{B}, \mathcal{U} \) and \( f \in \bigcup_{X \in \mathcal{U}^*} \mathbb{2}^X, Q'(A, \dot{B}, \mathcal{U}, f) \) has the \( \mathcal{C}^+ \)-c.c.

**Proof.** For conditions \( p \) and \( q \) in \( \mathbb{Q}'(A, \dot{B}, \mathcal{U}, f) \), if \( f_p = f_q, X_p = X_q \) and \( \text{dom}(N_p) = \text{dom}(N_q) \), then \( \langle f_p, X_p, N'' \rangle \) is a common extension of \( p \) and \( q \), where \( N'' \) has the domain \( \text{dom}(N_p) \) and \( N''(M) = N_p(M) \cup N_q(M) \). Therefore \( \{ p \in Q'; f_p = f \ & \ X_p = X \ & \ \text{dom}(N_p) = \mathfrak{N} \} \) is centered for every \( f \in \mathbb{P}, X \in [A \otimes B(1)]^{<\omega} \) and a finite \( \mathcal{E} \)-chain \( \mathfrak{N} \) of countable transitive elementary submodels. \( \square \)
3.2 Proof of Theorem 3.1

To prove Theorem 3.1, we assume that the ground model $\mathbf{V}$ is $\mathbf{L}$. Let $S_0$ and $S_1$ be stationary on $\omega_2$ with $S_0 \cap S_1 = \emptyset$ and $S_0 \cup S_1 = \text{Cof}(\omega_1) \cap \omega_2$, where $\text{Cof}(\omega_1) = \{\alpha < \text{On}; \text{cf}(\alpha) = \omega_1\}$. Then $\mathbf{V}$ satisfies $\diamondsuit_{\omega_2}(S_1)$. Let $\{D_\alpha; \alpha \in S_1\}$ be a diamond sequence, i.e. for any subset $E$ of $\omega_2$, $\{\alpha < S_1; E \cap \alpha = D_\alpha\}$ is stationary. We define a countable support iteration $\langle P_\alpha, Q_\alpha; \alpha < \omega_2 \rangle$ (and pick a $P_\alpha$-generic filter $G \upharpoonright \alpha$ over $\mathbf{V}$ for $\alpha < \omega_2$ recursively) as follows:

Stage $2\alpha$ with $2\alpha \notin \text{Cof}(\omega_1)$ Construct an ultrafilter base $\langle X_\alpha; \alpha < \omega_2 \rangle$ (e.g. using a $\sigma$-centered Mathias forcing).

Stage $2\alpha + 1$ Construct to force $\text{MA}$ by a book-keeping argument.

Stage $\alpha \in S_0$ Let $Q_\alpha := P(\mathcal{U}((X_\xi; \xi < \alpha)))$, where $\mathcal{U}((X_\xi; \xi < \alpha))$ is the ultrafilter generated by $\langle X_\xi; \xi < \alpha \rangle$. (We notice that

\[ \models_{P_\alpha} \mathcal{U}((X_\xi; \xi < \alpha)) \text{ is an ultrafilter} \]

if $\alpha < \omega_2$ has the cofinality $\omega_1$.)

Stage $\alpha \in S_1$ If $D_\alpha$ codes some $\langle \dot{f}, \dot{A}, \dot{B} \rangle$, where

- $\dot{f}$ is a $P_\alpha$-name for a condition of $P(\mathcal{U}((X_\xi; \xi < \alpha)))$,
- $\dot{A}$ is a $P_\alpha$-name for a family of infinite subsets of $\omega$, and
- $\dot{B}$ is a $P_\alpha \dot{P}(\mathcal{U}((X_\xi; \xi < \alpha)))$-name for a family of infinite subsets of $\omega$,

