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<th>Title</th>
<th>Geometric Properties of Generalized Fractional Integral Operator (Study on Applications for Fractional Calculus Operators in Univalent Function Theory)</th>
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<tr>
<td>Author(s)</td>
<td>Choi, Jae Ho; Kim, Yong Chan; Ponnusamy, S.</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2004), 1363: 21-32</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/25302">http://hdl.handle.net/2433/25302</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Geometric Properties of Generalized Fractional Integral Operator

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September 11, 2002

Abstract

Let $A$ be the class of normalized analytic functions in the unit disk $\Delta$ and define the class

$$\mathcal{P}(\beta) = \{f \in A : \exists \varphi \in \mathbb{R} \text{ such that } \text{Re}[e^{i\varphi}(f'(z) - \beta)] > 0, z \in \Delta\}.$$ 

In this paper we find conditions on the number $\beta$ and the nonnegative weight function $\lambda(t)$ such that the integral transform

$$V_{\lambda}(f)(z) = \int_{0}^{1} \lambda(t) \frac{f(tz)}{t} dt$$

is convex of order $\gamma$ ($0 \leq \gamma \leq 1/2$) when $f \in \mathcal{P}(\beta)$. Some interesting further consequences are also considered.

Key Words. Gaussian hypergeometric function, integral transform, convex function, starlike function, fractional integral

2000 Mathematics Subject Classification. 30C45, 33C05

1. Introduction and Preliminaries

Let $A$ denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Also let $S$, $S^*(\gamma)$ and $\mathcal{K}(\gamma)$ denote the subclasses of $A$ consisting of functions which are univalent, starlike of order $\gamma$ and convex of order $\gamma$ in $\Delta$, respectively. In particular, the classes $S^*(0) = S^*$ and $\mathcal{K}(0) = \mathcal{K}$ are the familiar ones of starlike and convex functions in $\Delta$, respectively. We note that for $0 \leq \gamma < 1$,

$$f(z) \in \mathcal{K}(\gamma) \iff zf'(z) \in S^*(\gamma)$$

and $f \in S^*(\gamma)$ if and only if $\text{Re}(zf(z)/f(z)) > \gamma$ for $z \in \Delta$. 
Let $a$, $b$ and $c$ be complex numbers with $c \neq 0, -1, -2, \ldots$. Then the Gaussian/classical hypergeometric function $\, _2F_1(a, b; c; z) \equiv F(a, b; c; z)$ is defined by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \left\{ \begin{array}{ll} 1 & (n = 0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}). \end{array} \right.$$ 

The hypergeometric function $F(a, b; c; z)$ is analytic in $\Delta$ and if $a$ or $b$ is a negative integer, then it reduces to a polynomial. For functions $f_j(z)$ ($j = 1, 2$) of the forms

$$f_j(z) := \sum_{n=1}^{\infty} a_{j,n} z^n \quad (a_{j,1} := 1; j = 1, 2),$$

let $(f_1 \ast f_2)(z)$ denote the Hadamard product or convolution of $f_1(z)$ and $f_2(z)$, defined by

$$(f_1 \ast f_2)(z) := \sum_{n=1}^{\infty} a_{1,n} a_{2,n} z^n \quad (a_{j,1} := 1; j = 1, 2).$$

For $f \in \mathcal{A}$, the following special case gives rise to a natural convolution operator $H_{a,b,c}$ defined by

$$H_{a,b,c}(f)(z) := z F(a, b; c; z) \ast f(z).$$

Note that this is a three-parameter family of operators and contains as special cases several of the known linear integral or differential operators studied by a number of authors. In fact, this operator was considered first time in this form by Hoholov [7] and has been studied extensively by Ponnusamy [11], Ponnusamy and Rønning [14] and many others [2, 8, 5]. For example, by letting $H_{1,b,c}(f) = \mathcal{L}(b, c)(f)$, we get the operator $\mathcal{L}(b, c)(f)$ discussed by Carlson and Shaffer [4]. Clearly, $\mathcal{L}(b, c)$ maps $\mathcal{A}$ onto itself, and $\mathcal{L}(c, b)$ is the inverse of $\mathcal{L}(b, c)$, provided that $b \neq 0, -1, -2, \ldots$. Furthermore, $\mathcal{L}(b, b)$ is the unit operator and

$$(1.1) \quad \mathcal{L}(b, c) = \mathcal{L}(b, e) \mathcal{L}(e, c) = \mathcal{L}(e, c) \mathcal{L}(b, c) \quad (c, e \neq 0, -1, -2, \ldots).$$

