A Summary of "Combinatorial Belyi Cuspidalization and Arithmetic Subquotients of the Grothendieck-Teichmüller Group"

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This is a summary of the paper [7], entitled "Combinatorial Belyi Cuspidalization and Arithmetic Subquotients of the Grothendieck-Teichmüller Group".

First, let us fix notations. Let G be a profinite group; $H \subseteq G$ a closed subgroup. Then we shall denote by $C_G(H)$ the *commensurator* of $H \subseteq G$, i.e.,

$$C_G(H) \stackrel{\text{def}}{=} \{g \in G \mid H \cap g \cdot H \cdot g^{-1} \text{ is of finite index in } H \text{ and } g \cdot H \cdot g^{-1}\}.$$

For a connected Noetherian scheme S, we shall write Π_S for the étale fundamental group of S, relative to a suitable choice of basepoint. Write $X \stackrel{\text{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}} \backslash \{0, 1, \infty\}$, i.e., the projective line over the field of algebraic numbers $\overline{\mathbb{Q}}$ [in the field of complex numbers], minus the three points "0", "1", "∞"; $G_{\mathbb{Q}} \stackrel{\text{def}}{=} \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Let p be a prime number. In [5], §3 [cf. [5], Corollary 3.7], S. Mochizuki developed the theory of Belyi cuspidalization and applied this theory to reconstruct the decomposition groups of the closed points of a hyperbolic orbicurve of strictly Belyi type over a p-adic local field, i.e., a finite extension of the field of p-adic numbers $\mathbb{Q}_p$ [cf. [5], Definition 3.5; [5], Remark 3.7.2]. The technique of cuspidalization has many applications to anabelian geometry, and the theory of Belyi cuspidalization is one of them.

In [7], we develop a certain combinatorial version [which is closely related to the Grothendieck-Teichmüller group GT] of the theory of Belyi cuspidalization developed in [5], §3. We shall refer to this as the theory of *combinatorial Belyi cuspidalization*. Let us recall that GT may be regarded as a closed subgroup of the outer automorphism group of Π_X [cf. [4], Definition 1.11, (i); [4], Remark 1.11.1], and that the natural outer action of $G_{\mathbb{Q}}$ on Π_X determines natural inclusions

$$G_{\mathbb{Q}} \ \subseteq \ \text{GT} \ \subseteq \ \text{Out}(\Pi_X).$$

In this context, we recall the following famous open question:

Question : Is the inclusion $G_{\mathbb{Q}} \ \subseteq \ \text{GT}$ bijective?

At the time of writing of the paper [7], the author does not know

whether or not the inclusion $G_{\mathbb{Q}} \ \subseteq \ \text{GT}$ *is a bijection.*

However, in [7], as an application of the theory of *combinatorial Belyi cuspidalization*, we obtain a partial solution to a certain p-adic tempered analogue of the above Question [cf. Theorem A below]. Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$. Then we shall write

- $\mathbb{C}_p$ for the p-adic completion of $\overline{\mathbb{Q}}_p$;
- $X_{\mathbb{C}_p} \stackrel{\text{def}}{=} \mathbb{P}^1_{\mathbb{C}_p} \setminus \{0, 1, \infty\}$;
- $\Pi^{\text{tp}}_{X_{\mathbb{C}_p}}$ for the tempered fundamental group of $X_{\mathbb{C}_p}$, relative to a suitable choice of basepoint [cf. [1], §4, for the definition and basic properties of tempered fundamental groups].

Let us recall that [for suitable choices of basepoints] $\Pi_{X_{\mathbb{C}_p}}$ [hence also Π_X] may be regarded as the profinite completion of $\Pi^{\text{tp}}_{X_{\mathbb{C}_p}}$, and $\Pi^{\text{tp}}_{X_{\mathbb{C}_p}}$ may be regarded as a subgroup of Π_X [cf. [1], §4.5]. Then the operation of passing to the profinite completion induces a natural homomorphism

$$\text{Out}(\Pi^{\text{tp}}_{X_{\mathbb{C}_p}}) \to \text{Out}(\Pi_X).$$

It follows immediately from the fact that the normalizer of $\Pi^{\text{tp}}_{X_{\mathbb{C}_p}}$ in Π_X is $\Pi^{\text{tp}}_{X_{\mathbb{C}_p}}$ [cf. [1], Corollary 6.2.2; [3], Lemma 6.1, (ii)], that this natural homomorphism is *injective*. Thus, we shall use this natural injection to regard $\text{Out}(\Pi^{\text{tp}}_{X_{\mathbb{C}_p}})$ as a subgroup of $\text{Out}(\Pi_X)$. Write

$$\text{GT}^{\text{tp}}_p \stackrel{\text{def}}{=} \text{GT} \cap \text{Out}(\Pi^{\text{tp}}_{X_{\mathbb{C}_p}}) \subseteq \text{Out}(\Pi_X).$$

Then the following holds [cf. [7], Corollary B]:

Theorem A (Natural surjection from GT^{tp}_p to $G_{\mathbb{Q}_p}$). *One may construct a surjection*

$$\text{GT}^{\text{tp}}_p \twoheadrightarrow G_{\mathbb{Q}_p} \stackrel{\text{def}}{=} \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$$

whose restriction to $G_{\mathbb{Q}_p}$ is the identity automorphism.