such that

\[ \mathbf{V}[G \upharpoonright \alpha] \models \langle \dot{f}[G \upharpoonright \alpha], \dot{A}[G \upharpoonright \alpha], \dot{B}[G \upharpoonright \alpha] \rangle \models \mathcal{U}(\langle X_\xi; \xi < \alpha \rangle) \text{ forms an } (\omega_1, \alpha) \text{-gap } \]

then let $Q_\alpha := Q'(\dot{A}[G \upharpoonright \alpha], \dot{B}[G \upharpoonright \alpha], \mathcal{U}(\langle X_\xi; \xi < \alpha \rangle), \dot{f}[G \upharpoonright \alpha])$. Otherwise, let $Q_\alpha := \{1\}$. We write $G \upharpoonright \omega_2$ by $G$.

We note that $P_{\omega_2}$ is proper because proper-ness is closed under countable support iterations. So the following lemma indicates that it does not collapse cardinals. To show it, we use the following definition (see [16], [17] or [18]).

Definition 3.6. (Shelah) For a forcing notion $\mathbf{X}$, $\mathbf{X}$ satisfies the $\aleph_2$-properness isomorphism condition ($\aleph_2$-pic) if for all (some) large enough regular cardinal $\theta$, $\alpha < \beta < \omega_2$, countable elementary submodels $N_\alpha, N_\beta$ of $H(\theta)$ and a function $\pi: N_\alpha \rightarrow N_\beta$ satisfying that
\* \* \* \* 

- \( \alpha \in N_\alpha, \beta \in N_\beta, N_\alpha \cap \omega_2 \subseteq \beta, N_\alpha \cap \alpha = N_\beta \cap \beta, X \in N_\alpha \cap N_\beta, \)

- \( \pi \) is an isomorphism, \( \pi(\alpha) = \pi(\beta), \) and \( \pi \lceil (N_\alpha \cap N_\beta) \) is identity,

if \( p \in X \cap N_\alpha, \) then there exists an \((N_\alpha, X)\)-generic condition \( q \) which is a common extension of \( p \) and \( \pi(p) \) such that

\[
q \forces X \pi"(\dot{G} \cap N_\alpha) = \dot{G} \cap N_\beta .
\]

**Lemma 3.7.** \( \mathbb{P}_{\omega_2} \) has the \( \aleph_2 \)-c.c.

**Proof.** Shelah has shown the following facts about the \( \aleph_2 \)-pic (see [17]):

- Under CH, any \( \aleph_2 \)-pic forcing notion has the \( \aleph_2 \)-chain condition and preserves \( \aleph_1 \).

- Under CH, \( \aleph_2 \)-pic-ness is closed under countable support iterations.

- If a forcing notion is proper and has size \( \leq \aleph_1 \), it has the \( \aleph_2 \)-pic.

Therefore it suffices to show that all \( Q'(A, \dot{B}, U, f) \) have the \( \aleph_2 \)-pic. (I refer to the proof of Lemma 6 in [18] for the argument below.)

Let \( \theta \) be a large enough regular cardinal, and \( \alpha < \beta < \omega_2 \), countable elementary submodels \( N_\alpha, N_\beta \) of \( H(\theta) \) and a function \( \pi : N_\alpha \rightarrow N_\beta \) satisfy the assumptions of \( \aleph_2 \)-pic. And let \( p \in Q'(A, B, U, f) \cap N_\alpha \). Because of \( N_\alpha \cap \mathcal{P}(\omega) = N_\beta \cap \mathcal{P}(\omega) \) and \( N_\alpha = N_\beta \), it is proved that \( \pi(p) \) is a condition of \( Q'(A, \dot{B}, U, f), \) \( f_p = f_{\pi(p)}, X_p = X_{\pi(p)}, \) and \( \text{dom}(N_p) = \text{dom}(N_{\pi(p)}) \). So \((f_p, X_p, N')\) is a common extension of \( p \) and \( \pi(p) \), where \( \text{dom}(N') = \text{dom}(N_p) \) and for \( M \in \text{dom}(N_p), N'(M) = N_p(M) \cup N_{\pi(p)}(M) \). We put

\[
q := \langle f_p, X_p, \dot{N}' \cup \{ N_\alpha \cap H(c^+), N_\alpha \cap H(c^+) \} \rangle .
\]

