Gravity and Higher Spin States in the IIB Matrix Model and the Effective Field Theory

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Abstract

To construct quantum gravity theory is one of the most critical issues in particle physics. While string theory can describe the graviton, its perturbative formulation has difficulty in treating the quantum aspects of spacetime. This problem is believed to be solved by constructing the non-perturbative formulation of string theory, and the IIB matrix model is one of the most promising candidates. There have been a lot of works pointing out that the IIB matrix model contains the spacetime and gravity. On the other hand, the physical meaning of the degrees of freedom in it is not fully understood.

In many works, matrices are interpreted as noncommutative coordinates. There it has been reported that the matrix model contains the graviton with this interpretation despite the lack of general covariance. On the other hand, when one regards matrices as derivative operators on curved spacetimes, gravitational fields can be described more directly. This is called the operator interpretation. It is in question, however, whether the model with it is positive-definite, since the model contains infinitely many fields that are massless at tree-level, and the presence of higher spin gauge symmetries is not trivial. The structure of its effective field theory remains to be studied as well.

There are many aspects left to be discussed. In this thesis, we study the IIB matrix model with the focus on the gravitational and higher spin fields. After reviewing the IIB matrix model, we discuss how the gravitational force should be described by the model in the noncommutative treatment, which is somewhat unnatural. Next, we move to the operator interpretation. We see that the naive reduction of the massless fields leads to inconsistency. However, we show that the model possesses higher spin symmetries for them, and that mass terms for them are induced when the supersymmetries are broken. These two results are partial evidence for the positive-definiteness of the matrix model.

We also study possible structures of the effective field theory for the IIB matrix model, by pursuing the consistent structures within the framework of field theory. It has been known that the effective action become a function of ordinary actions, which gives a theoretical origin for degenerate vacua. Since the direct analysis is quite difficult, we instead investigate how inflation can be realize with that action, and obtain the allowed form of inflaton potential. As another study, we attempt to construct the Lagrangian for a massive higher spin field in curved background, which is expected to emerge from the matrix model. There we show that a field whose spin is higher than 2 cannot be described by a consistent Lagrangian. It is likely that infinitely many higher spin fields are needed to be introduced. This is consistent with the analysis of the IIB matrix model.

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Chapter 1

Introduction

Nowadays, particle physics at the low-energy region we can access is well-described by the Standard Model (SM). It appears that, since the discovery of of the Higgs particles, we have not encountered the essential discrepancy between experimental observable values and the predictions by the model. Although some anomalies are being reported in several measurements such as muon g - 2 factor, they tend to be resolved as the experimental data is accumulated. Therefore it is natural to think that we need no drastic change in SM to describe the low-energy particle physics.

On the other hand, it is certain as well that SM or its slight extension is not a UV-complete theory. It has some problem to be solved involving the gravity.

One of the most important and difficult issues of field theory is to construct a theory of quantum gravity. It is a notorious problem that the general relativity is incompatible with the quantum field theory, in a sense that the action of graviton, which is obtained by expanding the Einstein-Hilbert action around a background, is non-renormalizable.

Confronting this fact, there have been two philosophy researchers take. First, one can assume that there is some UV completion over the quantum field theory and the theory of classical gravity. It is supposed to consist in Planck scale. In this direction, string theory has been considered to be such a UV completion and intensively discussed for a couple of decades. As a second option, one can suspect that the theory of classical gravity can be actually lifted to a quantum theory when we treat quantum fluctuation non-perturbatively. In such a standpoint, no theory beyond quantum field theory is needed to describe quantum gravity. While a lot of progress has been made in the both directions, the question which is the correct way is not settled as yet.

We take the former philosophy, and assume that quantum gravity should be described by string theory. So far we have much (but not sufficient) knowledge and understanding on string theory at the perturbative level. It is certain that string theory includes the massless spintwo mode that is identified to the graviton on a background target spacetime. On the other

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hand, its non-perturbative behavior is relatively less understood. It is critical to establish a non-perturbative formulation of string theory for describing quantum gravity.

We would like to stress that it is not sufficient just to take into account some non-perturbative objects such as D-branes. String-phenomenological models, intersecting D-branes for example, do not provide the mechanism for them to form. It is closely related to the spacetime geometry since the branes have basically large energy density. In order to evaluate the exact consistency of those models, we need a formulation of string theory that treats the dynamics of spacetime as well. Moreover, since "gravity" means the nontrivial structure of spacetime, a theory of quantum gravity must treat not only the quantum graviton on some classical background spacetime, but the quantum nature of spacetime itself. If string theory describes that, it goes outside a perturbative formulation. The above observation forces us to investigate a non-perturbative formulation of string theory.

It is hardly possible to deduce the appropriate form of the formulation directly from some least principles. Instead, we can take a more realistic approach, where we study and analyze the candidates for it.

In this thesis, we focus on one of the the most promising candidates, namely the IIB matrix model [1,2]. The dynamical valuables in this model are several matrices, and the action is written only with the trace of their products. The most important feature of the matrix model is that it does not require the existence of the spacetime at the starting point. Spacetimes emerge as a result of the dynamics of the matrices. Nevertheless, it contains rich physics, a part of which reproduces results we have in string theory and field theory. First of all, its partition function coincides with that of the ordinary Yang-Mills theory at large-N limit. On the other hand, it yields the action of type IIB string in a double-scaling limit. Moreover, there are diverse works reporting many interesting aspects of it. It reproduces the Hamiltonian of light-cone string field theory [3], or it induces (3+1) expanding universe [4], or it realizes some special non-commutative spacetimes [5], and so on. Many of those studies suggest that the IIB matrix model describes gravity and spacetimes at the quantum level, going beyond perturbative string theory.

However, we still have a fundamental question to be settled. What should we interpret the physical meanings of matrices as? Up to today, there is no consensus about the answer for that question. Some works treat the expectation values of them as the coordinates in the flat spacetime [6–17]. Other works treat them as momenta or its generalization [18–22]. Further other works treat them as noncommutative coordinates [23–26,43]. The relation among those interpretations, and whether they are related to each other in the first place, are unclear. Accordingly, it is still in question how the gravitational degrees of freedom (DoF) are actually contained in the IIB matrix model.

Among several interpretations, the most sophisticated one is called the operator interpre-

tation [27]. With this interpretation, we regard matrices as derivative operators on curved spacetimes. There the matrix model can describe the curved spacetime emergent from the dynamics of matrices, and possesses manifest diffeomorphism (diffeo) invariance. This fact implies that the matrix model directly contains gravitational DoF with this interpretation. While there has been several works on its DoF and equations of motion (EoM) for them [28–33], the positivity and symmetries of the model largely remains to be studied. In particular, the matrix model contains infinitely many fields which are massless at the classical level. While most of them seem to be higher spin fields,¹ the existence of their higher spin gauge symmetries is not trivial. The structure of radiative corrections has to be discussed as well. We see them as an interesting problem to investigate.

On the other hand, we should pay attention in the study of the matrix model to how it connects to the theory describing our current universe. In principle, the appropriate Planck scale theory should reproduce SM as the low-energy effective field theory. The connection between high- and low-energy physics is a challenging problem, which is related to the construction of a theory of quantum gravity.

Therefore, in this thesis, we will study both the IIB matrix model and aspects of field theory that can be related to the matrix model as its UV completion.

In the study on the matrix model, before we treat the operator interpretation, we first treat it with one of the popular interpretation, where matrices represent coordinates for noncommutative geometry. There we investigate how we can describe the gravitational force, and point out the necessity of taking average of the noncommutativity in order to reproduce ordinary gravitational interactions. Although it might be possible to treat the gravity in the matrix model with this interpretation, we see that the formulation is rather awkward.

Then, as one of the main parts of this thesis, we study the matrix model with the operator interpretation, with an expectation that it is a model of Planck scale to describe quantum gravity. Concretely, we investigate the topics mentioned above. We will see the minimality of the original operator interpretation, existence of higher spin gauge symmetries, and generation of mass terms by the radiative correction. The latter two results are partial evidence of positivity of the matrix model.

As for the study on field theory, we analyze two problems. One is on a possible mechanism of inflation. The IIB matrix model suggests that its low-energy effective field theory [31, 32] have naturally-tuned coupling constants [33], forcing the Higgs potential to have degenerate minima. It is related to the principle called the multiple point criticality principle [34]. There is works [35, 36] which claim that this mechanism and Higgs inflation [37] can be compatible with an introduction of a novel conformal factor and the frame transformation. We generalize this statement and determine the form of the inflaton (or Higgs) potential allowed by Planck

¹we refer to spin larger than 2 as higher spin.

scale physics, the matrix model for example.