Also, we note that $\mathcal{L}(b, b)f(z) = f(z)$, $\mathcal{L}(2,1)f(z) = zf'(z)$,

$$\mathcal{K}(\gamma) = \mathcal{L}(1,2)\mathcal{S}^*(\gamma) \quad (0 \leq \gamma < 1),$$

$$\mathcal{S}^*(\gamma) = \mathcal{L}(2,1)\mathcal{K}(\gamma) \quad (0 \leq \gamma < 1)$$

and the Ruscheweyh derivatives [16] of $f(z)$ are $\mathcal{L}(n+1,1)f(z)$, $n \in \mathbb{N} \cup \{0\}$. For $\beta < 1$, we define

$$\mathcal{P}(\beta) = \{ f \in \mathcal{A} : \exists \varphi \in \mathbb{R} \text{ such that } \Re[e^{\beta\varphi}(f'(z) - \beta)] > 0, \ z \in \Delta \}.$$ 

Throughout this paper we let $\lambda : [0, 1] \to \mathbb{R}$ be a nonnegative function with the normalization $\int_0^1 \lambda(t) \, dt = 1$. For certain specific subclasses of $f \in \mathcal{A}$, many authors considered the geometric properties of the integral transform of the form

$$V_{\lambda}(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} \, dt.$$
Generalized Fractional Integral Operator

More recently, starlikeness of this general operator $V_{\lambda}(f)$ was discussed by Fournier and Ruscheweyh [6] by assuming that $f \in \mathcal{P}(\beta)$. The method of proof is the duality principle developed mainly by Ruscheweyh [17]. This result was later extended by Ponnusamy and Rønning [15] by means of finding conditions such that $V_{\lambda}(f)$ carries $\mathcal{P}(\beta)$ into starlike functions of order $\gamma$, $0 \leq \gamma \leq 1/2$ and was further generalized in [3].

In this paper, we find conditions on $\beta$, $\gamma$ and the function $\lambda(t)$ such that $V_{\lambda}(f)$ carries $\mathcal{P}(\beta)$ into $\mathcal{K}(\gamma)$. As a consequence of this investigation, a number of new results are established. The following lemma is the key for the proof of our main results.

1.3. Lemma. Let $\Lambda(t)$ be a real valued monotone decreasing function on $[0, 1]$ satisfying $\Lambda(1) = 0$, $t\Lambda(t) \to 0$ for $t \to 0^+$ and

$$-\frac{t\Lambda'(t)}{(1+t)(1-t)^{1+2\gamma}} = \frac{\lambda(t)}{(1+t)(1-t)^{1+2\gamma}}$$

is decreasing on $(0, 1)$ where

$$\Lambda(t) = \int_{t}^{1} \frac{\lambda(s)}{s} ds.$$

If $\beta = \beta(\lambda, \gamma)$ is given by

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = -\int_{0}^{1} \gamma(1+t)\lambda(t) \frac{1}{(1-\gamma)(1+t)^2} dt$$

then $V_{\lambda}(\mathcal{P}_{\beta}) \subset \mathcal{K}(\gamma)$, $0 \leq \gamma \leq \frac{1}{2}$, where $V_{\lambda}(f)$ is defined above.

Proof. Proof of this lemma quickly follows from the work of Ponnusamy and Rønning [15] and therefore, we omit the details. \qed

2. Main Results

In order to apply Lemma 1.3 with $\gamma \in [0, \frac{1}{2}]$ it suffices to show that

$$u(t) = -\frac{t\Lambda'(t)}{(1+t)(1-t)^{1+2\gamma}}$$

is decreasing on the interval $(0, 1)$ where $\Lambda(t) = \int_{t}^{1} \frac{\lambda(s)}{s} ds$. Taking the logarithmic derivative of $u(t)$ and using the fact that $\Lambda'(t) = -\frac{\lambda(t)}{t}$, we have

$$\frac{u'(t)}{u(t)} = \frac{\lambda'(t)}{\lambda(t)} + \frac{2(\gamma + (1 + \gamma)t)}{1 - t^2}$$

and therefore, $u(t)$ is decreasing on $(0, 1)$ if and only if

$$(2.1) \quad (1 - t^2)\lambda'(t) + 2(\gamma + (1 + \gamma)t) \lambda(t) \leq 0.$$
From now on, we define

