Spiral Conditions for Splines and Their Applications to Curve Design

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Abstract

Walton & Meek obtained a $G^2$ fair curve by adding a spiral segment to one end of an existing curve ([1]). The added segments are commonly quadratic, T-cubic, general cubic and PH quintic spirals. We derive the larger regions for the end points of two-parameter general cubic and PH quintic spirals.

1 Introduction

Spirals have several advantages of containing neither inflection points, singularities nor curvature extrema. Such curves are useful in the design of fair curves. Walton & Meek ([1]) considered a $G^2$ curve design with spiral segments.

The object of this paper is to examine their methods and obtain, in some cases, larger reachable regions for the end points of quadratic, T-cubic, general cubic and PH quintic spiral segments starting from the origin. The added segment passes through the origin and is constrained by its beginning unit tangent vector $(1, 0)$ and its beginning curvature. Sections 2-3 treat the cases (i) starts a non-inflection point with a radius of curvature, $r$, and continues with a curvature of increasing magnitude up to a given non-inflection point, or (ii): a segment starts a non-inflection point with a radius of curvature, $r$, and continues with a curvature of decreasing magnitude up to a given non-inflection point. Figure 1 shows that T-cubic spirals are more flexible than quadratic ones. Sections 4-5 treat the two cases: (iii) a segment starts an inflection point with a curvature of increasing magnitude up to a given radius of curvature, $r$, with a given ending unit tangent vector $(\cos \theta, \sin \theta)$, or (iv) starts a non-inflection point with a given radius of curvature, $r$, and continues with a curvature of decreasing magnitude up to an inflection point with a given ending unit tangent vector $(\cos \theta, \sin \theta)$. Sections 2-5 consider the spiral curve $z(t) = (x(t), y(t)), 0 \leq t \leq 1$ and obtain the reachable regions for $(\xi, \eta) = z(1)/r$. Its signed curvature $\kappa(t)$ is given by

$$\kappa(t) = z'(t) \times z''(t)/\|z'(t)\|^3 \quad (1.1)$$
where "×" and \( \| \bullet \| \) mean the cross product of two vectors and the Euclidean norm, respectively.

2 Quadratic spirals

This section first treats the cases (i) and (ii) for quadratic spirals whose unit tangent vector are not fixed. Require the following conditions for \( 0 < \theta < \pi/2 \):

\[
z(0) = (0, 0), \quad z'(0) \parallel (1, 0), \quad z'(1) \parallel (\cos \theta, \sin \theta), \quad \kappa(0) = 1/r
\]  

(2.1)

to obtain

\[
x'(t) = u_0(1 - t) + \left( u_0^2 t/r \right) \cot \theta, \quad y'(t) = u_0^2 t/r \quad (u_0 > 0)
\]  

(2.2)

Then, with \( t = 1/(1 + s) \)

\[
\kappa'(t) = \frac{3r^2(1 + s)^4 \{ rs(\tan \theta - u_0) \cot \theta + u_0(r \sin \theta \cos \theta - u_0) \csc^2 \theta \}}{(r^2 s^2 + 2rsu_0 \cot \theta + u_0^2 \csc^2 \theta)^{5/2}}
\]  

(2.3)

Hence, the curvature is monotone increasing if \( 0 < u_0 \leq (r/2) \sin 2\theta \) and monotone decreasing if \( u_0 \geq r \tan \theta \). Note with \( z = \tan \theta \),

\[
(\xi, \eta) \left( = \frac{z(1)}{r} \right) = \left( \frac{u_0}{2r} \left( 1 + \frac{u_0}{rz} \right), \frac{1}{2} \left( \frac{u_0}{r} \right)^2 \right)
\]  

(2.4)

Since \( u_0 = r\sqrt{2\eta} \) and \( z = \sqrt{2\eta}/(\sqrt{2}\xi - \sqrt{\eta}) \), we obtain the necessary and sufficient condition for the existence of a unique quadratic segment for given \( (\xi, \eta) \):

**Lemma 2.1** The system of equations (2.4) has a unique solution \( (u_0, z) \) satisfying \( u_0, z > 0 \) if \( \sqrt{2}\xi > \sqrt{\eta} \).

Now we derive the condition for the curvature to be monotone increasing or decreasing.