Another field-theoretic issue we study is to construct a formulation of massive higher spin fields (MHSF). In the analysis of the matrix model, the higher spin fields acquire their mass terms when the supersymmetries get broken. Since there is no signal of supersymmetry at low-energy scale (concretely TeV scale), it is natural to expect that the effective field theory of the matrix model contains MHSF in the curved spacetime, although it might be too heavy to be detected. It is worthwhile to describe them directly in the framework of field theory. In the flat spacetime, the formulation to describing MHSF is well-known [38,39]. On the other hand, the analysis for those in curved backgrounds is not enough as yet. Therefore, we attempt to construct a consistent Lagrangian for MHSF in the curved spacetimes.

In addition to the two studies, we have one more investigation on a field-theoretic problem: regularization of chiral gauge theory. Although it is rather independent topic in this thesis and thus we will put it in the appendix, it is indirectly related to the IIB matrix model. There are several reports [40,41] that the IIB matrix model can yield the chiral gauge theory which is defined on background branes as a classical solution. It should be connected to the UV behavior of chiral gauge theory. Therefore, in an attempt to grasp that connection, we propose a possible gauge-invariant regularization of chiral gauge theory by taking advantage of a lattice formulation called the domain-wall fermion [42], which can be seen as a fermion on a brane. This work is in progress.

This thesis is organized as follows. In Chapter 2 we introduce the IIB matrix model, focusing on how it reproduces results of theories with a spacetime. In Chapter 3, we investigate the gravitational interaction in the matrix model with the noncommutative interpretation, based on [43]. As a more promising treatment, in Chapter 4, we explain the operator interpretation of the matrix model, which is the main approach of our study on the model. Subsequently, we present one of the main analysis of the thesis in Chapter 5, which is on the stability of the matrix model with the operator interpretation. This chapter 5, which is on a model of inflation and on a construction of a theory of MHSF in the curved background, respectively. These chapters are based on our papers [46] and [47], respectively. Finally, we summarize and conclude the thesis in Chapter 8. As an appendix, we also report an ongoing study on a regularization of chiral gauge theory in Appendix A. This is rather an independent part of the thesis, based on an ongoing work [48].

Chapter 2

A Review of the IIB Matrix Model

In this chapter, we have a brief review of the IIB matrix model with the focus on how the spacetime emerges from it. First, we will show the notion of *large-N reduction*, by which a field theory gets equivalent to the corresponding matrix model.¹ Then, we will introduce the IIB matrix model as a special case. Various aspects of it are discussed, including the reproduction of type-IIB string theory and emergence of spacetimes.

2.1 Large-N reduction of a field theory

As a simple example, consider a Euclidean self-interacting scalar field theory described with the following action:

$$S = \frac{1}{g_0^2 \Lambda_0^{6-d}} \int d^d x \operatorname{Tr} \left(\frac{1}{2} (\partial_\mu \phi(x))^2 + \frac{m^2}{2} \phi(x)^2 + \frac{1}{3!} \phi(x)^3 \right).$$
(2.1)

Here $\phi(x)$ represents a scalar field which takes the value of a $N \times N$ hermitian matrix. We can regard it as a field in the adjoint representation of U(N), although it is unimportant what DoF the matrix denotes. Λ_0 is the energy scale specific to the theory. g_0 is the dimensionless coupling constant. To calculate the physical quantity, we need to extract its Feynman rules:

$$\langle \phi(k)_{ij}\phi(-k)_{kl} \rangle = g_0^2 \Lambda_0^{6-d} \frac{\delta_{il}\delta_{jk}}{k^2 - m^2},$$
 (2.2)

¹In this thesis, we use the term "matrix model" to mean the model which is described by matrices without the space nor time.

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Figure 2.1: The propagator and vertex in the double line notation. The arrows lie in the direction from the row-index toward colomn-index. Indices connected by a line are contracted. In the corresponding matrix model (Eq.(2.7)), the indices denotes the frow of partial momenta $p_{\mu}^{(i)}$.



The propagator and vertex can be represented in the double-line notation as Fig.(2.1). By using them we can calculate amplitudes, green functions or the free energy. The diagrams are classified into two types: the planer and non-planer diagrams. Examples of them among the vacuum diagrams are the ones in Fig.(2.2), B_1 and B_2 . Their loop integrations are given by

$$B_1 = V \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} g_0^2 N^3 \Lambda_0^{6-d} \frac{1}{k_1^2 + m^2} \frac{1}{k_2^2 + m^2} \frac{1}{(k_1 + k_2)^2 + m^2},$$
 (2.4)

$$B_2 = V \int \frac{d^a k_1}{(2\pi)^d} \frac{d^a k_2}{(2\pi)^d} g_0^2 N \Lambda_0^{6-d} \frac{1}{k_1^2 + m^2} \frac{1}{k_2^2 + m^2} \frac{1}{(k_1 + k_2)^2 + m^2},$$
(2.5)

with V denoting the volume of the spacetime. Note that the non-planar diagram is suppressed compared to the planar one by $1/N^2$. This is because that it consists of one index line while the planar one has three index lines. Since each line represents the contraction of the indices, a closed line gives the factor of N. Then we can take the following limit:

 $N \to \infty, \quad g_0 \to 0 \quad \text{with} \quad \lambda := g_0^2 N \quad \text{fixed.}$ (2.6)

 λ is called the 't Hooft coupling constant. This is a large-N limit.² In gengeral, vacuum planar diagrams with *n*-loops (in the viewpoint of field theory) are $O(N^2 \lambda^{n-1})$, while non-planar ones

²There are various variations of the limit where N goes to infinity. They are referred to as large-N limits as well, according to the context.



Figure 2.2: Examples of vacuum diagrams. (B1) A planar diagram. This gives a leading contribution in the large-N limit. (B2) A non-planar diagram. This is suppressed by the factor $1/N^2$ compared to the planar, hence it drops in the limit.

are $O(N^k \lambda^{n-1})$ with $k \leq 0$. Thus, in the large-N limit, only planar diagrams do contribute to physical quantities.

Next, we show that the planar diagrams in the above theory can be reproduced from a matrix model, the action of which is given by

$$S = \frac{1}{g_0^2 \Lambda_0^6} \operatorname{Tr} \left(-\frac{1}{2} [P_\mu, \Phi] [P^\mu, \Phi] + \frac{m^2}{2} \Phi^2 + \frac{1}{3!} \Phi^3 \right).$$
(2.7)

Here, Φ is a $N \times N$ hermitian matrices and P_{μ} is a hermitian matrices which serves like an external field. We assume that its eigenvalues distribute in \mathbb{R}^d uniformly. Then we can diagonalize it by some unitary transformation $\Phi \to U\Phi U^{-1}$:

$$P_{\mu} = \text{diag}(\cdots, p_{\mu}^{(i)}, p_{\mu}^{(i+1)}, p_{\mu}^{(i+2)}, \cdots).$$
 (2.8)

Let us calculate the same diagrams as the above ones in this matrix model. With P_{μ} diagonalized, Eq.(2.7) is written as

$$S = \frac{1}{g_0^2 \Lambda_0^6} \operatorname{Tr}\left(\frac{1}{2} \sum_{i,j} \left((p_\mu^{(i)} - p_\mu^{(j)})^2 + m^2 \right) |\Phi_{ij}|^2 + \frac{1}{3!} \sum_{i,j,k} \Phi_{ij} \Phi_{kl} \Phi_{li} \right).$$
(2.9)

The propagator and vertex is given by

$$\langle \Phi_{ij} \Phi_{kl} \rangle = g_0^2 \Lambda_0^6 \frac{\delta_{jk} \delta_{li}}{(p^{(i)} - p^{(j)})^2 + m^2},$$

(3-pt. vertex) = $\frac{-1}{g_0^2 \Lambda_0^6} \delta_{jk} \delta_{lm} \delta_{ni}.$ (2.10)

By comparing the propagator with Eq.(2.2), we can regard that a matrix element Φ_{ij} has the momentum $p_{\mu}^{(i)} - p_{\mu}^{(j)}$. Index lines transfer the partial momenta $p_{\mu}^{(i)}$. We represent as \tilde{B}_1 and

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 B_2 the diagrams corresponding to B_1 and B_2 . They are calculated as:

$$\tilde{B}_1 = \sum_{i,j,k} g_0^2 \Lambda_0^6 \frac{1}{(p^{(i)} - p^{(j)})^2 + m^2} \frac{1}{(p^{(j)} - p^{(k)})^2 + m^2} \frac{1}{(p^{(k)} - p^{(i)})^2 + m^2}$$
(2.11)

$$\tilde{B}_2 = \sum_i g_0^2 \Lambda_0^6 \frac{1}{m^2} \frac{1}{m^2} \frac{1}{m^2}.$$
(2.12)

Note that \tilde{B}_2 contains no momenta because the diagram (the same as B_2) consists of one line, hence every propagator has the vanishing momentum $p_{\mu}^{(i)} - p_{\mu}^{(i)} = 0$. Although \tilde{B}_2 takes a different form from B_2 , they drop in the both theory, in the large-N limit.