\begin{align}
\varphi(1-t) &= 1 + \sum_{n=1}^{\infty} b_n(1-t)^n \quad (b_n \geq 0) \\
\lambda(t) &= C t^{b-1} (1-t)^{c-a-b} \varphi(1-t)
\end{align}

and

\begin{align}
\varphi(1-t) &= 0 \\
\varphi'(1-t) &= 0
\end{align}

where \( C \) is a normalized constant so that \( \int_0^1 \lambda(t) \, dt = 1. \) For \( f \in \mathcal{A}, \) Balasubramanian et al. [2] defined the operator \( P_{a,b,c} \) by

\[
P_{a,b,c}(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} \, dt,
\]

where \( \lambda(t) \) is given by (2.3).

Special choices of \( \varphi(1-t) \) led to various interesting geometric properties concerning certain well-known operators. Observe that \( \Lambda(t) = \Lambda(t) = 0 \) and (2.1) is equivalent to

\[
(c - a - 3 - 2\gamma)t^2 + (c - a - b - 2\gamma)t + 1 - b \geq -t(1 - t^2) \frac{\varphi'(1-t)}{\varphi(1-t)}
\]

and this inequality may be rewritten in a convenient form as

\begin{align}
D(t^2 + t) + (1-b)(1-t^2) + t(1-t) &\geq -t(1 - t^2) \frac{\varphi'(1-t)}{\varphi(1-t)} \\
\end{align}

where \( D = c - a - b - 1 - 2\gamma. \) In view of (2.2), \( \varphi(1-t) > 0 \) and \( \varphi'(1-t) \geq 0 \) on \( (0,1), \) so that the right hand side of the inequality (2.4) is nonpositive for all \( t \in (0,1). \) If we assume that \( 0 \leq \gamma \leq 1/2, \ a > 0, \ 0 < b \leq 1 \) and \( c \geq a + b + 2\gamma + 1, \) then the left hand side of the inequality (2.4) clearly is nonnegative for all \( t \in (0,1). \) Thus, the inequality (2.4) holds for all \( t \in (0,1). \) In conclusion, from Lemma 1.3, we have the following theorem and techniques as in the proofs of [5, Theorem 1] and [13, 15, 8] show that the value \( \beta \) in Theorem 2.5 is sharp.

2.5. **Theorem.** Let \( 0 \leq \gamma \leq 1/2, \ a > 0, \ 0 < b \leq 1 \) and \( c \geq a + b + 2\gamma + 1, \) and let \( \lambda(t) \) be given by (2.3). Define \( \beta = \beta(a,b,c,\gamma) \) by

\[
\beta - \frac{1}{2} = -\int_0^1 \lambda(t) \frac{1 - \gamma(1+t)}{(1-\gamma)(1+t)^2} \, dt.
\]

If \( f(z) \in \mathcal{P}(\beta), \) then \( P_{a,b,c}(f)(z) \in \mathcal{K}(\gamma). \) The value of \( \beta \) is sharp.

2.6. **Corollary.** Let \( 0 \leq \gamma \leq 1/2, \ 0 < a \leq 1, \ 0 < b \leq 1 \) and \( c \geq a + b + 2\gamma + 1. \) Suppose that \( \varphi(1-t) \) and \( \varphi(a) \) are defined by

\begin{align}
\varphi(1-t) &= F(c-a,1-a;c-a-b+1;1-t) \\
\end{align}

and

\[
\lambda(t) = C t^{b-1} (1-t)^{c-a-b} \varphi(1-t)
\]

where \( C \) is a normalized constant so that \( \int_0^1 \lambda(t) \, dt = 1. \) For \( f \in \mathcal{A}, \) Balasubramanian et al. [2] defined the operator \( P_{a,b,c} \) by

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where \( \lambda(t) \) is given by (2.3).

