Title

Solutions to a Nearly Simple Harmonic Vibration Equation by Means of N-Fractional Calculus (Study on Applications for Fractional Calculus Operators in Univalent Function Theory)

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Solutions to a Nearly Simple Harmonic Vibration
Equation by Means of N-Fractional Calculus

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Abstract

In this paper, solutions to a nearly simple harmonic vibration equation are discussed by means of N-fractional calculus, and some investigation of the solutions are reported.

Keywords: N-Fractional Calculus, Simple Harmonic Vibration Equation.

Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) [1 Vol. 1]

Let \( D = \{ D_-, D_+ \} \), \( C = \{ C_-, C_+ \} \),
\( C_- \) be a curve along the cut joining two points \( z \) and \( -\infty + i \text{Im}(z) \),
\( C_+ \) be a curve along the cut joining two points \( z \) and \( \infty + i \text{Im}(z) \),
\( D_+ \) be a domain surrounded by \( C_- \), \( D_+ \) be a domain surrounded by \( C_+ \).

(Here \( D \) contains the points over the curve \( C \)).

Moreover, let \( f = f(z) \) be a regular function in \( D(z \in D) \),
\[ f_\nu = (f)_\nu = \frac{\Gamma(v+1)}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{v+1}} d\xi \quad (\nu \in \mathbb{Z}^-), \]
\[ (f)_{-m} = \lim_{\nu \to -m} (f)_\nu \quad (m \in \mathbb{Z}^+), \]
where \(-\pi \leq \arg(\zeta - z) \leq \pi \) for \( C_- \), \( 0 \leq \arg(\xi - z) \leq 2\pi \) for \( C_+ \),
\( \zeta \neq z \), \( z \in C \), \( \nu \in \mathbb{R} \), \( \Gamma \); Gamma function,
then \( (f)_\nu \) is the fractional differintegration of arbitrary order \( \nu \) (derivatives of order \( \nu \) for \( \nu > 0 \), and integrals of order \( -\nu \) for \( \nu < 0 \),) with respect to \( z \) of the function \( f \), if \( |(f)_\nu| < \infty \).

(II) On the fractional calculus operator \( N^\nu \) [3]

Theorem A. Let fractional calculus operator (Nishimoto's Operator) \( N^\nu \) be
\[ N^\nu = \left( \frac{\Gamma(v+1)}{2\pi i} \int_C \frac{d\xi}{(\xi - z)^{v+1}} \right) (\nu \in \mathbb{Z}^-) \] [Refer to (1)],
\[ N^{-m} = \lim_{\nu \to -m} N^\nu \quad (m \in \mathbb{Z}^+), \]
and define the binary operation \( \circ \) as
\[ N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta(N^\alpha f) \quad (\alpha, \beta \in \mathbb{R}), \]
then the set
\[ \{ N^\nu \} = \{ N^\nu \mid \nu \in \mathbb{R} \} \]
is an Abelian product group (having continuous index \( \nu \)) which has the inverse transform operator \( (N^\nu)^{-1} = N^{-\nu} \) to the fractional calculus operator \( N^\nu \), for the function \( f \) such that
\[ f \in F = \{ f; 0 = |f| < \infty, \nu \in \mathbb{R} \} \text{, where } f = f(z) \text{ and } z \in C. \text{ (vis. } -\infty < \nu < \infty \}. \]
(For our convenience, we call $ N^\beta \circ N^\alpha $ as product of $ N^\beta $ and $ N^\alpha $.)

**Theorem B.** "F.O.G. \{N^\nu\} is an "Action product group which has continuous index $ \nu " for the set of $ F $ (F.O.G.; Fractional calculus operator group) [3].

§ 1. General solution to nearly simple harmonic vibration equations

**Theorem 1.** Let $ \varphi \in p^\alpha = \{ \varphi : \| \varphi \| < \infty, \nu \in \mathbb{R} \} $, then the homogeneous fractional order differintegral equation (nearly simple harmonic vibration equation for $ |\epsilon| << 1 $)

$$ \varphi_{2+\epsilon} + \varphi \cdot \omega^2 = 0 \quad (\omega \neq 0, \quad \varphi = \varphi(t), \quad t, \epsilon \in \mathbb{R}, \quad \epsilon \approx -2) \quad (0) $$

has the following general solutions.