As for \tilde{B}_1 , we make a transformation of the variables:

$$l_{\mu}^{(i)} \equiv p_{\mu}^{(i)} - p_{\mu}^{(j)}, \quad l_{\mu}^{(j)} \equiv p_{\mu}^{(j)} - p_{\mu}^{(k)}.$$
(2.13)

From the assumption of uniform distribution for $\left\{p_{\mu}^{(i)}\right\}_{i}$, the values of $l_{\mu}^{(i)}$ and $l_{\mu}^{(j)}$ distribute uniformly as well. \tilde{B}_{1} is rewritten as

$$\tilde{B}_1 = \sum_{l^{(i)}, l^{(j)}, k} g_0^2 \Lambda_0^6 \frac{1}{(l^{(i)})^2 + m^2} \frac{1}{(l^{(j)})^2 + m^2} \frac{1}{(l^{(i)})^2 + m^2} \frac{1}{(l^{(i)} + l^{(j)})^2 + m^2}.$$
(2.14)

There is one summation of the index which keeps $l_{\mu}^{(i)}$ and $l_{\mu}^{(j)}$ fixed. We have represented it as Σ_k . In the large-N limit, we can replace the summations over $l_{\mu}^{(i)}$ and $l_{\mu}^{(j)}$ with the integrations over continuous variables:

$$\Lambda_0^d \frac{1}{N} \sum_{l^{(i)}} f(l^{(i)}) \to \int \frac{d^d k}{(2\pi)^d} f(k), \qquad (2.15)$$

With Λ_0 required to give the quantity the appropriate dimension. Through the replacement we treat it as the UV cutoff scale. Using Eq.(2.15) and $\sum_k 1 = N$, the diagram takes the following form:

$$\tilde{B}_1 = \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} g_0^2 N^3 \Lambda_0^{6-2d} \frac{1}{k_1^2 + m^2} \frac{1}{k_2^2 + m^2} \frac{1}{(k_1 + k_2)^2 + m^2},$$
(2.16)

$$\tilde{B}_2 = g_0^2 N \frac{\Lambda_0^6}{m^6}.$$
(2.17)

 \tilde{B}_1 coincides with Eq.(2.4) when we identify the volume of the matrix model (Eq.(2.7)) to $1/\Lambda_0^d$. This coincidence occurs in general planar diagrams of arbitrary loops. In this sense, The matrix model reproduces the quantities derived from the field theory (Eq.(2.1)) in the large-N limit.

Generically, the physical quantities of a matrix field theory in the large-N limit can be reproduced in the corresponding matrix model. It can be shown by focusing on the planar diagrams [49]. Thus, the matrix model is referred to as the large-N reduction of the original field theory. The reduction means that the information of spacetimes can partially be absorbed into the matrix, which originally takes values in the internal spaces, gauge group for example. Due to the fact that the spacetime information is included in P_{μ} , whose eigenvalues serve as momenta, the treatment of the present matrix model is called the momentum interpretation. In this treatment, the spacetime in the coordinate space emerges as below. In the basis diagonalizing P_{μ} , we can introduce a set of parameters $\{x^{\mu}\} \in \mathbb{R}^d$ to rewrite the trace of the commutator as

$$\operatorname{Tr}\left(-[P_{\mu}, \Phi]^{2}\right) = \operatorname{Tr}\left(e^{iPx}[iP_{\mu}, \Phi]e^{-iPx}e^{iPx}[iP^{\mu}, \Phi]e^{-iPx}\right)$$
$$= \operatorname{Tr}\left(\partial_{\mu}(e^{iPx}\Phi e^{-iPx})\partial^{\mu}(e^{iPx}\Phi e^{-iPx})\right).$$
(2.18)

By using a parameter-dependent matrix $\Phi(x) \equiv e^{iPx} \Phi e^{-iPx}$, Eq.(2.7) turns into the following form:

$$S = \frac{1}{g_0^2 \Lambda_0^6} \operatorname{Tr} \left(\frac{1}{2} (\partial_\mu \Phi(x))^2 + \frac{m^2}{2} \Phi(x)^2 + \frac{1}{3!} \Phi(x)^3 \right).$$
(2.19)

Here, under the assumption that we can separate the trace over DoF corresponding to P_{μ} , it can be replaced to the integration over x^{μ} due to the uniformity of eigenvalues of P_{μ} . As a result, we obtain Eq.(2.1).

In the above example, we had to introduce P_{μ} as an external matrix. Contrary to this, the large-N reduction of gauge theory requires no such matrices, and the model contains only dynamical matrices. This fact can be easily seen in the Pure Yang-Mills theory. Consider the following action:

$$S = \frac{1}{g^2 \Lambda^{4-d}} \int d^d x \frac{1}{4} \operatorname{Tr}(F_{\mu\nu}^2),$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu],$$
(2.20)

where A_{μ} is a gauge field in the adjoint representation of U(N). The large-N reduction of this system is represented as

$$S = \frac{1}{g^2 \Lambda^4} \operatorname{Tr} \left(-\frac{1}{4} ([P_\mu, A_\nu] - [P_\nu, A_\mu] + [A_\mu, A_\nu])^2 \right).$$

= $\frac{1}{g^2 \Lambda^4} \operatorname{Tr} \left(-\frac{1}{4} [P_\mu + A_\mu, P_\nu + A_\nu]^2 \right).$ (2.21)

By redefining the matrix as $P_{\mu} + A_{\mu} \rightarrow A_{\mu}$,

$$S = \frac{1}{g^2 \Lambda^4} \text{Tr} \left(-\frac{1}{4} [A_{\mu}, A_{\nu}]^2 \right).$$
 (2.22)

Thus the spacetime information is completely absorbed into the gauge field matrix. It is

remarkable that Eq.(2.22) is the same form as that obtained from the dimensional reduction of the original action (Eq.(2.20)). Conversely, starting from the matrix model (Eq.(2.22)), the equation of motion for A_{μ} is

$$[A^{\nu}, [A_{\nu}, A_{\mu}]] = 0. \tag{2.23}$$

As a solution of this, a diagonal momenta matrix exists:

$$A_{\mu} = P_{\mu} = \text{diag}(\cdots, p_{\mu}^{(i)}, p_{\mu}^{(i+1)}, p_{\mu}^{(i+2)}, \cdots).$$
(2.24)

When we expand the action around this classical solution, *i.e.* substituting $A_{\mu} = P_{\mu} + A_{\mu}$, we obtain Eq.(2.21). From this viewpoint, spacetimes emerge from classical solutions of the matrix model, and the fluctuation around them describes the field theory on the spacetime. This concept is the standard approach of analyzing the matrix model.