Special choices of \( \varphi(1-t) \) led to various interesting geometric properties concerning certain well-known operators. Observe that \( \Lambda(t) = \Lambda(t) = 0 \) and (2.1) is equivalent to

\[
(c - a - 3 - 2\gamma)t^2 + (c - a - b - 2\gamma)t + 1 - b \geq -t(1 - t^2) \frac{\varphi'(1-t)}{\varphi(1-t)}
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and this inequality may be rewritten in a convenient form as

\begin{align}
D(t^2 + t) + (1-b)(1-t^2) + t(1-t) &\geq -t(1 - t^2) \frac{\varphi'(1-t)}{\varphi(1-t)} \\
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where \( D = c - a - b - 1 - 2\gamma. \) In view of (2.2), \( \varphi(1-t) > 0 \) and \( \varphi'(1-t) \geq 0 \) on \( (0,1), \) so that the right hand side of the inequality (2.4) is nonpositive for all \( t \in (0,1). \) If we assume that \( 0 \leq \gamma \leq 1/2, \ a > 0, \ 0 < b \leq 1 \) and \( c \geq a + b + 2\gamma + 1, \) then the left hand side of the inequality (2.4) clearly is nonnegative for all \( t \in (0,1). \) Thus, the inequality (2.4) holds for all \( t \in (0,1). \) In conclusion, from Lemma 1.3, we have the following theorem and techniques as in the proofs of [5, Theorem 1] and [13, 15, 8] show that the value \( \beta \) in Theorem 2.5 is sharp.

2.5. **Theorem.** Let \( 0 \leq \gamma \leq 1/2, \ a > 0, \ 0 < b \leq 1 \) and \( c \geq a + b + 2\gamma + 1, \) and let \( \lambda(t) \) be given by (2.3). Define \( \beta = \beta(a,b,c,\gamma) \) by

\[
\beta - \frac{1}{2} = -\int_0^1 \lambda(t) \frac{1 - \gamma(1+t)}{(1-\gamma)(1+t)^2} \, dt.
\]

If \( f(z) \in \mathcal{P}(\beta), \) then \( P_{a,b,c}(f)(z) \in \mathcal{K}(\gamma). \) The value of \( \beta \) is sharp.

2.6. **Corollary.** Let \( 0 \leq \gamma \leq 1/2, \ 0 < a \leq 1, \ 0 < b \leq 1 \) and \( c \geq a + b + 2\gamma + 1. \) Suppose that \( \varphi(1-t) \) and \( \varphi(a) \) are defined by

\begin{align}
\varphi(1-t) &= F(c-a,1-a;c-a-b+1;1-t) \\
\end{align}
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and

\[ C = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}, \]

respectively. Define \( \beta = \beta(a,b,c,\gamma) \) by

\[ \frac{\beta - \frac{1}{2}}{1 - \beta} = -C \int_0^1 (1-t)^{c-a-b-1} \left( \frac{1 - \gamma(1+t)}{(1 - \gamma)(1+t)^2} \right) \varphi(1-t) \, dt. \]

If \( f(z) \in \mathcal{P}(\beta) \), then \( H_{a,b,c}(f)(z) \) defined by

\[ H_{a,b,c}(f)(z) := C \int_0^1 (1-t)^{c-a-b-2} \varphi(1-t)f(tz) \, dt. \]

belongs to \( \mathcal{K}(\gamma) \). The value of \( \beta \) is sharp.

**Proof.** The integral representation for \( H_{a,b,c}(f)(z) \) has been obtained in [2, 8]. By (2.7) and (2.8), it follows that the corresponding operator \( P_{a,b,c}(f)(z) \) equals \( H_{a,b,c}(f)(z) \). Note that the assumption implies that \( 0 < a \leq 1 \) and \( c - a > 0 \) and \( c - a - b + 1 > 0 \) from which the nonnegativity of \( \varphi(1-t) \) on \((0, 1)\) is clear. Now, the desired result follows from Theorem 2.5.

Setting \( a = 1 \) in Corollary 2.6, we obtain

2.10. **Corollary.** Let \( 0 \leq \gamma \leq 1/2 \), \( 0 < b \leq 1 \) and \( c \geq b + 2\gamma + 2 \). Also let

\[ \beta(1,b,c,\gamma) = 1 - \frac{1 - \gamma}{2[1 - F(2,b;c,-1) - \gamma(1 - F(1,b;c,1))]} \]

If \( \beta(1,b,c,\gamma) \leq \beta < 1 \) and \( f(z) \in \mathcal{P}(\beta) \), then \( L(b,c)f(z) \in \mathcal{K}(\gamma) \).