**Case (i) (increasing curvature):** Note

\[
u_0 - \frac{r \sin 2\theta}{2} = \frac{2r\sqrt{\eta} \{ \sqrt{2}\xi^2 - 3\xi\sqrt{\eta} + \sqrt{2}\eta(1 + \eta) \}}{2(\xi^2 - \sqrt{2}\xi\eta + \eta^2)}
\]  

(2.5)

Hence, the unique quadratic spiral with a curvature of increasing magnitude exists if

\[
\xi, \eta > 0, \sqrt{2}\xi^2 - 3\xi\sqrt{\eta} + \sqrt{2}\eta(1 + \eta) \leq 0
\]  

(2.6)

**Case (ii) (decreasing curvature):** Note

\[
u_0 - r \tan \theta = (2r\sqrt{\eta}(\xi - \sqrt{2\eta}))/\sqrt{(2\xi - \sqrt{\eta})}
\]  

(2.7)

Since \( \kappa(1) = r^2 \sin^2 \theta/u_0^3(> 0) \), the unique quadratic spiral with a curvature of decreasing magnitude exists if

\[
\xi \geq \sqrt{2\eta} (> 0)
\]  

(2.8)

Thus we have
**Theorem 2.1** The reachable region of increasing one is given by (2.6) (where the equality means $\kappa'(1) = 0$) and the reachable region for a quadratic spiral of decreasing curvature magnitude is given by (2.8) (where the equality means $\kappa'(0) = 0$).

### 3 T-cubic spirals

The section first treats the cases (i) and (ii) for two-parameter T-cubic spirals of the form: $z'(t) = (u(t)^2 - v(t)^2, 2u(t)v(t))$ with linear $u(t), v(t)$ where the unit tangent vectors are not fixed. Require (2.1) for $0 < \theta < \pi$ to obtain

$$u(t) = u_0(1 - t) + \left(\frac{tu_0^3}{2r}\right) \cot(\theta/2), \quad v(t) = tu_0^3/(2r) \quad (u_0 > 0) \quad (3.1)$$

Then, with $t = 1/(1 + s)$

$$\kappa'(t) = \frac{128r^3(1 + s)^5 \{rs(2r \tan \frac{\theta}{2} - u_0^2) \sin \theta + u_0^2(r \sin \theta - u_0^2)\}}{(1 - \cos \theta) \left(4r^2s^2 + 4rsu_0^2 \cot \frac{\theta}{2} + u_0^4 \csc^2 \frac{\theta}{2}\right)^3} \quad (3.2)$$

Hence, the curvature is monotone increasing if $0 < u_0 \leq \sqrt{r \sin \theta}$ and monotone decreasing if $u_0 \geq \sqrt{2r \tan(\theta/2)}$. Easily we obtain with $z = \tan \theta/2$

$$\xi = \frac{u_0^3}{12r} \left\{4 + \frac{2u_0^2}{rz} + \frac{u_0^4}{r^2} \left(1 - 1\right)\right\}, \quad \eta = \frac{u_0^6}{6r^2} \left(1 + \frac{u_0^2}{rz}\right) \quad (3.3)$$

Here, we note the necessary and sufficient condition for the existence of a unique T-cubic segment for given $(\xi, \eta)$:

**Lemma 3.1** The system of equations (3.3) has a unique solution $(u_0, z)$ satisfying $u_0, z > 0$ if $\sqrt{6}\xi > (2 - 3\eta)\sqrt{\eta}$.

Proof of lemma. As in [1], a change of variables: $u_0^2/r = g$ reduces (3.3) to

$$6\xi = 2g + g^2/z + (1 - z^2)g^3/(2z^2), \quad 6\eta = g^2 + g^3/z \quad (3.4)$$

Eliminate $z$ to obtain

$$\phi(g) \left(= g^6 - 3g^4 + 12\xi g^3 - 36\eta^2\right) = 0, \quad 0 < g < \sqrt{6\eta} \quad (3.5)$$

Then,

$$(1 + m)^6\phi \left(\sqrt{6\eta}/(1 + m)\right) = 12\eta\sqrt{\eta}\sum_{i=0}^{6} a_i m^i, \quad m > 0 \quad (3.6)$$

where

$$(a_6, a_5, a_4) = -3\sqrt{\eta}(1, 6, 15), \quad (a_3, a_2, a_1, a_0)$$

$$= 6\sqrt{6} \left(\xi - 10\lambda, 3(\xi - 3\lambda), 3(\xi - 2\lambda), \xi - 2\lambda(1 - 9\lambda^2)\right), \quad \lambda = \sqrt{\eta}/6 \quad (3.7)$$
Since $2\lambda(1 - 9\lambda^2) < 2\lambda < 3\lambda < 10\lambda$, $a_0 \leq 0$ implies $a_i < 0, 1 \leq i \leq 6$. Combine Descartes' rule of signs and intermediate value of theorem to give the desired result if $a_0 > 0$, i.e., $\xi > 2(1 - 9\lambda^2)\lambda = (2 - 3\eta)\sqrt{\eta/6}$.