2.2 The IIB matrix model and emergence of type-IIB string theory

Among various matrix models, we intensively treat *the IIB matrix model* and its deformation in this thesis. Its action is given by

$$S_{IIB} = -\alpha \operatorname{Tr} \left(\frac{1}{4} [A_{\mu}, A_{\nu}]^2 + \frac{1}{2} \bar{\Psi}_{\alpha} (\Gamma^{\mu})_{\alpha\beta} [A_{\mu}, \Psi_{\beta}] \right),$$

$$\alpha = \frac{N}{g^2 \Lambda_0^4}.$$
 (2.25)

Here, A_{μ} ($\mu = 1, \dots, 10$) and Ψ_{α} ($\alpha = 1, \dots, 16$) are $N \times N$ bosonic and fermionic hermitian matrices, respectively. a is a ten-dimensional vector index, and α is a spinor index. Ψ_{α} is assumed to satisfy conditions for a ten-dimensional Majorana-Weyl spinor. Eq.(2.25) is the large-N reduction of ten-dimensional super Yang-Mills theory:

$$S_{\rm SYM} = \frac{N\Lambda_0^6}{g^2} \int d^{10}x \left(\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}\bar{\psi}D\psi\right).$$
(2.26)

In this section, we re-interpret matrices as DoF which have spacetime information directly in the form of the coordinates. In other words, we will interpret eigenvalues of matrices as coordinates of the target spacetime. This interpretation is called *the coordinate interpretation*. With this interpretation, we reproduce the action of type-IIB string theory from the IIB matrix model. First, we show that a form of actions for string theory, namely Schild action can be obtained from the classical limit of a particular matrix model. Then, we derive the model as an effective theory of the IIB matrix model. Consider the following action for string theory:

$$S_{\rm NG} = -T \int d^2 \sigma \left[\sqrt{-\frac{1}{2} \Sigma^2} + i \epsilon^{ab} \partial_a X^{\mu} (\bar{\theta}^1 \Gamma_{\mu} \partial_b \theta^1 + \bar{\theta}^2 \Gamma_{\mu} \partial_b \theta^2) + \epsilon^{ab} \bar{\theta}^1 \Gamma^{\mu} \partial_a \theta^1 \bar{\theta}^2 \Gamma_{\mu} \partial_b \theta^2 \right],$$
(2.27)

where θ^1 and θ^2 are ten-dimensional Majorana-Weyl spinors, and they have the same chirality. $\Sigma^{\mu\nu}$ is defined as

$$\Sigma^{\mu\nu} = \epsilon^{ab} \Pi^{\mu}_{a} \Pi^{\nu}_{b},$$

$$\Pi^{\mu}_{a} = \partial_{a} X^{\mu} - i\bar{\theta}^{1} \Gamma^{\mu} \partial_{a} \theta^{1} + i\bar{\theta}^{2} \Gamma^{\mu} \partial_{a} \theta^{2}.$$
 (2.28)

Note that θ^2 in the above action has been analytically-continued from that in the ordinary GS action as $\theta^2 \rightarrow i\theta^2$ This operation is necessary in order to make the path integral well-defined. Eq.(2.27) has $\mathcal{N} = 2$ super symmetry

$$\delta_{\text{SUSY}}\theta^{1} = \epsilon^{1},$$

$$\delta_{\text{SUSY}}\theta^{2} = \epsilon^{2},$$

$$\delta_{\text{SUSY}}X^{\mu} = i\bar{\epsilon}^{1}\Gamma^{\mu}\theta^{1} - i\bar{\epsilon}^{2}\Gamma^{\mu}\theta^{2},$$
(2.29)

and kappa symmetry

$$\delta_{\kappa}\theta^{1} = (1+\tilde{\Gamma})\kappa^{1},$$

$$\delta_{\kappa}\theta^{2} = (1-\tilde{\Gamma})\kappa^{2},$$

$$\delta_{\kappa}X^{\mu} = i\bar{\theta}^{1}\Gamma^{\mu}(1+\tilde{\Gamma})\kappa^{1} - i\bar{\theta}^{2}\Gamma^{\mu}(1-\tilde{\Gamma})\kappa^{2},$$
(2.30)

$$\tilde{\Gamma} = \frac{1}{2\sqrt{-\frac{1}{2}\Sigma^2}} \Sigma^{\mu\nu} \Sigma_{\mu\nu}.$$
(2.31)

Let us gauge-fix the kappa symmetry by imposing the conditions $\theta^1 = \theta^2 = \psi$. The action changes its form as

$$\tilde{S}_{\rm GS} = -T \int d^2 \sigma \left[\sqrt{-\frac{1}{2} \sigma^2} + 2i \epsilon^{ab} \partial_a X^{\mu} \bar{\psi} \Gamma_{\mu} \partial_b \psi \right], \qquad (2.32)$$

$$\sigma^{\mu\nu} = \epsilon^{ab} \partial_a X \mu \partial_b X^{\nu}. \tag{2.33}$$

 $\tilde{S}_{\rm GS}$ is equivalent to the following action (up to the normalization of fields) through the introduction of a Poisson bracket:

$$S_{\text{Schild}} = \int d^2 \sigma \left[\alpha \sqrt{g} (\frac{1}{4} \{ X^{\mu}, X^{\nu} \}^2 - \frac{i}{2} \bar{\psi} \Gamma_{\mu} \{ X^{\mu}, \psi \}) + \beta \sqrt{g} \right], \qquad (2.34)$$

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$$\{f,g\} = \frac{1}{\sqrt{g}} \epsilon^{ab} \partial_a f \partial_b g. \tag{2.35}$$

 α and β are constants. In this action, the transformation laws for $\mathcal{N} = 2$ super symmetries are written as

$$\delta^{(1)}\psi = -\frac{1}{2\sqrt{-\frac{1}{2}\sigma^2}}\sigma_{\mu\nu}\Gamma^{\mu\nu}\epsilon^{(1)},$$

$$\delta^{(1)}X^{\mu} = 4i\bar{\epsilon}^{(1)}\Gamma^{\mu}\psi,$$

$$\delta^{(2)}\psi = \epsilon^{(2)},$$

$$\delta^{(2)}X^{\mu} = 0.$$
(2.36)

The quantum string theory based on Eq.(2.35) is formally defined by the following path integral:

$$Z = \int \mathcal{D}\sqrt{g}\mathcal{D}X\mathcal{D}\psi e^{-S_{\text{Schild}}}.$$
(2.37)

This system is, in fact, equivalent ro the classical limit of the system whose action and partition function are given by

$$S_{\rm mat} = \alpha {\rm Tr} \left(-\frac{1}{4} [A_{\mu}, A_{\nu}]^2 - \frac{1}{2} \bar{\Psi} \Gamma^{\mu} [A_{\mu}, \Psi] \right) + \beta {\rm Tr} 1, \qquad (2.38)$$

$$Z_{\rm mat} = \sum_{n=0}^{\infty} \int dA d\Psi e^{-S_{\rm mat}}.$$
(2.39)

Here, A_{μ} and Ψ are $n \times n$ hermitian matrices. The equivalence is shown as below. It is natural that the contribution of large n configurations is dominant, and that eigenvalues of the matrices are distributed uniformly. Therefore, it is consistent to consider the classical limit, where we replace matrices with c-numbers and their commutators with the Poisson brackets, just in the ordinary quantum mechanics. The trace is converted to the inner-product in the functional space. In the present case, we take the space of smooth functions on the world-sheet, and defined their inner-product as the integration over it:

$$[F,G] \longrightarrow i\{f(\sigma), g(\sigma)\},$$

$$\operatorname{Tr}[\cdots] \longrightarrow \frac{1}{2\pi} \int d^2 \sigma \sqrt{g}.$$
 (2.40)

Note that the Poisson bracket satisfies the following conditions:

$$\int d^2 \sigma \sqrt{g} \{f, h\} = 0,$$

$$\int d^2 \sigma \sqrt{g} f\{h, k\} = \int d^2 \sigma \sqrt{g} h\{k, f\}.$$
 (2.41)

These come from the vanishing trace of commutators and cyclicity of tripple products. Applying the above replacement, Eq.(2.39) turns into Eq.(2.37). The summation over n has been converted to the functional integral over the volume element $\int \mathcal{D}\sqrt{g}$. It is remarkable that Eq.(2.39) is described only the mere matrices and thus well-defined. Indeed, it has been shown that Eq.(2.38) is a matrix regularization of world-sheet.³

On the other hand, Eq.(2.38) is understood as the effective action of the IIB matrix model as well. In studying Eq.(2.25), we assume that the eigenvalues of A_a are distributed in the range of

$$-\pi\Lambda < (\text{the eigen values of } A_{\mu}) < \pi\Lambda.$$
 (2.42)

 Λ is the parameter characterizing the energy scale of the system. We have to consider its interpretation with care as mentioned later. In the approach where the eigenvalues of A_{μ} are the spacetime corrdinates, Λ is regarded as the IR cutoff. In the model, consider the following background:

$$A_{\mu} = P_{\mu} = \text{diag}(\cdots, p_{\mu}^{(i)}, p_{\mu}^{(i+1)}, \cdots), \qquad (2.43)$$

with $\{p_{\mu}^{(i)}\}$ distributed uniformly. In the continuum limit (large-N limit), this background is interpreted as the flat spacetime.