(i) \[ \varphi = \sum_{n=0}^{m} a_n e^{A(n,\epsilon)\omega^{m(1+\epsilon)} t} + \cos B(n,\epsilon) \omega^{m(1+\epsilon)} t + i \sin B(n,\epsilon) \omega^{m(1+\epsilon)} t, \quad (1) \]
where

$ A(n,\epsilon) = \cos \beta(n,\epsilon) \quad (2), \quad B(n,\epsilon) = \sin \beta(n,\epsilon) \quad (3) \]

and

$$ \beta(n,\epsilon) = \pi(1+2n) / (2+\epsilon) \quad (4) $$

for $ \epsilon \in \mathbb{R} $.

(ii) \[ \varphi = \sum_{n=0}^{m} a_n e^{G(n,\epsilon)\omega^{r(\epsilon)} t} \times \cos H(n,\epsilon) \omega^{r(\epsilon)} t + i \sin H(n,\epsilon) \omega^{r(\epsilon)} t, \quad (5) \]
where

$ G(n,\epsilon) = \cos \pi(\frac{1}{2}+n) r(\epsilon) \quad (6), \quad H(n,\epsilon) = \sin \pi(\frac{1}{2}+n) r(\epsilon) \quad (7) \]

and

$$ r(\epsilon) = \sum_{k=0}^{\infty} (-\epsilon/2)^k \quad (8) $$

for $ |\epsilon| < 2 $.

(iii) \[ \varphi = \sum_{n=0}^{m} a_n e^{P(n,\epsilon)\omega^{l-(\epsilon/2)} t} \times \cos Q(n,\epsilon) \omega^{l-(\epsilon/2)} t + i \sin Q(n,\epsilon) \omega^{l-(\epsilon/2)} t, \quad (9) \]
where

$ P(n,\epsilon) = \cos \pi(\frac{1}{2}+n) \left(1-\frac{\epsilon}{2}\right), \quad (10) \]

$ Q(n,\epsilon) = \sin \pi(\frac{1}{2}+n) \left(1-\frac{\epsilon}{2}\right), \quad (11) \]

for $ |\epsilon| << 1 $.
Where \( a_n \) is an arbitrary constant correspond to \( \beta(n, \epsilon) \) and
\[
m \text{ is finite when } \epsilon \text{ is a rational number, and}
m \text{ is infinite when } \epsilon \text{ is an irrational number.}
\]

**Note 1.** We must call (9) as approximate (or almost) general solution to equation (0), because it is not general solution in the strict sense.

**Proof of (i)**

Set
\[
\varphi = e^{\lambda t},
\]
then operate \( N^{2+\epsilon} \) to the both sides of (12), we have then
\[
N^{2+\epsilon} \varphi = \varphi_{2+\epsilon} = \lambda^{2+\epsilon} e^{\lambda t}.
\]

Therefore, we have
\[
\lambda^{2+\epsilon} + \omega^2 = 0,
\]
from (13), (12) and (0).

Hence
\[
\lambda = (-\omega^2)^{1/(2+\epsilon)} = e^{i\pi(1+2n)/(2+\epsilon)}
\]
\[
= \gamma(n, \epsilon) \quad (n = 0, 1, 2, \ldots, m).
\]

Then letting
\[
\beta(n, \epsilon) = \pi(1+2n) / (2+\epsilon)
\]
we have
\[
\gamma(n, \epsilon) = e^{i\beta(n, \epsilon)} \omega^{2/(2+\epsilon)}
\]
\[
= \{ A(n, \epsilon) + iB(n, \epsilon) \} \omega^{2/(2+\epsilon)}
\]
where \( A(n, \epsilon) \) and \( B(n, \epsilon) \) are the ones shown by (2) and (3), respectively.

We have then a particular solution
\[
\varphi = e^{\gamma(n, \epsilon)} \approx e^{(A(n, \epsilon) + iB(n, \epsilon)) \omega^{2/(2+\epsilon)}}
\]
\[
= \{ A(n, \epsilon) + iB(n, \epsilon) \} \omega^{2/(2+\epsilon)}
\]
to equation (0).