As before, we can prove that \( q \) is also a condition of \( Q'(A, \dot{B}, U, f) \) and an \((N_\alpha, Q'(A, \dot{B}, U, f))\)-generic. So it is true that \( q \forces Q'(A, \dot{B}, U, f) \pi"(\dot{G} \cap N_\alpha) = \dot{G} \cap N_\beta \) because the compatibility in \( Q'(A, \dot{B}, U) \) is simply decided by \( f_p, X_p \) and \( \text{dom}(N_p) \) for any \( p \in Q'(A, \dot{B}, U) \). \( \square \)

In \( V[G], c = \aleph_2 \) and MA holds. By the standard Löwenheim-Skolem argument (see also [9]), since we iterate \( \mathbb{P}((X_\xi; \xi < \alpha)) \) stationary many times, it follows that \( m(\mathbb{P}((X_\alpha; \alpha < \omega_2))) = \aleph_2 \) in \( V[G], \) hence \( \mathbb{P}(U((X_\alpha; \alpha < \omega_2))) \) is \( \omega_2 \)-Baire. So it suffices to show that \( \mathbb{P}(U((X_\alpha; \alpha < \omega_2))) \) adds no \((\omega_1, \omega_2)\)-gaps.

Assume not, i.e. in \( V \) there are \( \mathbb{P}_{\omega_2} \)-names \( \dot{f}, \dot{A} \) and a \( \mathbb{P}_{\omega_2} \) name \( \dot{B} \) such that \( \dot{f}[G](= f) \in \mathbb{P}(U((X_\alpha; \alpha < \omega_2))), \dot{A}[G](= A) \subseteq \mathcal{P}(\omega) \) and

\[
f \forces \mathbb{P}(U((X_\alpha; \alpha < \omega_2))) \pi"(\dot{A}, \dot{B}[G]) \text{ forms an \((\omega_1, \omega_2)\)-gap } .
\]
We may consider \( \dot{B}[G] \) as a \( \mathcal{P}(U((X_\alpha; \alpha < \omega_2))) \)-name for a function from \( \omega_2 \) into \( \mathcal{P}(\omega) \), i.e.

\[
f \Vdash_{\mathcal{P}(U((X_\alpha; \alpha < \omega_2)))} \forall \alpha < \beta < \omega_2 (\dot{B}[G](\alpha) \subseteq \dot{B}[G](\beta)) \quad \& \forall a \in A \forall \alpha < \omega_2 (a \perp \dot{B}[G](\alpha)) \ .
\]

We note that \( \mathcal{P}(U((X_\alpha; \alpha < \omega_2))) \) does not add new reals.

**Claim 3.8.** \( C(\dot{A}, \dot{B}, \mathcal{U}(\langle X_\alpha; \alpha < \omega_2 \rangle), \dot{f}) := \{ \alpha \in \text{Cof}(\omega_1) \cap \omega_2; \)

\[
\mathcal{V}[G \upharpoonright \alpha] = " \ f \in \mathcal{P}(U((X_\xi; \xi < \alpha))), \ \mathcal{A} \subseteq \mathcal{V}[G \upharpoonright \alpha], \text{ and} \ f \Vdash_{\mathcal{P}(U((X_\xi; \xi < \alpha)))} \(" \dot{A}, \dot{B}[G] \upharpoonright \alpha \) \text{ forms an } (\omega_1, \alpha) \)-gap " \}

is \( \omega_1 \)-club.