**Proof.** Putting \( a = 1 \) in (2.9) it follows that

\[ \frac{\beta - \frac{1}{2}}{1 - \beta} = -\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \frac{1 - \gamma(1+t)}{(1 - \gamma)(1+t)^2} \, dt = \frac{\Gamma(c)}{(1 - \gamma)(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \left\{ \frac{\gamma}{1+t} - \frac{1}{(1+t)^2} \right\} \, dt = \frac{1}{1 - \gamma} \left[ \gamma F(1,b;c,1) - F(2,b;c,1) \right] \]

where the last step follows from the Euler integral representation. Solving the last equation gives the number \( \beta(1,b,c,\gamma) \) given by (2.11). The desired conclusion follows from Corollary 2.6.

2.12. **Theorem.** Let \(-1 < a \leq 2, 0 \leq \gamma \leq 1/2 \) and \( p \geq 2(1 + \gamma) \). Suppose that \( \beta = \beta(a,p,\gamma) \) is given by

\[ \frac{\beta - \frac{1}{2}}{1 - \beta} = -\frac{(1+a)^p}{\Gamma(p)} \int_0^1 t^a(\log(1/t))^{p-1} \frac{1 - \gamma(1+t)}{(1 - \gamma)(1+t)^2} \, dt. \]
Then, for \( f \in \mathcal{P}(\beta) \), the Hadamard product function \( \Phi_p(a; z) \ast f(z) \) defined by

\[
\Phi_p(a; z) \ast f(z) = \left( \sum_{n=1}^{\infty} \frac{(1+a)^p}{(n+a)^p} z^n \right) \ast f(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 (\log 1/t)^{p-1} t^{a-1} f(tz) dt
\]

belongs to \( \mathcal{K}(\gamma) \). The value of \( \beta \) is sharp.

**Proof.** To obtain this theorem, we choose \( \phi(1-t) \) and \( \lambda(t) \) in Theorem 2.5 as

\[
\phi(1-t) = \left( \frac{\log(1/t)}{1-t} \right)^{p-1} = \left( \frac{-\log(1-(1-t))}{1-t} \right)^{p-1},
\]

and

\[
\lambda(t) = \frac{(1+a)^p}{\Gamma(p)} t^a (1-t)^{(p-1)} \phi(1-t),
\]

respectively. The desired conclusion follows from Theorem 2.5 and the hypotheses. \( \square \)

Our final application concerns the integral operator studied by Ponnusamy [12], Ponnusamy and Rønning [13] and later by Balasubramanian, Ponnusamy and Vuorinen [2]. Define

\[
\lambda(t) = \begin{cases} 
(a+1)(b+1) \left( \frac{t^a(1-t^{b-a})}{b-a} \right) & \text{for } b \neq a, a > -1, b > -1, \\
(a+1)^2 t^a \log(1/t) & \text{for } b = a, a > -1. 
\end{cases}
\]

With this \( \lambda(t) \), we have an integral transform

\[
G_f(a, b; z) := \left( \sum_{n=1}^{\infty} \frac{1+a}{n+a} \frac{1+b}{n+b} z^n \right) \ast f(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt.
\]

In view of symmetry between \( a \) and \( b \), without loss of generality, we assume that \( b > a \) in the case \( b \neq a \). Note that in the limiting case \( b \to \infty \) \( (b \neq a) \), \( G_f(a, b; z) \) reduces to a well-known Bernardi operator given by

\[
G_f(a, \infty; z) := \left( \sum_{n=1}^{\infty} \frac{1+a}{n+a} z^n \right) \ast f(z) = \frac{1+a}{z^a} \int_0^1 t^{a-1} f(t) dt \equiv \mathcal{L}(a+1, a+2) f(z).
\]

**2.14. Theorem.** Let \( b > -1, a > -1 \) be such that any one of the following conditions holds:

(i) \(-1 < a \leq 0 \) and \( a = b \)

(ii) \(-1 < a \leq 0 \) and \( b > a \) with \(-1 < b \leq 2 \).

Suppose that \( \lambda(t) \) is defined by (2.13) and \( \beta \) is given by

\[
\frac{\beta - \frac{1}{2}}{1 - \beta} = - \int_0^1 \frac{\lambda(t)}{(1+t)^2} dt.
\]
If \( f \in \mathcal{P}(\beta) \), then the function \( G_f(a, b; z) \) is convex in \( \Delta \). The value of \( \beta \) is sharp.