Inversely (20) satisfies equation (0) clearly. Therefore, we have (1) from
\[
\varphi = \sum_{n=0}^{m} a_n \cdot \varphi_{|n|}
\]
as the general solution to equation (0), where \( a_n \) is an arbitrary constant correspond to \( \beta(n, \epsilon) \).

**Proof of (ii)**

For \(|\epsilon| < 2\), we have
\[
\frac{1}{2+\epsilon} = \frac{1}{2} r(\epsilon),
\]
where
\[
r(\epsilon) = \sum_{k=0}^{\infty} (-\epsilon/2)^k.
\]
We have then
\[ \beta(n, \epsilon) = \pi(\frac{1}{2} + n)r(\epsilon) \] (23)
from (22) and (4).
Therefore, we have (5) from (23) and (1).

**Proof of (111)**
For \( |\epsilon| << 1 \), we have
\[ r(\epsilon) = \sum_{k=0}^{\infty} (-\epsilon/2)^k = 1 - \frac{\epsilon}{2} \] (24)
then
\[ \beta(n, \epsilon) = \pi(\frac{1}{2} + n)(1 - \frac{\epsilon}{2}) \] (25)
Therefore, we have (9) from (25) and (5).

**§ 2. Investigation for \( \varphi|_{(\epsilon)} \)**

Here we investigate the solutions \( \varphi|_{(\epsilon)} \) of the case (111) in § 1.

**Theorem 2.** When \( \omega > 0 \),
\[ \varphi|_{(\epsilon)} = e^{-(\frac{1}{2})(2r+1)\pi/4} \left[ \cos Q + (-1)^r i \sin Q \right] \] (26)
is convergent for
\[ \begin{cases} 0 < -\epsilon << 1 & \text{for } r = 2k \\ 0 < \epsilon << 1 & \text{for } r = 2k + 1 \end{cases} \] (27)
where
\[ Q = \omega(1 - \frac{\epsilon}{2} \log \omega) \] (28)
\[ \varphi|_{(\epsilon)} = e^{iP(n, \epsilon)\omega^{-1}t} \left[ \cos Q(n, \epsilon)\omega^{-1}t + i \sin Q(n, \epsilon)\omega^{-1}t \right] \] (29)
and \( k = 0, 1, 2, \cdots \).

**Proof.** (1) Investigation for \( \varphi|_{(0)} \)

When \( |\epsilon| << 1 \), we have (29) from § 1. (20) having § 1. (10) and (11), since
\[ \varphi|_{(\epsilon)} = e^{iP(n, \epsilon)Q(n, \epsilon)\omega^{-1}t} \] (30)
In the case of \( n = 0 \), we have
\[ P(0, \epsilon)\omega^{-1}t = (\epsilon \frac{\pi}{4})\omega \] (30)
and
\[ Q(0, \epsilon)\omega^{-1}t = \omega(1 - \frac{\epsilon}{2} \log \omega) = Q \] (31)
from (10) and (11) respectively, because we have
\[
\cos \frac{\pi}{2} \left( 1 - \frac{\epsilon}{2} \right) = \sin \frac{\epsilon \pi}{4} = \sum_{k=0}^{\infty} (-1)^k \frac{(\epsilon \pi/4)^{2k+1}}{(2k+1)!} \quad (|\epsilon \pi/4| < \infty),
\]

\[
\sin \frac{\pi}{2} \left( 1 - \frac{\epsilon}{2} \right) = \cos \frac{\epsilon \pi}{4} = \sum_{k=0}^{\infty} (-1)^k \frac{(\epsilon \pi/4)^{2k}}{(2k)!} \quad (|\epsilon \pi/4| < \infty),
\]

and
\[
\omega ^{1-(\epsilon^2/2)} = \omega \cdot e^{\epsilon \log \omega^{-1/2}} \approx \omega \frac{(\epsilon \log \omega^{-1/2})^{k}}{k!} \infty \quad (|\epsilon \log \omega^{-1/2}| < \infty).
\]

Therefore, we have
\[
\varphi \big|_{(0)} \approx e^{(\epsilon \pi/4)\omega t} \left[ \cos Qt + i \sin Qt \right], \quad (|\epsilon| << 1).
\]

The solution (35) is divergent for \( \epsilon > 0 \) and is convergent (damping form) for \( \epsilon < 0 \) when \( \omega > 0 \). And we have
\[
Q = \omega \left( 1 - \frac{\epsilon}{2} \log \omega \right) > \omega \quad (\epsilon < 0, \omega > 1).
\]