Before proceeding further, we have more comments on the continuum limit. Since the size of the matrices N is a sort of cutoffs, we have to take its limit in the coordination with that of Λ , which has been introduced as an explicit energy scale. That simultaneous limit is called the double scaling limit. The proper way to take it is read off from the original super Yang-Mills theory. Eq.(2.26) has the classical energy scale $m = \Lambda/g^{1/3}$. The double scaling limit is required to keep both m and the coefficient α in Eq.(2.25) finite.⁴ From the conditions

$$\alpha = \frac{N}{g^2 \Lambda^4} : \text{fixed}, \quad m^6 = \frac{\Lambda^6}{g^2} : \text{fixed}, \tag{2.44}$$

we obtain

$$\frac{N}{\Lambda^{10}}$$
: fixed. (2.45)

³A matrix regularization is the regularization where the functions defined on a manifold with the Poisson structure are regularized with the approximation by the matrices. The algebraic structure of the matrices is determined by demanding that it coincides with that of the original functional space in the large-N limit.

⁴In [1], A classical solution of the IIB matrix model is regarded as D-string and the authors identified the coupling constant as $\alpha \sim 1/(g_s \alpha')$. The motivation of regarding the solution as D-string is based on the fact that it is a BPS object.

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Eventually, the double scaling limit is defined as below:

$$\begin{split} \Lambda &\longrightarrow \infty, \\ N &\sim \Lambda^{10} &\longrightarrow \infty, \\ g &\sim \Lambda^3 &\longrightarrow \infty. \end{split} \tag{2.46}$$

Returning to the derivation of Eq.(2.38), we operate the path integral defined with Eq.(2.25). When we integrate around the flat background Eq.(2.43), The bosonic and fermionic contributions from the non-diagonal components are canceled. On the other hand, nontrivial contributions from the diagonal components do not exist at least one-loop level. The partition function is represented as

$$Z_0(N) \propto \int \prod_{i=1}^N \prod_{\mu=0}^9 \frac{dA_{\mu ii}}{N!} \prod_{\gamma=1}^{16} d\Psi_{\gamma ii} dc_{ii} db_{ii}.$$
 (2.47)

c and b are the Faddeev-Popov ghost and anti-ghost. The integration over the bosonic part $dA_{\mu ii}$ symply yields the width of the eigenvalue-distribution:

$$\int \prod_{i=1}^{N} \prod_{\mu=0}^{9} dA_{\mu i i} \propto \Lambda^{10N}.$$
(2.48)

The integration over the ghosts $db_{ii}dc_{ii}$, and the one over the fermionic part $d\Psi_{\gamma ii}$ gives respectively

$$\int \prod_{i=1}^{N} dc_{ii} c db_{ii} \propto \alpha^{\frac{N}{2}}, \qquad (2.49)$$

$$\int \prod_{i=1}^{N} \prod_{\gamma=1}^{16} d\Psi_{\gamma i i} \propto \alpha^{2N}.$$
(2.50)

It is easily understood through the dimensional analysis. Therefore, we obtain the following partition function:

$$Z_0(N) \propto \frac{1}{N!} (\Lambda^{10} \alpha^{\frac{5}{2}})^N$$
 (2.51)

After the above analysis as a preliminary, consider a background below:

$$A_{\mu} = \begin{pmatrix} \hat{A}_{\mu} & & & \\ & p_{\mu}^{(1)} & & \\ & & p_{\mu}^{(2)} & \\ & & & \ddots \end{pmatrix}.$$
(2.52)

Here A_{μ} is an $n \times n$ hermitian matrix. The partition function around the background is represented as Z. The effective action for the $n \times n$ block is expected to be obtained by subtracting the loop correction of totally flat spacetime from the correction around Eq.(2.52). Therefore we define the effective action as

$$S_{n \times n \text{ eff}} \equiv -\log\left(\frac{Z}{Z_0(N)}\right). \tag{2.53}$$

In the path integral of Z, the contribution from the $(N-n) \times (N-n)$ block, which lives in the right-bottom sector in the matrices, is nothing other than $Z_0(N-n)$. On the other hand, the contribution from the non-diagonal block yields correction terms of the same form as those in the super Yang-Mills theory. Combining them together, the resulting effective action is given by

$$S_{n \times n \text{ eff}} = -\log\left(\frac{Z_0(N-n)}{Z_0(N)}\right) - \alpha \operatorname{Tr}\left(\frac{1}{4}[\hat{A}_{\mu}, \hat{A}_{\nu}]^2\right) + \cdots$$
$$= \operatorname{const.} \times n \log\left(\frac{\Lambda^{10}\alpha^{\frac{5}{2}}}{N}\right) - \alpha \operatorname{Tr}\left(\frac{1}{4}[\hat{A}_{\mu}, \hat{A}_{\nu}]^2\right) + \cdots$$
$$= \operatorname{const.} \times n - \alpha \operatorname{Tr}\left(\frac{1}{4}[\hat{A}_{\mu}, \hat{A}_{\nu}]^2\right) + \cdots$$
(2.54)

The explicit terms in the above action is identical to Eq.(2.38). We have trancate in the dots higher order correction terms of A_{μ} , Tr(\hat{F}^4) for example. They are the divergent terms which already appear in the super Yang-Mills theory. In the exact treatment, we have to add to Eq.(2.25) the counterterms for them. Although Eq.(2.26) is non-renormalizable action perturbatively, we can assume the existence of a nontrivial fixed point of renormalization group flow, due to the super symmetry. Accordingly, The correction terms included in Eq.(2.54) is observed to be controllable or ot have no effect in the double scaling limit Eq.(2.46).

We have seen that the effective action of the IIB matrix model is identified to a form of the action of type-IIB string theory (Eq.(2.38)). It is necessary to compare them at the level of partition function. In Eq.(2.39), there is summation over n, the size of the matrices. It is valid that we have calculate the effective action with n fixed, since the background has a particular value of n as a classical solution. Yet, the path integral itself includes contributions from blocks of verious size. In this sense, starting from Eq.(2.25), we have been able to reproduce Eq.(2.39). The constant term proportional to n in Eq.(2.54) is understood as a chemical potential. Since it is natural that the configurations with large n are dominant, it reproduces the path integral of string theory Eq.(2.37).

As above, we have had a brief introduction of the IIB matrix model and explanation of how it reproduce string theory. One of the essential points is that the matrix model reproduce the variety of physics only at the large-N limit, while the original model with finite N is free from the UV divergent and the path integral is well-defined. It suggests that the matrix model is non-pertubatively regularized theory. Another critical point is the emergence of spacetimes. In the both case of large-N-reducing gauge theory, and of the reproducing string theory, the notion of spacetime has naturally emerged from the matrix model. These observation leads to the expectation that the IIB matrix model (or its modification) describes the quantum spacetime and gravity beyond perturbative string theory.

On the other hand, it should be paid attention that we have re-interpreted the physical meaning of matrices in order to derive the string action from the large-N reduction of super Yang-Mills theory. Here, the change in the interpretation of the dynamical variables leads to the change of physics they describe. It might imply some profound relationship between different theories. Otherwise, If we take the pessimistic standpoint, it might be just a problem of interpretations and some of them be physically senseless or unhealthy.

Although the issue cannot be settled so far, we will attempt in the following to find out the really appropriate mechanism of emergent spacetime and physics on it.

Chapter 3

Graviton Exchange in the Matrix Model with the Noncommutative Interpretation

Before studying the matrix model with the interpretation of our main interest, we first take a noncommutative version of the momentum interpretation, the noncommutative interpretation.

This is one of the most well-studied interpretation. It is well known that there is a close relation between the matrix model and noncommutative (NC) field theory [52–54]: The fluctuation of matrices around some noncommutative classical solution is equivalent to the field of NC field theory, where the product of fields is defined by the star product. Remarkably, the NC U(1) gauge field is uniformly coupled with all the matters. Such a special behavior of the U(1)gauge field reminds us the property of gravity [55–57]. In fact, the couplings between the U(1)gauge field and the matters can be expressed by the effective metric made of the gauge field, at the leading order with respect to the noncommutativity (see Section 3.1 for the details). Thus, the U(1) gauge field can be viewed as the fluctuation of the metric. Moreover, it is suggested that UV/IR mixing emerging at one-loop calculation of the U(1) NC field can be understood in terms of induced or emergent gravity [5, 23, 58–60]¹ (for a review, see also [24, 62]).