For $\xi > (2 - 3\eta)\sqrt{\eta/6}$, we obtain the condition for the curvature to be monotone increasing or decreasing.

**Case (i) (increasing curvature):** Note

$$u_0^2 - r \sin \theta = rg(g^6 + 3g^4 + 24\eta g + 36\eta^2)/(g^6 + g^4 - 12\eta g + 36\eta^2) \leq 0 \quad (3.7)$$

which requires $g^6 + 3g^4 + 24\eta g + 36\eta^2 \leq 0$. Since $\phi(g) = 0$, it is reduced to

$$\psi(g) = g^4 + 6\xi g - 12\eta \leq 0 \quad (3.8)$$

Since $\xi > (2 - 3\eta)\sqrt{\eta/6}$, the unique positive zero $c$ of $\psi(g)$ is less than $\sqrt{6\eta}$. The condition for $\phi(c) \geq 0$ is equivalent to the one for the equations of $\psi(g)$ and $\phi(g) - m$, $m \geq 0$ to have the common zero. Mathematica helps us reduce their Sylvester's resultant to

$$m^4 + 36\left\{3\xi^2 + 4\eta(1 + \eta)\right\} m^3 + 432\left\{9\xi^4 + 3\xi^2\eta(8 + 9\eta) + 2\eta^2(9 + 14\eta + 9\eta^2)\right\}$$

$$+ 3888\left\{12\xi^6 + 3\xi^4(3 + 16\xi + 24\xi^2) + 4\xi^2\eta^2(9 + 32\eta + 27\eta^2) + 16\eta^3(3 + 5\eta + 5\eta^2$$

$$+ 3\eta^3)\right\} m + 46656\eta^2 J(\xi, \eta) \quad (3.9)$$

where

$$J(\xi, \eta) = 36\xi^6 + 3\xi^4(9 + 16\eta + 36\eta^2) - 12\xi^2\eta(6 + 19\eta - 8\eta^2 - 9\eta^3) + 4\eta^2(3 + 2\eta + 3\eta^2)^2$$

Thus, Descartes' rule of signs implies that the unique T-cubic spiral with a curvature of increasing magnitude exists if

$$J(\xi, \eta) \leq 0, \ \xi, \eta > 0 \quad (3.10)$$

**Case (ii) (decreasing curvature):** Note

$$u_0^2 - 2r \tan(\theta/2) = 3gr(2\eta - g^2)/(6\eta - g^2) \geq 0 \quad (3.11)$$

which requires $0 < g \leq \sqrt{2\eta}$. Since

$$\phi(0) < 0, \ \phi(\sqrt{2\eta}) = 4\eta\sqrt{\eta}\left\{6\xi - (6 - \eta)\sqrt{2\eta}\right\}$$

Lemma 3.1 requires $\phi(\sqrt{2\eta}) \geq 0$, i.e.,

$$6\xi \geq (6 - \eta)\sqrt{2\eta}, \eta > 0 \quad (3.12)$$

Since $\kappa(1) = (16r^3/u_0^8)\sin^4(\theta/2) > 0$, the reachable region for the unique T-cubic spiral with a curvature of decreasing magnitude is given by (3.12). Hence we have
Theorem 3.1 The reachable region for a $T$-cubic spiral of increasing curvature magnitude is given by (3.10) (where the equality means $\kappa'(1) = 0$) and the reachable region of decreasing one is given by (3.12) (where the equality means $\kappa'(0) = 0$).

Fig. 1 (Cases (i) (heavy dots)-(ii) (light dots)). Regions for quadratic (left) and $T$-cubic (right) spirals.

4 General cubic spirals

This section treats cases (iii) - (iv) for general cubic two-parameter spirals where the ending tangent vector is fixed.