Then, letting
\[
T_Q = 2\pi / Q ; \text{ period of the function } \cos Qt ,
\]
and
\[
T_\omega = 2\pi / \omega ; \text{ period of the function } \cos \omega t ,
\]
we have
\[
T_Q < T_\omega \quad (Q > \omega) \quad \text{when } \epsilon < 0.
\]

That is, we have that "the period \( T_Q \) of \( \cos Qt = \cos \omega(1 - \frac{\epsilon}{2} \log \omega)t \) is smaller than the one \( T_\omega \) of \( \cos \omega t \)" when \( \epsilon < 0, \omega > 1 \).

(II) Investigation for \( \varphi \big|_{(1)} \)

In the case of \( n = 1 \), we have
\[
P(1, \epsilon) \approx -(\epsilon \pi/4) \omega ,
\]
and
\[
Q(1, \epsilon) \approx -\omega (1 - \frac{\epsilon}{2} \log \omega) = -Q ,
\]
from (10) and (11) respectively, because we have
\[
\cos \frac{3\pi}{2} \left( 1 - \frac{\epsilon}{2} \right) = -\sin \frac{3\epsilon \pi}{4} = -\sum_{k=0}^{\infty} (-1)^k \frac{(\epsilon 3\pi/4)^{2k+1}}{(2k+1)!} \quad (|\epsilon 3\pi/4| < \infty),
\]

\[
\sin \frac{3\pi}{2} \left( 1 - \frac{\epsilon}{2} \right) = -\cos \frac{3\epsilon \pi}{4} = -\sum_{k=0}^{\infty} (-1)^k \frac{(\epsilon 3\pi/4)^{2k}}{(2k)!} \quad (|\epsilon 3\pi/4| < \infty),
\]
and (34).

Therefore, we have
\[
\varphi \big|_{(0)} \approx e^{-(\epsilon 3\pi/4)\omega t} \left[ \cos Qt - i \sin Qt \right], \quad (|\epsilon| << 1).
\]

The solution (41) is convergent (damping form) for \( \epsilon > 0 \) and is divergent for \( \epsilon < 0 \) when \( \omega > 0 \). And we have
\[
Q = \omega \left( 1 - \frac{\epsilon}{2} \log \omega \right) < \omega \quad (\epsilon > 0, \omega > 1).
\]

Then, in this case we have
\[
T_Q > T_\omega \quad (Q < \omega) \quad \text{when } \epsilon > 0, \omega > 1.
\]
That is, we have that "the period $T_0$ of $\cos Qt = \cos \omega(1 - \frac{1}{2} \log \omega)t$ is larger than the one $T_\omega$ of $\cos \omega t$" when $\varepsilon > 0$, $\omega > 1$.

(III) Investigation for $\varphi|_{(2)}$

In the case of $n=2$, we have

$$ P(2, \varepsilon) \approx (\varepsilon \pi/4) \omega $$

and

$$ Q(2, \varepsilon) \approx \omega(1^{\varepsilon} - 2\log\omega) = Q $$

from (10) and (11) respectively, because we have

$$ \cos \frac{5\pi}{2}(1-\frac{\varepsilon}{2}) \approx \sum_{k=0}^{\infty}(-1)^k \frac{(\varepsilon \pi/4)^{2k+1}}{(2k+1)!} $$

(1 $\epsilon 5\pi I < \infty$),

$$ \sin \frac{5\pi}{2}(1 - \varepsilon) = \cos \frac{\varepsilon \pi/4} = \sum_{k=0}^{\infty}(-1)^k \frac{(\varepsilon \pi/4)^{2k}}{(2k)!} $$

(49)

and (34).

Therefore, we have

$$ \varphi|_{(2)} \sim e^{(\varepsilon \pi/4)\omega t} [\cos Qt + i \sin Qt] $$

(47)

The solution (47) is divergent for $\varepsilon > 0$ and is convergent (damping form) for $\varepsilon < 0$, $\omega > 0$. And we have

$$ Q = \omega(1 - \frac{1}{2} \log \omega) > \omega \quad (\varepsilon < 0, \omega > 1). $$

Therefore we have

$$ T_0 < T_\omega \quad (Q > \omega) \quad \text{when} \quad \varepsilon < 0, \omega > 1. $$

That is, we have that "the period $T_0$ of $\cos Qt = \cos \omega(1 - \frac{1}{2} \log \omega)t$ is smaller than the one $T_\omega$ of $\cos \omega t$" when $\varepsilon < 0$, $\omega > 1$.