While such successful results are present, it is not yet clear whether the mechanism rigorously reproduces the real gravity. For example, we have not yet obtained the explicit diffeoinvariance within the degrees of freedom of matrices.² At a first glance, it seemingly does not work because of the off-shell degrees of freedom; the U(1) gauge field has only four, while a graviton does ten. However, still there is the possibility because the off-shell degrees of freedom are unphysical. Therefore, it is quite necessary to check whether the emergent gravity scenario can actually explain the results of the ordinary gravity. As an important check, it is interesting

¹Recently, another mechanism of emergent gravity was also discussed on a specific background [61].

²On the other hand, the diffeomorphism can be explicitly seen in the operator interpretation.

and meaningful to calculate a two-body scattering amplitude of test particles exchanging the NC U(1) gauge field, and to compare it with that of the usual graviton exchange. In this paper, we perform such analysis, and see that the NC U(1) gauge theory correctly reproduces the usual graviton exchange *if* the noncommutativity is appropriately averaged and the test particles are massless. Although this result shows a partial success of the mechanism, it may also indicate the necessity of considering another framework in order to produce the correct four-dimensional gravity. If such a new framework is actually found, we can get deep understating between the matrix model and gravity. As the most promising candidate, we will consider the operator interpretation in the later chapters.

3.1 A brief review of emergent gravity in the NC interpretation

Before going into the calculation of the scattering amplitude, let us briefly review the emergent gravity scenario starting from the matrix model with noncommutative interpretation. We consider the following action:

$$S = S_{\text{IIB}} + S_{\Phi}$$

$$= -\text{Tr}\left(\frac{(2\pi)^2}{4\Lambda^4} [A_a, A_b]^2\right) - \frac{(2\pi)^2}{g^2\Lambda^4} \text{Tr}\left(\frac{1}{2}[A_a, \Phi]^2\right)$$

$$= -\text{Tr}\left(\frac{(2\pi)^2}{4\Lambda^4} \delta^{ac} \delta^{bd}[A_a, A_b][A_c, A_d]\right) - \frac{(2\pi)^2}{g^2\Lambda^4} \text{Tr}\left(\frac{1}{2} \delta^{ab}[A_a, \Phi][A_b, \Phi]\right)$$
(3.1)

where S_{IIB} is the bosonic part of the IIB matrix model, with indices running as $a = 1, \dots, 10.^3$ Λ represents a cut-off scale, and P_a and Φ are $N \times N$ hermitian matrices. Note that δ^{ab} stands for the ten-dimensional flat metric. In the following discussion, we consider fluctuations around a specific background \bar{P}_a . We interpret \bar{P}_a as the derivatives with respect to the coordinate. This is called the momentum interpretation of the matrix model. In this model, \bar{P}_a 's are determined by the classical equation of motion:

$$\left[A^{b}, [A_{b}, A_{a}]\right] + \left[\Phi, [A_{a}, \Phi]\right] = 0.$$
(3.2)

Among the various solutions, we consider the following one that gives the four dimensional NC spacetime:

$$[\bar{P}_{\mu}, \bar{P}_{\nu}] = i\Lambda_{\rm NC}^2 \times \tilde{\theta}_{\mu\nu} \mathbf{1} \quad (\mu, \nu = 1, \cdots, 4),$$

$$\bar{P}_i = 0 \quad (i = 5, \cdots, 10), \ \Phi = 0,$$
(3.3)

where $\theta_{\mu\nu}$ is an antisymmetric dimensionless constant, and $\Lambda_{\rm NC}$ is a NC scale. This is the well-known noncommutative geometry called the 4D Moyal-Weyl plane \mathbb{R}^4_{θ} . By considering the

³From now on, we re-label the vector indices by latin letters.

fluctuation around the solution as $A_{\mu} \rightarrow P_{\mu} + A_{\mu}$, and using the well-known correspondence between matrix and function on the NC spacetime [52], the action takes the form

$$S = -\operatorname{Tr}\left(\frac{(2\pi)^2}{4\Lambda^4}\delta^{\mu\alpha}\delta^{\nu\beta}[\bar{P}_{\mu} + A_{\mu}, \bar{P}_{\nu} + A_{\nu}][\bar{P}_{\alpha} + A_{\alpha}, \bar{P}_{\beta} + A_{\beta}]\right)$$
$$-\frac{(2\pi)^2}{g^2\Lambda^4}\operatorname{Tr}\left(\frac{1}{2}\delta^{\mu\nu}[\bar{P}_{\mu} + A_{\mu}, \Phi][\bar{P}_{\nu} + A_{\nu}, \Phi]\right)$$
$$= \frac{\Lambda_{\mathrm{NC}}^4}{\Lambda^4}\left\{\frac{1}{4}N\tilde{\theta}^{\mu\nu}\tilde{\theta}_{\mu\nu} + \frac{1}{4}\int d^4x\sqrt{|\tilde{\theta}|}\left(\delta^{\mu\nu}\delta^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}\right)_{\star} + \frac{1}{2g^2}\int d^4x\sqrt{|\tilde{\theta}|}\delta^{\mu\nu}\left(\left(\partial_{\mu}\Phi - i[A_{\mu}, \Phi]\right)\left(\partial_{\nu}\Phi - i[A_{\nu}, \Phi]\right)\right)_{\star}\right\}, \quad (3.4)$$

where we have neglected the fluctuation of P_i 's $(i = 5, \dots 10,)$ for simplicity. Here, $\tilde{\theta} = \det(\tilde{\theta}_{\mu\nu})$, $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} - i \left[A_{\mu}, A_{\nu}\right]_{\star}$, and \star represents the Moyal product defined by

$$(f \star g)(x) = \exp\left(\frac{i}{2}\Lambda_{\rm NC}^{-2}\theta^{\mu\nu}\partial^{(y)}_{\mu}\partial^{(z)}_{\nu}\right)f(y)g(z)\Big|_{y=z=x},\tag{3.5}$$

where $\theta^{\mu\nu} = (\tilde{\theta}^{-1})^{\mu\nu}$. Furthermore, note that we can always eliminate $(\Lambda_{\rm NC}^4/\Lambda^4)|\tilde{\theta}|^{1/2}$ by the field redefinition $\Phi \to (\Lambda^2/\Lambda_{\rm NC}^2)|\tilde{\theta}|^{-1/4}\Phi$ in the last term of Eq.(3.4). The above argument is the usual interpretation of the matrix model as NC field theory. On the other hand, it was also argued that A_{μ} can be interpreted as the fluctuation of the four dimensional spacetime metric in the semi-classical limit [55–57]. Here, 'semi-classical' means that we should keep the lowest order terms in $\Lambda_{\rm NC}^{-2}\theta^{\mu\nu}$, and neglect higher order terms. In this approximation, noncommutativity gets switched off, and commutator turns into the Poisson bracket as

$$[f,g]_{\star} \sim i\{f,g\}, \quad \{f,g\} \equiv \Lambda_{\rm NC}^{-2} \times \theta^{\mu\nu} \partial_{\mu} f \partial_{\nu} g. \tag{3.6}$$

Then, Eq.(3.4) now becomes

$$S\Big|_{\text{semi}} = \frac{\Lambda_{\text{NC}}^4}{4\Lambda^4} \int d^4x \sqrt{|\tilde{\theta}|} \left(\sqrt{|G|} G^{\mu\nu}\right) \tilde{\theta}_{\mu\alpha} \tilde{\theta}_{\nu\beta} \left(\sqrt{|G|} G^{\alpha\beta}\right) + \int d^4x \, \frac{1}{2g^2} \sqrt{|G|} G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi, \quad (3.7)$$

where

$$\sqrt{|G|}G^{\mu\nu} = \delta^{\mu\nu} + \Lambda_{\rm NC}^{-2} \times (\theta^{\alpha\mu}\partial_{\alpha}A^{\nu} + \theta^{\alpha\nu}\partial_{\alpha}A^{\mu}) + \Lambda_{\rm NC}^{-4} \times \mathcal{O}(A^2), \tag{3.8}$$