Case (iii) (increasing curvature): Require for fixed $0 < \theta < \pi/2$:

\[ z(0) = (0,0), \quad z'(0) \parallel (1,0), \quad \kappa(0) = 0, \quad z'(1) \parallel (\cos \theta, \sin \theta), \quad \kappa(1) = 1/r \quad (4.1) \]

to obtain

\[ x(t) = \left\{ \frac{qrt}{6 \sin \theta} \right\} \left\{ q \left\{ (3 - 2t)t + m(3 - 3t + t^2) \right\} + t^2 \sin 2\theta \right\}, \quad y(t) = qrt^3 \sin \theta / 3 \quad (4.2) \]

A symbolic manipulator helps us obtain

\[
\left\{ x'(t)^2 + y'(t)^2 \right\}^{5/2} \kappa'(t) = \left[ \frac{q^5 r^4}{4(1+s)^5 \sin 2\theta} \right] \sum_{i=0}^{5} b_i s^i, \quad t = 1/(1+s) \quad (4.3)
\]

where

\[ b_0 = 4 \left\{ 3q \cos \theta - (4 + m) \sin \theta \right\} \sin \theta, \quad b_1 = 2 \left\{ 6q^2 - q(5 - 4m) \sin 2\theta - 10m \sin^2 \theta \right\} \]

\[ b_2 = 2q \left\{ (-2 + 13m)q - 2m(4 - m) \sin 2\theta \right\}, \quad b_3 = 2mq \left\{ (-3 + 10m)q - 2m \sin 2\theta \right\} \]

\[ b_4 = 5m^3 q^2, \quad b_5 = m^3 q^2 \]

Hence, we obtain a sufficient spiral condition, i.e., $b_i \geq 0, 0 \leq i \leq 5$:

Lemma 4.1 The general cubic segment $z(t), 0 \leq t \leq 1$ of the form (4.2) is a spiral satisfying (4.1) if $m > 3/10$ and

\[ q \geq q(m, \theta) \left( = \text{Max} \left\{ \frac{(4 + m) \tan \theta}{3}, \quad \frac{2m(4 - m) \sin 2\theta}{13m - 2}, \quad \frac{2m \sin 2\theta}{10m - 3}, \quad \frac{1}{6} \left\{ (5 - 4m) \cos \theta + \sqrt{60m + (5 - 4m)^2 \cos^2 \theta} \right\} \sin \theta \right\} \right) \quad (4.4) \]
where $q(m, \theta) = \{(4 + m)/3\} \tan \theta$ for $m \geq 2(\sqrt{6} - 1)/5(\approx 0.5797)$.

From (4.2), we have
\[
(\xi, \eta) = \left( q \{(1 + m)q + \sin 2\theta\} / (6 \sin \theta), q \sin \theta / 3 \right)
\] (4.5)

Solve (4.5) for $m, q$ to obtain
\[
(m, q) = \left( \left( -3\eta^2 + 2\xi \sin^3 \theta - \eta \sin \theta \sin 2\theta \right) / (3\eta^2), 3\eta / (\sin \theta) \right)
\] (4.6)

Note $q \geq \{(4 + m)/3\} \tan \theta$ and $m \geq 2(\sqrt{6} - 1)/5$ to obtain the reachable region (indicated by heavy dots in Fig. 2 (left)) for the end points of the general cubic spiral where
\[
\frac{3(3 + 2\sqrt{6})\eta^2 + 5\eta \sin \theta \sin 2\theta}{10 \sin^3 \theta} \leq \xi \leq \frac{27\eta^3 - 9\eta^2 \sin \theta \tan \theta + 2\eta \sin^4 \theta}{2\sin^4 \tan \theta}
\] (4.7)

In addition,
\[
\frac{\eta}{\xi} \leq \frac{3 \sin 2\theta}{(1 + m)(4 + m) + 6 \cos^2 \theta}
\] (4.8)

Note that $m = 1$ and $\kappa'(1) = 0$ (i.e., $q = (4 + m)/3 \tan \theta$) are fixed in Walton & Meek([1]) where the reachable region reduces to a single point:
\[
(\xi, \eta) = \left( q \{(1 + m)q \cos \theta + 2\sin \theta\} / (6 \sin \theta), (1 + m)q^2 / 6 \right)
\] (4.13)

Case (iv) (decreasing curvature): Require for fixed $0 < \theta < \pi/2$:
\[
z(0) = (0, 0), \quad z'(0) \parallel (1, 0), \quad \kappa(0) = 1/r, \quad z'(1) \parallel (\cos \theta, \sin \theta), \quad \kappa(1) = 0
\] (4.10)

Then, transformation, i.e., rotation, shift, reflection with respect to $y$-axis and change of variable $t$ with $1 - t$ to (4.2) gives
\[
x(t) = \left( qrt / (6 \sin \theta) \right) \left[ q t \{3 - (2 - m)t \} \cos \theta + 2(3 - 3t + t^2) \sin \theta \right]
\] (4.11)
\[
y(t) = \left( q^2rt^2 / 6 \right) \{3 - (2 - m)t \}
\]