**Proof.** Clearly, as in the proof of Theorem 2.5, it suffices to verify the inequality (2.1) for the \( \lambda(t) \) defined by (2.13). Now, for the \( \lambda(t) \) given by (2.13), we have

\[
\lambda'(t) = \begin{cases} \frac{(a+1)(b+1)}{b-a}ta^{-1} (a - bt^{b-a}) & \text{for } b > a > -1, \\ (a+1)^2 (-1 + a \log(1/t)) t^{a-1} & \text{for } b = a > -1. \end{cases}
\]

**Case (i):** Let \( b = a > -1 \). If we substitute the \( \lambda(t) \) and the \( t\lambda'(t) \) expression in (2.1), the inequality (2.1) is seen to be equivalent to

(2.15) \[ -a \left(1 - t^2\right) \log(1/t) + 1 - t^2 - 2t^2 \log(1/t) \geq 0, \quad t \in (0, 1). \]

Clearly, as \(-1 < a \leq 0\), this inequality holds if it holds for \( a = 0 \). Substituting \( a = 0 \), this becomes

\[ 1 - t^2 - 2t^2 \log(1/t) \geq 0, \quad t \in (0, 1), \]

which, for \( t = e^{-x} \), is equivalent to

\[ e^{2x} \geq 1 + 2x, \quad x \geq 0. \]

Since this inequality holds for all \( x \geq 0 \), the inequality (2.15) holds for all \( t \in (0, 1) \) and the desired conclusion holds in this case.

**Case (ii):** Let \( b > a > -1 \). If we substitute the \( \lambda(t) \) given by (2.13) and the corresponding \( t\lambda'(t) \) expression in (2.1), the inequality (2.1) is seen to be equivalent to

(2.16) \[ (1 - t^2) (at^{a-1} - bt^{b-1}) + 2 (t^{a+1} - t^{b+1}) \leq 0 \]

which may be rewritten as

\[ \psi_t(a) - \psi_t(b) \leq 0, \quad t \in (0, 1), \]

where

\[ \psi_t(a) = a \left(1 - t^2\right) t^{a-1} + 2t^{a+1}. \]

For each fixed \( t \in (0, 1) \), we first claim that \( \psi_t(a) \) is an increasing function of \( a \). Differentiating \( \psi_t(a) \) with respect to \( a \), we find that

\[ \psi'_t(a) = t^{a-1} \left[1 - t^2 - 2t^2 \log(1/t) - a \left(1 - t^2\right) \log(1/t)\right]. \]

Using the previous case, namely the inequality (2.15), it follows that \( \psi'_t(a) \geq 0 \) for all \( a \in (-1, 0) \) and for \( t \in (0, 1) \). In particular, for \( b > a \) with \( b \in (-1, 0) \) and \( a \in (-1, 0) \), the inequality (2.16) holds.

When \( b > a \) with \( 0 \leq b \leq 2 \) and \( a \in (-1, 0] \), we have

\[ \psi_t(a) \leq \psi_t(0) = 2t \quad \text{for } t \in (0, 1). \]
Now, we claim that for $b > a$ with $0 \leq b \leq 2$ and $a \in (-1,0]$, the inequality
\[ 2t \leq \psi_t(b) = b \left(1 - t^2\right) t^{b-1} + 2t^{b+1} \]
holds for all $t \in (0,1)$. To verify this inequality, we rewrite it as
\[ 2 \left(t^{-b} - 1\right) \leq b \left(t^{-2} - 1\right) \quad \text{for } t \in (0,1) \]
which, for $t = 1 - x$, is equivalent to the inequality
\[ (2.17) \quad 2 \left((1 - x)^{-b} - 1\right) \leq b \left((1 - x)^{-2} - 1\right) \quad \text{for } x \in (0,1). \]
Since
\[ 2(b) = b(2) \quad \text{for all } n \geq 1, \]
a comparison of the coefficients of $x^n$ on both sides of the inequality (2.17) implies that (2.17) clearly holds. Thus, for $0 \leq b \leq 2$ and $a \in (-1,0]$ with $b > a$, we have
\[ \psi_t(a) \leq 2t \leq \psi_t(b) \quad \text{for } t \in (0,1) \]
and the proof is now complete. \[ \square \]

3. The Fractional Integral Operator

There are a number of definitions for fractional calculus operators in the literature. We use here the following definition due to Saigo [18] (see also [10, 19]).