(IV) Repeating the same procedure as (I) ~ (III), we have this theorem clearly.

Note. Notice that when $\omega > 0$, the solutions

$\varphi|_{(2)}(k = 0, 1, 2, \cdots)$ are convergent (damping form) for $\varepsilon < 0$, and

$\varphi|_{(2k+1)}$ are convergent (damping form) for $\varepsilon > 0$, respectively.

Theorem 3. When $\omega > 0$ the nearly simple harmonic vibration equation

$$ \varphi_{2*} + \varphi \cdot \omega^2 = 0 \quad \left\{ \begin{array}{l} \omega \neq 0, \quad \varphi = \varphi(t), \\ t \in \mathbb{R}, \quad |\varepsilon| < 1 \end{array} \right. $$

(49)

has converging almost general solutions

$$ \varphi \approx \sum_{k=0}^{p} a_{2k} \varphi|_{(2k)} \quad \text{when} \quad \varepsilon < 0 $$

(50)

and

$$ \varphi \approx \sum_{k=0}^{p} a_{2k+1} \varphi|_{(2k+1)} \quad \text{when} \quad \varepsilon > 0 $$

(51)

where

$p$ is finite when $\varepsilon$ is a rational number, and

$p$ is infinite when $\varepsilon$ is an irrational number.

Proof. It is clear from the proof of Theorem 2, since we have (9) as solutions to equation (49).
§ 3. Some Graphs for $\phi|_{(\omega)}$

To equation $\phi_{2*\epsilon} + \phi \cdot \omega^2 = 0 \ (0 < \epsilon << 1)$ we have a convergent (damping form) solution
\[ \phi|_{(1)} = e^{-(\omega \pi/4)\omega t} [\cos Qt - i \sin Qt], \ (\omega > 0). \]
To this function we have $T_0 (> T_*)$ as its period for $\epsilon > 0$,
since $Q = \omega (1 - \frac{1}{2} \log \omega) < \omega \ (\epsilon > 0, \omega > 1)$.

To equation $\phi_{2*\epsilon} + \phi \cdot \omega^2 = 0 \ (\epsilon = 0)$ we have the solution $\phi = \cos \omega t + i \sin \omega t$. The period of which is $T_\omega = 2\pi / \omega$.

To equation $\phi_{2*\epsilon} + \phi \cdot \omega^2 = 0 \ (0 < -\epsilon << 1)$ we have a convergent (damping form) solution
\[ \phi|_{(1)} = e^{(\omega \pi/4)\omega t} [\cos Qt + i \sin Qt], \ (\omega > 0). \]
To this function we have $T_0 (< T_*)$ as its period for $\epsilon < 0$,
since $Q = \omega (1 - \frac{1}{2} \log \omega) > \omega \ (\epsilon < 0, \omega > 1)$.

Fig. 1. Case of $\omega > 1$. 
In Fig. 1., the graphs of $\text{Re } \varphi|_{(0)}$ and $\text{Re } \varphi|_{(1)}$, for the case $\omega > 1$, are shown in which the portion of amplitude and the one of vibration are separated. When $0 < \omega < 1$, we have $T_Q (\text{of } \text{Re } \varphi|_{(1)}) < T_w (\epsilon > 0)$ and $T_Q (\text{of } \text{Re } \varphi|_{(0)}) > T_w (\epsilon < 0)$. When $\omega = 1$, we have $T_Q (\text{of } \text{Re } \varphi|_{(1)}) = T_w (\epsilon > 0)$ and $T_Q (\text{of } \text{Re } \varphi|_{(0)}) = T_w (\epsilon < 0)$ since we have $Q = \omega (\epsilon > 0, \epsilon < 0)$. (See Fig. 2 and Fig. 3, respectively.)