Note that Lemma 4.1 is valid under the above transformation, or directly
\[
\left\{ x'(t)^2 + y'(t)^2 \right\}^{5/2} \kappa'(t) = - \left[ q^5r^4 / \left\{ 4(1 + s)^5 \sin^2 \theta \right\} \right] \sum_{i=0}^{5} b_i s^{5-i}, \quad t = 1/(1 + s)
\] (4.12)

Note
\[
(\xi, \eta) = \left( q \{(1 + m)q \cos \theta + 2\sin \theta\} / (6 \sin \theta), (1 + m)q^2 / 6 \right)
\] (4.13)

Solve (4.13) for $m, q$ to obtain
\[
(m, q) = \left( \frac{2\eta - 3\xi^2 + 6\xi \eta \cot \theta - 3\eta^2 \cot^2 \theta}{3(\xi - \eta \cot \theta)^2}, 3(\xi - \eta \cot \theta) \right)
\] (4.14)
As in the above Case (iii), we obtain the reachable region (indicated by light dots in Fig. 2 (left)) for the end points of the general cubic spiral

\[ 27(\xi - \eta \cot \theta)^3 \geq 9 \tan \theta (\xi - \eta \cot \theta)^2 + 2\eta \tan \theta \]  

(4.15)

\[ 10\eta - 3(3 + 2\sqrt{6})(\xi - \eta \cot \theta)^2 \geq 0 \]

Note that \( m = 1 \) and \( \kappa'(0) = 0 \) (i.e., \( q = (4 + m)/3 \tan \theta \)) are fixed in Walton & Meek([1]) where the reachable region reduces to a single point:

\[ \xi = 40 \tan \theta /27, \quad \eta = 25 \tan^2 \theta /27 \]  

(4.16)

5 PH quintic spirals

This section treats cases (iii) and (iv) for two-parameter PH-quintic spiral segment of the form: \( z'(t) = (u(t)^2 - v(t)^2, 2u(t)v(t)) \). For later use, we note

\[ \left\{ u^2(t) + v^2(t) \right\}^3 \kappa'(t) = 2 \left\{ \left[ u(t)v''(t) - u''(t)v(t) \right] \left\{ u^2(t) + v^2(t) \right\} \right. 
\]

\[ -4 \left[ u(t)v''(t) - u''(t)v(t) \right] \left\{ u(t)u''(t) + v(t)v''(t) \right\} \right\} \]  

(5.1)

Case (iii) (increasing curvature): Require (4.1) for fixed \( 0 < \theta < \pi \) to obtain

\[ \frac{u(t)}{\sqrt{r}} = \frac{\sqrt{q}}{4 \sin \frac{\theta}{2}} \left[ q \left\{ m(1 - t) + 2t \right\} (1 - t) + 2t^2 \sin \theta \right], \quad \frac{v(t)}{\sqrt{r}} = \sqrt{q}t^2 \sin \frac{\theta}{2} \]  

(5.2)

A symbolic manipulator helps us obtain

\[ w(t) = \left[ q^3 r^2 / \left\{ 16(1 + s)^5 \sin^2 \frac{\theta}{2} \right\} \right] \sum_{i=0}^{5} c_i s^i, \quad t = 1/(1+s) \]  

(5.3)

where

\[ c_0 = 16 \left\{ q \sin \theta - (6 + m) \sin^2 \frac{\theta}{2} \right\}, \quad c_1 = 8 \left\{ 2q^2 - q(4 - 3m) \sin \theta - 14m \sin^2 \frac{\theta}{2} \right\} \]

\[ c_2 = 4q \left\{ (-2 + 9m)q - 3(4 - m) \sin \theta \right\}, \quad c_3 = 4mq \left\{ (-3 + 7m)q - 3m \sin \theta \right\} \]

\[ c_4 = m^2 q^2 (-2 + 7m), \quad c_5 = m^3 q^2 \]

Hence, we obtain a sufficient spiral condition \( c_i, 0 \leq i \leq 5 \) for \( z(t) \):