**Proof.** " \( \mathcal{V}[G \upharpoonright \alpha] = " \ f \in \mathcal{P}(U((X_\xi; \xi < \alpha))) \ & \mathcal{A} \subseteq \mathcal{V}[G \upharpoonright \alpha] " \) is upward closed with respect to \( \alpha \), and \( \omega_1 \)-closed-ness is trivial because for \( \alpha \in \text{Cof}(\omega_1) \cap \omega_2, \)

\[
\mathcal{V}[G \upharpoonright \alpha] \cap 2^\omega = \bigcup_{\xi < \alpha} \mathcal{V}[G \upharpoonright \xi] \cap 2^\omega.
\]

So we check that it is unbounded.

We note that in \( \mathcal{V}[G] \) for all \( x \in A^\perp \) and \( g \leq_{\mathcal{P}(U((X_\alpha; \alpha < \omega_2)))} f \), there are \( y_{x,g} \in \mathcal{P}(\omega) \), \( \tau_{x,g} \leq_{\mathcal{P}(U((X_\alpha; \alpha < \omega_2)))} g \) and \( \xi_{x,g} < \beta_{x,g} < \omega_2 \) such that

- \( x \notin y_{x,g}, \)
- \( y_{x,g} \in \mathcal{V}[G \upharpoonright \beta_{x,g}], \text{ and} \)
- \( \tau_{x,g} \Vdash_{\mathcal{P}(U((X_\alpha; \alpha < \omega_2)))} \" \dot{B}[G](\xi_{x,g}) = y_{x,g} " \).

Taking \( \alpha < \omega_2 \), we recursively construct \( \langle \gamma_\xi; \xi < \omega_1 \rangle \subseteq \text{Cof}(\omega_1) \cap \omega_2 \) such that

- \( \alpha \leq \gamma_0 \) and \( \gamma_\xi \leq \gamma_\eta \) for \( \xi \leq \eta < \omega_1, \)
- \( \mathcal{V}[G \upharpoonright \gamma_{\xi+1}] = " \forall x \in A^\perp \cap \mathcal{V}[G \upharpoonright \gamma_\xi] \forall g \in \mathcal{P}(U((X_\xi; \xi < \gamma_\xi))) \) with \( g \leq_{\mathcal{P}(U((X_\xi; \xi < \gamma_\xi)))} f \) \( (\tau_{x,g} \in \mathcal{P}(U((X_\xi; \xi < \gamma_\xi))) \ & \xi_{x,g} < \gamma_{\xi+1} " \), and
- \( \text{if } \eta \text{ is limit, then let } \gamma_\eta := \sup_{\xi < \eta} \gamma_\xi. \)

Then \( \sup_{\xi < \omega_1} \gamma_\xi \) is in \( C(\dot{A}, \dot{B}, \mathcal{U}(\langle X_\alpha; \alpha < \omega_2 \rangle), \dot{f}). \)

Since \( m(\mathcal{P}(U((X_\alpha; \alpha < \omega_2)))) = \kappa_2 \), by Lemma 2.6, in \( \mathcal{V}[G] \)

\[
f \Vdash_{\mathcal{P}(U((X_\alpha; \alpha < \omega_2)))} \" \dot{A} \otimes (\dot{B}[G]^+(f))^\mathcal{V} \quad \text{is not a union of countably many } K_1 \text{-homogeneous subsets } " .
\]
By Proposition 1.5, this is equivalent to

\[ f \Vdash_{P}( \mathcal{U}(X_{\alpha}; \alpha < \omega_{2})) \langle (\mathcal{A}, (\mathcal{B}[G]^{+}(f))^{\forall} \rangle \text{ is not countably separated} \]

(in \(\mathcal{V}[G]\)), i.e. for all \(P'(\mathcal{U}(X_{\alpha}; \alpha < \omega_{2}))\)-names \(\check{a} = (\dot{a}_{n}; n < \omega) \in (\mathcal{A}^{\alpha})^{\omega}\) and \(g \leq_{P'}(\mathcal{U}(X_{\alpha}; \alpha < \omega_{2})) f\), there are \(y_{\vec{c},g}, z_{\vec{c},g} \in \mathcal{P}(\omega)\), \(r_{\vec{c},g} \leq_{P'}(\mathcal{U}(X_{\alpha}; \alpha < \omega_{2})) g\) and \(\check{\xi}_{\vec{c},g} < \beta_{\vec{c},g} < \omega_{2}\) such that