The above theorem is proved by applying the theory of *combinatorial Belyi cuspidalization*, together with the following theorem [cf. [7], Theorem C], which is an application of E. Lepage's theory of resolution of non-singularities [cf. [2], Theorem 2.7]:

Theorem B (Determination of moduli of certain types of p-adic hyperbolic curves by data arising from geometric tempered fundamental groups). *Let $Y \to X_{\mathbb{C}_p}$ be a connected finite étale covering of $X_{\mathbb{C}_p}$; y, y' elements of $Y(\mathbb{C}_p)$. Write Y_y (respectively, $Y_{y'}$) for $Y \setminus \{y\}$ (respectively, $Y \setminus \{y'\}$); Π^{tp}_Y (respectively, $\Pi^{\text{tp}}_{Y_y}$, $\Pi^{\text{tp}}_{Y_{y'}}$) for the tempered fundamental group of Y (respectively, Y_y, $Y_{y'}$). Suppose that there exists an outer isomorphism of topological* *groups* $\Pi^{\mathrm{tp}}_{Y_y} \xrightarrow{\sim} \Pi^{\mathrm{tp}}_{Y_{y'}}$ *that fits into a commutative diagram of outer homomorphisms of topological groups*

$$\begin{array}{ccc} \Pi^{\mathrm{tp}}_{Y_y} & \xrightarrow{\sim} & \Pi^{\mathrm{tp}}_{Y_{y'}} \\ \downarrow & & \downarrow \\ \Pi^{\mathrm{tp}}_{Y} & = & \Pi^{\mathrm{tp}}_{Y}, \end{array}$$

where the vertical arrows are the outer surjections induced by the natural open immersions of hyperbolic curves. Then $y = y'$.

Finally, in [7], we consider yet another interesting class of closed subgroups of GT which act naturally on the field of algebraic numbers $\overline{\mathbb{Q}}$. For any field F and positive integer n, we shall write

$$F^{\times} \stackrel{\mathrm{def}}{=} F \setminus \{0\}, \quad \mu_n(F) \stackrel{\mathrm{def}}{=} \{x \in F^{\times} \mid x^n = 1\},$$
$$\mu(F) \stackrel{\mathrm{def}}{=} \bigcup_{m \geq 1} \mu_m(F), \quad F^{\times\infty} \stackrel{\mathrm{def}}{=} \bigcap_{m \geq 1} (F^{\times})^m.$$

In the remainder of the present manuscript, let K be a field. Then we shall say that the field K is *stably* $\times\mu$*-indivisible* if, for every finite extension L of K, $L^{\times\infty} \subseteq \mu(L)$. In fact, such fields exist in great abundance [cf. [7], Lemma D]. For instance, any *subfield* of an *abelian* or *pro-prime-to-p Galois* extension of a *finite extension* of $\mathbb{Q}$ or $\mathbb{Q}_p$ is *stably* $\times\mu$*-indivisible*.

Suppose that K is a *stably* $\times\mu$*-indivisible* field of characteristic 0. Let $\overline{K}$ be an algebraic closure of K. Write $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/K)$. Then we apply the theory of combinatorial Belyi cuspidalization to obtain the following [cf. [7], Corollary E]:

Theorem C (Natural homomorphism from the commensurator in GT of the absolute Galois group of a stably $\times\mu$-indivisible field to $G_{\mathbb{Q}}$). *Fix an embedding* $\overline{\mathbb{Q}} \hookrightarrow \overline{K}$. *In the following, we shall use this embedding to regard* $\overline{\mathbb{Q}}$ *as a subfield of* $\overline{K}$. *Thus, we obtain a homomorphism* $G_K \to G_{\mathbb{Q}}$ ($\subseteq$ GT) *[cf. the discussion at the beginning of the Introduction]. Suppose that this homomorphism* $G_K \to G_{\mathbb{Q}}$ *is injective. In the following, we shall use this injection* $G_K \hookrightarrow G_{\mathbb{Q}}$ *to regard* G_K *as a subgroup of* $G_{\mathbb{Q}}$*, hence also as a subgroup of* GT. *Then one may construct a natural surjection*

$$C_{\mathrm{GT}}(G_K) \twoheadrightarrow C_{G_{\mathbb{Q}}}(G_K) \ (\subseteq G_{\mathbb{Q}}).$$

whose restriction to $C_{G_{\mathbb{Q}}}(G_K)$ *is the identity automorphism.*

The above theorem is proved by applying the theory of *combinatorial Belyi cuspidalization*, together with the injectivity portion of the section conjecture for hyperbolic curves of genus 0 over a *stably* $\times\mu$*-indivisible* field of characteristic 0. This injectivity is a consequence of a certain weak version of the Grothendieck conjecture for hyperbolic curves of genus 0 over a stably $\times\mu$-indivisible field of characteristic 0 [cf. [7], Theorem F], which generalizes a result of J. Stix [cf. [6], Theorem A].

References

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