From this we can read the fluctuation of the metric $h^{\mu\nu} = -(G^{\mu\nu} - \delta^{\mu\nu})$ as

$$h^{\mu\nu} = \Lambda_{\rm NC}^{-2} \times \left(\theta^{\mu\alpha}\partial_{\alpha}A^{\nu} + \theta^{\nu\alpha}\partial_{\alpha}A^{\mu} + \frac{1}{2}\delta^{\mu\nu}\theta^{\alpha\beta}F_{\alpha\beta}\right) + \Lambda_{\rm NC}^{-4} \times \mathcal{O}(A^2)$$
(3.9)



Figure 3.1: One-loop diagrams needed for the computation of the effective action of the NC U(1) gauge field. (Source: [43], doi:10.1093/ptep/ptx036)

In the following analysis, we investigate the dynamics of A_{μ} which is quadratic in the effective action. In this sense, the $\mathcal{O}(A^2)$ terms in $h^{\mu\nu}$ are not necessarily as long as we expand the Einstein-Hilbert (EH) action at the linearized level because such terms give higher order contributions.⁴ From the above equations, one can actually see that Φ couples to A_{μ} in the covariant way, and that A_{μ} can be interpreted as the fluctuation of the metric. On the other hand, as for the bosonic part of IIB matrix model, its semi-classical action cannot be written in a covariant way. (See the first term in Eq.(3.7).) Although this is a big problem in the present formulation of emergent gravity, we simply drop the term in the following discussion. In other words, we will focus on the matrix model which does not contain $F_{\mu\nu}^2 \sim [P_{\mu}, P_{\nu}]^2$ term. Then as we will see below(3.14), such a term is not induced by the quantum correction due to the noncommutativity.

Although we have found that the U(1) gauge field can be understood as the fluctuation of the metric, we have not yet obtained the action for it. It was claimed that it is given by the induced EH action by considering the one-loop effective action of A_{μ} in the semi-classical limit [23].⁵ By calculating the scalar one-loop diagrams (Fig.3.1), we obtain the effective action of A_{μ} as a NC field theory [23]:⁶

$$e^{-\Gamma_{\Phi}} = \int \mathcal{D}\Phi e^{-S} \big|_{1\text{-loop without IIB action}},$$

$$\Gamma_{\Phi} = -\frac{1}{32\pi^2 g^2} \int \frac{d^4 p}{(2\pi)^4} \bigg[-\frac{1}{6} F_{\mu\nu}(p) F^{\mu\nu}(-p) \log\left(\frac{\Lambda^2}{\Lambda_{\text{eff}}^2}\right)$$
(3.11)

⁴On the other hand, the $\mathcal{O}(A^2)$ terms in $h^{\mu\nu}$ are necessarily when we expand the cosmological constant term.

⁶In this reference, calculation was done by adding the mass term for the scalar as a regulator, and then taking the massless limit. Furthermore, the following replacement is used as a regularization of the loop integral:

$$\int \frac{d^4p}{(2\pi)^4} \frac{f(p)}{[p^2 + \Delta^2]^2} \to \int_0^\infty d\alpha \alpha \int \frac{d^4p}{(2\pi)^4} f(p) e^{-\alpha(p^2 + \Delta^2) - 1/(\alpha \Lambda^2)}.$$
(3.10)

Therefore, to maintain the consistency, we also use this regularization scheme in the following calculation.

⁵Here, note that there exist two ways of obtaining such a semi-classical limit: One is to take the semi-classical limit after calculating the one-loop effective action as a NC field theory. The other is to take first the semi-classical limit at the tree-level action, and calculate the effective action as an ordinary field theory. However, we have checked that both of the approaches produce the same result.

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$$+\frac{1}{4}\theta^{\mu\nu}F_{\mu\nu}(p)\theta^{\lambda\rho}F_{\lambda\rho}(-p)\left(\Lambda_{\rm eff}^{4}-\frac{1}{6}p^{2}\Lambda_{\rm eff}^{2}+\frac{(p^{2})^{2}}{1800}\left(47-30\log\left(\frac{p^{2}}{\Lambda_{\rm eff}^{2}}\right)\right)\right)\right],$$
(3.12)

where $\Lambda_{\text{eff}}^{-2} = \Lambda^{-2} + \tilde{p}^2/(4\Lambda_{\text{NC}}^4)$, $\tilde{p}^{\mu} = \theta^{\mu\nu}p_{\nu}$, and Λ is the cutoff momentum for loop integral. We suppose \tilde{p}^2 and p^2 are the same scale, since $\theta^{\mu\nu}$ is dimensionless and expected to be O(1). When we focus on the IR regime,

$$\frac{p^2 \Lambda^2}{\Lambda_{\rm NC}^4} < 1, \tag{3.13}$$

Eq.(3.12) can be expanded as

$$\begin{split} \Gamma_{\Phi} &\sim -\frac{1}{32\pi^{2}g^{2}} \int \frac{d^{4}p}{(2\pi^{4})} \bigg[\frac{\Lambda^{4}}{4\Lambda_{\rm NC}^{4}} \theta^{\mu\nu} F_{\mu\nu}(p) \theta^{\lambda\rho} F_{\lambda\rho}(-p) - \frac{\Lambda^{4}}{\Lambda_{\rm NC}^{4}} \frac{\Lambda^{2}}{8\Lambda_{\rm NC}^{4}} \tilde{p}^{2} \theta^{\mu\nu} F_{\mu\nu}(p) \theta^{\lambda\rho} F_{\lambda\rho}(-p) \\ &- \frac{\Lambda^{2}}{24\Lambda_{\rm NC}^{4}} \left(F^{\mu\nu}(p) F_{\mu\nu}(-p) \tilde{p}^{2} + \theta^{\mu\nu} F_{\mu\nu}(p) \theta^{\lambda\rho} F_{\lambda\rho}(-p) p^{2} \right) \bigg] \\ &= -\frac{1}{32\pi^{2}g^{2}} \int d^{4}x \bigg[\frac{\Lambda^{4}}{4\Lambda_{\rm NC}^{4}} \theta^{\mu\nu} F_{\mu\nu} \theta^{\lambda\rho} F_{\lambda\rho} + \frac{\Lambda^{4}}{\Lambda_{\rm NC}^{4}} \frac{\Lambda^{2}}{8\Lambda_{\rm NC}^{4}} \theta^{\mu\nu} F_{\mu\nu}(\partial \circ \partial) \theta^{\lambda\rho} F_{\lambda\rho} \\ &+ \frac{\Lambda^{2}}{24\Lambda_{\rm NC}^{4}} \bigg(F^{\mu\nu}(\partial \circ \partial) F_{\mu\nu} + \theta^{\mu\nu} F_{\mu\nu} \Box \theta^{\lambda\rho} F_{\lambda\rho} \bigg) \bigg], \end{split}$$
(3.14)

where $\partial \circ \partial = \theta^{\mu\alpha}\theta^{\nu\beta}\delta_{\alpha\beta}\partial_{\mu}\partial_{\nu}$, and $\Box = \delta^{\mu\nu}\partial_{\mu}\partial_{\nu}$. This result should be compared with the EH action with the cosmological constant term where the metric is given by Eq.(3.9):

$$S_{G} = \frac{1}{16\pi^{2}} \int d^{4}x \sqrt{|G|} \left(-\frac{1}{2}\Lambda^{4} - \frac{\Lambda^{2}}{12}R[G] \right)$$

$$\sim -\frac{1}{32\pi^{2}g^{2}} \int d^{4}x \left[\frac{\Lambda^{4}}{4\Lambda_{\rm NC}^{4}} \theta^{\mu\nu}F_{\mu\nu}\theta^{\lambda\rho}F_{\lambda\rho} + \frac{\Lambda^{2}}{24\Lambda_{\rm NC}^{4}}F^{\mu\nu}\partial\circ\partial F_{\mu\nu} \right], \qquad (3.15)$$

where we have extracted the quadratic part in A_{μ} and rescaled it as $A_{\mu} \rightarrow A_{\mu}/g$. From Eq.(3.15), one can see that the terms $\theta^{\mu\nu}F_{\mu\nu}\Box\theta^{\lambda\rho}F_{\lambda\rho}$ and $\theta^{\mu\nu}F_{\mu\nu}(\partial\circ\partial)\theta^{\lambda\rho}F_{\lambda\rho}$ in Eq.(3.14) are absent in Eq.(3.15). The latter is, however, a higher-order term in $(\Lambda/\Lambda_{\rm NC})^4$. Here we consider the effects of the noncommutativity in its lowest order. In other words, we assume that $\Lambda_{\rm NC}$ is larger than the cut-off momentum Λ and we shall neglect this term in the following discussion.⁷ The above mismatch between Eqs.(3.14) and (3.15) originates in the path-integral measure: In the NC theory, it is induced from the flat metric in the functional space of Φ .

$$||\delta\Phi||^2 = \int d^4x \ \delta\Phi(x)^2,$$
 (3.16)

and this apparently violates the diffeomorphism invariance. If we use the diffeomorphism

⁷The following analysis works in parallel and gives qualitatively the same result even if we take this term into account.

transformation

$$x^{\mu} \to y^{\mu} = x^{\mu} - \theta^{\mu\nu} A_{\nu}, \qquad (3.17)$$

which is not realized in the NC U(1) gauge theory, we can make $h^{\mu\nu}$ traceless in the leadingorder in A_{μ} . In such coordinates, the one-loop effective action indeed matches the EH action. See Appendix B for the details.