(i) For example the nearly simple harmonic vibration equation

$$\frac{d^2\varphi}{dt^2} + \varphi \cdot \omega^2 = 0 \quad (\epsilon = 0.01 > 0) \quad (52)$$

has solution

$$\varphi|_{(0)} = e^{-0.01 \pi / 4} \left[ \cos Qt - i \sin Qt \right] \quad (\omega > 0), \quad (53)$$

whose amplitude is $e^{-0.01 \pi / 4}$ and the period is

$$T_Q = \frac{2\pi}{Q} = \frac{2\pi}{\omega (1 - 0.005 \cdot \log \omega)} > \frac{2\pi}{\omega} \approx T_w \quad (\omega > 1). \quad (54)$$

(ii) The equation

$$\frac{d^{1.99}\varphi}{dt^{1.99}} + \varphi \cdot \omega^2 = 0 \quad (\epsilon = -0.01 < 0) \quad (55)$$

has solution

$$\varphi|_{(0)} = e^{-0.01 \pi / 4} \left[ \cos Qt + i \sin Qt \right], \quad (\omega > 0), \quad (56)$$

whose amplitude is $e^{-0.01 \pi / 4}$ and the period is

$$T_Q = \frac{2\pi}{Q} = \frac{2\pi}{\omega (1 + 0.005 \cdot \log \omega)} < \frac{2\pi}{\omega} = T_w \quad (\omega > 1). \quad (57)$$

(iii) The equation

$$\varphi_2 + \varphi \cdot \omega^2 = 0 \quad (58)$$

has solution

$$\varphi = \cos \omega t + i \sin \omega t. \quad (59)$$

This solution is produced in the process in which $\epsilon$ changes its sign in the equation

$$\varphi_{2*\epsilon} + \varphi \cdot \omega^2 = 0. \quad (49)$$

Notice that; When $\omega > 0$,

$\text{Re } \varphi|_{(0)}$, having $n =$ even number, give the same form damping vibration curves as the one of $\text{Re } \varphi|_{(0)}$ for $\epsilon < 0$, and

$\text{Re } \varphi|_{(0)}$, having $n =$ odd number, give the same form damping vibration curves as the one of $\text{Re } \varphi|_{(0)}$ for $\epsilon > 0$.\
To equation $\varphi_{t+\epsilon} + \varphi \cdot \omega^2 = 0 \ (0 < \epsilon << 1)$ we have a convergent (damping form) solution

$\varphi \bigg|_{(1)} = e^{-(\epsilon \pi/4)\omega t} \cos Qt - i \sin Qt$, \ (\omega > 0).

To this function we have $T_0 (< T_\omega)$ as its period for $\epsilon > 0$, since $Q = \omega(1 - \frac{1}{2} \log \omega) > \omega$ \ (\epsilon > 0, \ 0 < \omega < 1).

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Fig. 2. Case of $0 < \omega < 1$. 

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To equation $\varphi_{t+\epsilon} + \varphi \cdot \omega^2 = 0 \ (\epsilon = 0)$ we have a solution

$\varphi = \cos \omega t + i \sin \omega t$. The period of which is $T_\omega = \frac{2\pi}{\omega}$.

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To equation $\varphi_{t+\epsilon} + \varphi \cdot \omega^2 = 0 \ (0 < -\epsilon << 1)$ we have a convergent (damping form) solution

$\varphi \bigg|_{(0)} = e^{(\epsilon \pi/4)\omega t} \cos Qt + i \sin Qt$, \ (\omega > 0).

To this function we have $T_0 (> T_\omega)$ as its period for $\epsilon < 0$, since $Q = \omega(1 + \frac{1}{2} \log \omega) < \omega$ \ (\epsilon < 0, \ 0 < \omega < 1).
To equation \( \varphi_{2\star t} + \varphi \cdot \omega^2 = 0 \) \((0 < \varepsilon < 1)\) we have a convergent (damping form) solution
\[
\varphi|_{(1)} = e^{-(3\pi/4)\omega t}[\cos Qt - i\sin Qt], \quad (\omega > 0).
\]
To this function we have \( T_\sigma = T_\omega \) as its period for \( \varepsilon > 0 \), since \( Q = \omega(1 - \frac{\varepsilon}{2} \log \omega) = \omega \) \((\varepsilon > 0, \ \omega = 1)\).

Fig. 3. Case of \( \omega = 1 \).
References


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