Lemma 5.1 The PH-quintic segment \( z(t), 0 \leq t \leq 1 \) of the form (5.3) is a spiral satisfying (4.1) if \( m > 3/7 \) and

\[ q \geq q(m, \theta) \left( = \text{Max} \left[ \frac{6 + m}{2} \tan \frac{\theta}{2}, \frac{3m(4 - m) \sin \theta}{9m - 2}, \frac{3m \sin \theta}{7m - 3}, \right. \right. \]

\[ \left. \left. \frac{1}{4} \left( (4 - 3m) \sin \theta + \sqrt{56m(1 - \cos \theta) + (4 - 3m)^2 \sin^2 \theta} \right) \right] \right) \]  

(5.4)

where \( q(m, \theta) = \{(6 + m)/2\} \tan(\theta/2) \) for \( m \geq 2(-3 + \sqrt{30})/7(\approx 0.707) \).
With help of Mathematica (if necessary),

\[
(i) \quad \xi = \frac{q \{(2 + 3m + 3m^2)q^2 + 2q(3 + m) \sin \theta + 24 \cos \theta(1 - \cos \theta)\}}{120(1 - \cos \theta)}
\]

\[
(ii) \quad \eta = \frac{q \{(3 + m)q + 12 \sin \theta\}}{60}
\]

Unlike in general cubic spirals, it is not easy to solve for \(m\), \(q\) and so the reachable region (indicated by heavy dots in Fig. 2 (right)) is numerically determined. In addition,

\[
\frac{\eta}{\xi} \leq \frac{4(42 + 9m + m^2 + 24 \cos \theta) \sin \theta}{4(42 + 9m + m^2) \cos \theta + 3(48 + 56m + 50m^2 + 13m^3 + m^4 + 32 \cos^2 \theta)}
\]

Note that \(m = 1\) and \(\kappa'(1) = 0\) (i.e., \(q = \{(6 + m)/2\} \tan (\theta/2)\)) are fixed in [1] where the reachable region reduces to a single point:

\[
\xi = 7(69 + 26 \cos \theta + 6 \cos 2\theta) \sec^2 (\theta/2) \tan (\theta/2)/240
\]

\[
\eta = 7(13 + 6 \cos \theta) \tan^2 (\theta/2)/60
\]

**Case (iv) (decreasing curvature):** Require (4.8) for fixed \(0 < \theta < \pi\). Then, transformation, i.e., rotation, shift, reflection with respect to \(y\)-axis and change of variable \(t\) with \(1 - t\) to (5.2) gives

\[
\frac{u(t)}{\sqrt{r}} = \frac{\sqrt{q}}{4 \sin \frac{\theta}{2}} \left[ qt \{2 - (2 - m)t\} \cos \frac{\theta}{2} + 4(1-t)^2 \sin \frac{\theta}{2} \right], \quad \frac{v(t)}{\sqrt{r}} = \frac{qt\sqrt{q}}{4} \{2 - (2 - m)t\}
\]

Then, note that Lemma 5.1 remains valid under the transformation, i.e., rotation, shift, reflection and change of variable, or directly

\[
w(t) = -\left[q^3r^2/\{16(1 + s)^5 \sin^2 (\theta/2)\}\right] \sum_{i=0}^{5} c_i s^{5-i}, \quad t = 1/(1 + s)
\]

With help of Mathematica (if necessary),

\[
(i) \quad \xi = \frac{q \{24 + \{-24 + (2 + 3m + 3m^2)q^2\} \cos \theta + 2q(3 + m) \sin \theta\}}{120(1 - \cos \theta)}
\]

\[
(ii) \quad \eta = \frac{q^2 \{2(3 + m) \sin \theta + q(2 + 3m + 3m^2)(1 + \cos \theta)\}}{120 \sin \theta}
\]

The heavy and light dotted regions correspond to the cases (iii) and (iv), respectively. Note that \(m = 1\) and \(\kappa'(0) = 0\) (i.e., \(q = \{(6 + m)/2\} \tan (\theta/2)\)) are fixed in Walton & Meek ([1]) where the region reduces to a single point:

\[
\xi = 7(26 + 75 \cos \theta) \sec^2 (\theta/2) \tan (\theta/2)/240, \quad \eta = 147 \tan^2 (\theta/2)/40
\]

Walton & Meek’s points by (4.9), (4.16) for general cubics and (5.7), (5.11) for PH-quintic are denoted in black discs.
Fig. 2 (Cases (iii)(light dotted region)-(iv)(heavy dotted region)). Regions for general cubic (left) and PH-quintic (right) spirals for $\theta = \pi/4(\text{lower}), \pi/3(\text{upper})$.

6 Numerical Examples

Fig. 3. Vase profiles with $G^2$ cubic Bézier spiral segments and their shaded renditions.

References