- \((z_{\vec{c},g}, y_{\vec{c},g}) \in \mathcal{A} \otimes \hat{\mathcal{B}}[G]^{+}(f) \cap \mathcal{V}[G | \beta_{\vec{c},g}]\), and
- \(r_{\vec{c},g} \Vdash_{P'}(\mathcal{U}(X_{\alpha}; \alpha < \omega_{2})) \langle \check{B}[G](\check{\xi}_{\vec{c},g}) = \check{y}_{\vec{c},g} \& \forall n < \omega(z_{\vec{c},g} \notin r_{\vec{c},g} \forall \check{y}_{\vec{c},g} \in (\mathcal{C}^{*})^{\forall} \check{\dot{c}}_{n}) \rangle \).

So by an argument similar to the one of the previous claim,

\[ C'(\mathcal{A}, \mathcal{B}, \mathcal{U}(\langle \check{X}_{\alpha}; \alpha < \omega_{2} \rangle), f') := \{ \alpha \in \text{Cof}(\omega_{1}) \cap \omega_{2}; \]

\[ \mathcal{V}[G | \alpha] \models " f \in \mathbb{P}(\mathcal{U}(X_{\xi}; \xi < \alpha)), \]

\[ \mathcal{A} \subset \mathcal{V}[G | \alpha], \text{ and} \]

\[ f \Vdash_{P'}(\mathcal{U}(X_{\xi}; \xi < \alpha)) \langle (\mathcal{A}, (\mathcal{B}[\mathcal{G}]^{\forall}(f))^{\forall}) \rangle \text{ is not countably separated} "\]

is \(\omega_{1}\)-club. Thus by the diamond sequence; there exists

\[ \alpha \in C(\mathcal{A}, \mathcal{B}, \mathcal{U}(\langle \check{X}_{\alpha}; \alpha < \omega_{2} \rangle), f) \cap C'(\mathcal{A}, \mathcal{B}, \mathcal{U}(\langle \check{X}_{\alpha}; \alpha < \omega_{2} \rangle), f') \]

such that \(D_{\alpha}\) codes \(\langle f', \mathcal{A}, \mathcal{B} | \alpha \rangle \). So \(Q_{\alpha} = Q'(\mathcal{A}, (\mathcal{B}[\mathcal{G}] | \alpha), \mathcal{U}(\langle X_{\xi}; \xi < \alpha \rangle), f)\) and \(\mathcal{G}(\alpha)\) is \(Q_{\alpha}\)-generic over \(\mathcal{V}[G | \alpha]\). Then \(\mathcal{X}' := \cup\{ X_{p}; p \in \mathcal{G}(\alpha) \}\) is uncountable \(K_{0}\)-homogeneous. We note that for all \(p \in \mathcal{G}(\alpha), f_{p}\) is in \(\mathbb{P}(\mathcal{U}(X_{\alpha}; \alpha < \omega_{2}))\).

So by the same argument at the end of the proof of Main Lemma ??, there are a fusion \(f\) of \(\langle f_{p}; p \in \mathcal{G}(\alpha) \rangle\) and uncountable \(K_{0}\)-homogeneous \(\mathcal{X} \subset \mathcal{A} \otimes \hat{\mathcal{B}}[G]^{+}(f)\) such that \(\hat{f} \Vdash_{\mathbb{P}(\mathcal{U}(X_{\alpha}; \alpha < \omega_{2}))} " b \in \mathcal{B}^{+} \) for all \((a, b) \in \mathcal{X}\), which is a contradiction and completes the proof of Theorem 3.1.

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References