For $\lambda > 0$, $\mu, \nu \in \mathbb{R}$, the fractional integral operator $I^{\lambda,\mu,\nu}$ is defined by
\[ I^{\lambda,\mu,\nu}f(z) = \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0' (z - \zeta)^{\lambda-1} F(\lambda + \mu, -\nu; \lambda; 1 - \frac{\zeta}{z}) f(\zeta) d\zeta, \]
where $f(z)$ is taken to be an analytic function in a simply-connected region of the $z$-plane containing the origin with the order
\[ f(z) = \mathcal{O}(|z|^\epsilon) \quad (z \to 0) \]
for $\epsilon > \max\{0, \mu - \nu\} - 1$, and the multivaluedness of $(z - \zeta)^{\lambda-1}$ is removed by requiring that $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

In [10], Owa et al. considered the normalized fractional integral operator by defining $J^{\lambda,\mu,\nu}$ by
\[ J^{\lambda,\mu,\nu}f(z) = \frac{\Gamma(2 - \mu)\Gamma(2 + \lambda + \nu)}{\Gamma(2 - \mu + \nu)} z^\mu I^{\lambda,\mu,\nu}f(z), \quad \min\{\lambda + \nu, -\mu + \nu, -\mu\} > -2. \]
Clearly, $J^{\lambda,\mu,\nu}$ maps $A$ onto itself and for $f \in A$
\[ (3.1) \quad J^{\lambda,\mu,\nu}f(z) = \mathcal{L}(2, 2 - \mu) \mathcal{L}(2 - \mu + \nu, 2 + \lambda + \nu) f(z). \]
Generalized Fractional Integral Operator

A function \( f(z) \in A \) is said to be in the class \( \mathcal{R}(\alpha, \gamma) \) if
\[
(f * s_{\alpha})(z) \in S^{*}(\gamma) \quad (0 \leq \alpha < 1; \ 0 \leq \gamma < 1).
\]
Here \( s_{\alpha}(z) = z/(1-z)^{2(1-\alpha)} \) (0 \leq \alpha < 1) denotes the well-known extremal function for the class \( S^{*}(\alpha) \). Note that
\[
\mathcal{R}(\alpha, \gamma) = \mathcal{L}(1, 2-2\alpha)S^{*}(\gamma)
\]
and \( \mathcal{R}(\alpha, \alpha) \equiv \mathcal{R}(\alpha) \) is the subclass of \( A \) consisting of prestarlike functions of order \( \alpha \) which was introduced by Suffridge [21]. In [20], it is shown that \( \mathcal{R}(\alpha) \subset S \) if and only if \( \alpha \leq 1/2 \).

Our result in this section is to obtain a univalence criterion for the operator \( J_{\lambda, \mu, \nu} \).

3.3. Theorem. Let \( 0 \leq \gamma \leq 1/2, \ 0 < \mu < 2, \ \lambda \geq 2(1+\gamma) - \mu \) and \( \mu - 2 < \nu \leq \mu - 1 \). Define \( \beta = \beta(\lambda, \mu, \nu, \gamma) \) by
\[
\beta = 1 - \frac{1 - \gamma}{2[1 - F(2, 2 - \mu + \nu; 2 + \lambda + \nu; -1) - \gamma(1 - F(1, 2 - \mu + \nu; 2 + \lambda + \nu; -1))]}.
\]
If \( f(z) \in \mathcal{P}(\beta) \), then \( J_{\lambda, \mu, \nu} f(z) \in \mathcal{R}(\mu/2, \gamma) \).

Proof. Making use of (1.1) and (3.1), we note that
\[
J_{\lambda, \mu, \nu} f(z) = \mathcal{L}(2, 2-\mu)\mathcal{L}(2-\mu+\nu, 2+\lambda+\nu)f(z) = \mathcal{L}(1, 2-\mu)\mathcal{L}(2, 1)\mathcal{L}(2-\mu+\nu, 2+\lambda+\nu)f(z).
\]
By using Corollary 2.10, we obtain
\[
\mathcal{L}(2 - \mu + \nu, 2 + \lambda + \nu)f(z) \in \mathcal{K}(\gamma).
\]
Since \( 0 \leq \mu < 2 \), from (1.2), (3.2) and (3.4), we have \( J_{\lambda, \mu, \nu} f(z) \in \mathcal{R}(\mu/2, \gamma) \) and we complete the proof. \( \square \)

Taking \( \mu = 2\gamma \) in Theorem 3.3, we get

3.5. Corollary. Let \( 0 \leq \gamma \leq 1/2, \ \lambda \geq 2 \) and \( 2(\gamma - 1) < \nu \leq 2\gamma - 1 \). Define \( \beta = \beta(\lambda, \nu, \gamma) \) by
\[
\beta = 1 - \frac{1 - \gamma}{2[1 - F(2, 2 - 2\gamma + \nu; 2 + \lambda + \nu; -1) - \gamma(1 - F(1, 2 - 2\gamma + \nu; 2 + \lambda + \nu; -1))]}.
\]
If \( f(z) \in \mathcal{P}(\beta) \), then \( J_{0, \lambda, \nu} f(z) \in \mathcal{R}(\gamma) \subset S \).

Proof. If we put \( \mu = 2\gamma \) in Theorem 3.3, then
\[ J_{0, \lambda, \nu} f(z) \in \mathcal{R}(\gamma, \gamma) = \mathcal{R}(\gamma). \]
Since \( \gamma \leq 1/2 \), we have \( \mathcal{R}(\gamma) \subset S \) and therefore, the proof is completed. \( \square \)
3.6. Remark. In [2], Balasubramanian et al. found the conditions on the number $\beta$ and the function $\lambda(t)$ such that $P_{a,b,c}(f)(z) \in S^{*}(\gamma)$ ($0 \leq \gamma \leq 1/2$). Since

$$J^{\lambda,\mu,\nu}f(z) = P_{1-\mu,2\lambda,-\nu+2}(f)(z)$$

with

$$\phi(1-t) = F(\lambda + \mu, -\nu; \lambda; 1-t)$$

and

$$C = \frac{\Gamma(2-\mu)\Gamma(2+\lambda+\nu)}{\Gamma(\lambda)\Gamma(2-\mu+\nu)},$$

it is easy to find that the condition on $\beta$ and $\lambda(t)$ such that $J^{\lambda,\mu,\nu}f(z) \in S^{*}(\gamma)$.

Finally, by using Lemma 1.3 again, we investigate convexity of the operator $J^{\lambda,\mu,\nu}$.

3.7. Theorem. Let $0 \leq \gamma \leq 1/2$, $0 < \lambda \leq 1 + 2\gamma$, $2 < \mu < 3$ and $\nu > \mu - 2$. Define $\beta = \beta(\lambda, \mu, \nu, \gamma)$ by

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = -\frac{\Gamma(2-\mu)\Gamma(2+\lambda+\nu)}{\Gamma(\lambda)\Gamma(2-\mu+\nu)} \int_0^1 t(1-t)^{\lambda-1}(1-\gamma(1+t))F(\lambda + \mu, -\nu; \lambda; 1-t) dt.$$

If $f(z) \in \mathcal{P}(\beta)$, then $J^{\lambda,\mu,\nu}f(z) \in \mathcal{K}(\gamma)$. The value of $\beta$ is sharp.

Proof. Let $0 \leq \gamma \leq 1/2$, $0 < \lambda \leq 1 + 2\gamma$, $2 < \mu < 3$, $\nu > \mu - 2$, and let

$$\lambda(t) = \frac{\Gamma(2-\mu)\Gamma(2+\lambda+\nu)}{\Gamma(\lambda)\Gamma(2-\mu+\nu)} t(1-t)^{\lambda-1}F(\lambda + \mu, -\nu; \lambda; 1-t).$$

Then we can easily see that $\int_0^1 \lambda(t) dt = 1$, $\Lambda(t) = \int_0^t \lambda(s) ds/s$ is monotone decreasing on $[0, 1]$ and $\lim_{t \to 0^+} t\Lambda(t) = 0$. Also we find that the function $u(t) = \lambda(t)/(1+t)(1-t)^{1+2\gamma}$ is decreasing on $(0, 1)$, where $\lambda(t)$ is given by (3.8). Hence, $t\Lambda'(t)/(1+t)(1-t)^{1+2\gamma} = -u(t)$ is increasing on $(0, 1)$. From Lemma 1.3, we obtain the desired result.

Acknowledgment: The authors thank Prof. R. Balasubramanian for his help in the proof of Theorem 2.14. The work of the second author was supported by grant No. R05-2001-000-00020-0 from the Basic Research Program of the Korea Science and Engineering Foundation, while the work of the third author was supported by a Sponsored Research project (Ref No. DST/MS/092/98) from the Department of Science and Technology (India).

References

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