In spite of the mismatch, the similarity between Eq.(3.14) and Eq.(3.15) is impressive, and it is meaningful to study whether the NC U(1) gauge theory can actually describe the real gravity. In the following, we in particular consider the amplitude of the graviton exchange between two scalars.

3.2 Does noncommutative U(1) gauge field actually describe gravity ?

As a first step, we compute the two-body scattering amplitude of the scalar particles exchanging the U(1) gauge field whose action is given by Eq.(3.14). In the following discussion, we put $32\pi^2 g^2 = 1$ for simplicity, and drop the first term in Eq.(3.14) because it corresponds to the cosmological constant (CC) term, which we assume to be canceled by some mechanism. Adding a gauge fixing term and rewriting them in terms of A_{μ} , we have

$$\Gamma_{\Phi}|_{O(\Lambda^{2}) \text{ without CC term}} + \frac{1}{\alpha} \int \frac{d^{4}p}{(2\pi)^{4}} \frac{\Lambda^{2}}{12\Lambda_{\rm NC}^{4}} \tilde{p}^{2} p^{\mu} A_{\mu}(p) p^{\nu} A_{\nu}(-p) = \frac{\Lambda^{2}}{12\Lambda_{\rm NC}^{4}} \int \frac{d^{4}p}{(2\pi)^{4}} A^{\mu}(p) \left[\tilde{p}^{2} \left(p^{2} \delta_{\mu\nu} - \left(1 - \frac{1}{\alpha} \right) p^{\mu} p^{\nu} \right) + 2p^{2} \tilde{p}^{\mu} \tilde{p}^{\nu} \right] A^{\nu}(-p),$$
(3.18)

where α represents the gauge freedom. From this, one can read off the propagator of A_{μ} as

$$D_{\mu\nu}(p) = \frac{6\Lambda_{\rm NC}^4}{\Lambda^2} \frac{1}{\tilde{p}^2 p^2} \bigg[\delta_{\mu\nu} - (1-\alpha) \frac{p_\mu p_\nu}{p^2} - \frac{2}{3} \frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2} \bigg].$$
(3.19)

Supposing A_{μ} to propagate with Eq.(3.19), the two-body scattering amplitude of the scalar can be calculated in the semi-classical approximation. In the following discussion, we take the Feynman gauge $\alpha = 1$. From Eqs.(3.7) and (3.8) we can read the interaction between Φ and A_{μ} as

$$\mathcal{L}_{\text{Int}} = \Lambda_{\text{NC}}^{-2} \times \theta^{\alpha\mu} \partial_{\alpha} A^{\nu} \partial_{\mu} \Phi \partial_{\nu} \Phi \tag{3.20}$$

from which we can read the vertex as



Figure 3.2: A scattering of test particles exchanging the U(1) filed or graviton. In the former case, we read its propagator from the one-loop effective action Eq.(3.14). (Source: [43], doi:10.1093/ptep/ptx036)

We can now compute the two-body scattering amplitude (see Fig.3.2). Along with on-shell conditions for the scalar

$$p^{2} = q^{2} = 0, \quad (p+k)^{2} = p^{2}, \quad (q-k)^{2} = q^{2},$$
 (3.22)

we obtain

$$\mathcal{M}_A = \frac{6}{\Lambda^2} \frac{1}{k^2} \frac{(p \cdot \tilde{k})(q \cdot \tilde{k})}{\tilde{k}^2} \left[(4p \cdot q + k^2) - \frac{8}{3} \frac{(p \cdot \tilde{k})(q \cdot \tilde{k})}{\tilde{k}^2} \right] + (s \text{ channel}) + (u \text{ channel}). \quad (3.23)$$

This should be compared with the scattering amplitude calculated from the ordinary gravity system:

$$S = S_G + S_{\rm gf} + S_{\Phi}, \tag{3.24}$$

$$S_{G} = \frac{1}{2G_{N}} \int d^{4}x \sqrt{|(\delta+h)|} R[\delta+h] \Big|_{\text{quadratic part in }h},$$

= $\int d^{4}x \Big[\frac{1}{8} \partial_{\lambda} h_{\mu\nu} \partial^{\lambda} h^{\mu\nu} - \frac{1}{4} \partial^{\mu} h_{\mu\nu} \partial_{\lambda} h^{\lambda\nu} + \frac{1}{4} \partial^{\mu} h_{\mu\nu} \partial^{\nu} h - \frac{1}{8} \partial_{\mu} h \partial^{\mu} h \Big],$ (3.25)

$$S_{\rm gf} = \frac{1}{4} \int d^4x \left(\partial^\nu h_{\mu\nu} - \frac{1}{2} \partial_\mu h \right) \left(\partial^\lambda h^\mu_{\ \lambda} - \frac{1}{2} \partial^\mu h \right), \qquad (3.26)$$

$$S_{\Phi} = \int d^4x \sqrt{|(\delta+h)|} \left[\frac{1}{2} (\delta^{\mu\nu} - h^{\mu\nu}) \partial_{\mu} \Phi \partial_{\nu} \Phi \right] \Big|_{0\text{th and 1st order of } h}$$
$$= \int d^4x \left[\frac{1}{2} \partial_{\mu} \Phi \partial^{\mu} \Phi - \frac{1}{2} h^{\mu\nu} \partial_{\mu} \Phi \partial_{\nu} \Phi + \frac{1}{4} h \partial_{\mu} \Phi \partial^{\mu} \Phi \right], \qquad (3.27)$$

where the graviton is gauge-fixed in the de Donder gauge (harmonic gauge) which leads to the

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following propagator of graviton:

$$D^{(h)}_{\mu\nu\lambda\rho}(k) = \frac{2}{k^2} (\delta_{\mu\lambda}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\lambda} - \delta_{\mu\nu}\delta_{\lambda\rho}).$$
(3.28)

From the straightforward calculation, we obtain

$$\mathcal{M}_G = 2G_N\left(\frac{2(p \cdot q)^2}{k^2} + p \cdot q\right) + (s \leftrightarrow t) + (t \leftrightarrow u) = -G_N\left(\frac{su}{t} + \frac{tu}{s} + \frac{ts}{u}\right), \qquad (3.29)$$

where s, t and u are the Mandelstam variables. This does not match Eq.(3.23) although both of them lead to the inverse-square law. In particular, $\theta^{\mu\nu}$ explicitly remains in Eq.(3.23), and we need some mechanism to eliminate it. Because $\theta^{\mu\nu}$ is a moduli parameter which specifies the classical solution, it is natural to take a kind of average over it. For example, let us consider the average over the direction of $\theta^{\mu\nu}$ with the 'absolute value of $\theta^{\mu\nu}$ ' being fixed:

$$\theta^{\mu\alpha}\theta^{\nu\beta}\delta_{\alpha\beta} = \delta^{\mu\nu}.\tag{3.30}$$

We assume that $\theta^{\mu\nu}$ distributes in the Lorentz covariant manner:

$$\theta^{\mu\nu} \to \theta^{\mu\nu}_M = M^{\mu}_{\ \alpha} M^{\nu}_{\ \beta} \theta^{\alpha\beta}, \tag{3.31}$$

where M^{μ}_{α} is an element of SO(4). This is compatible with the assumption (3.30). Then the average over the direction of $\theta^{\mu\nu}$ yields Lorentz covariant quantities:

$$\int_{SO(4)} dM \ \theta_M^{\mu\nu} \theta_M^{\lambda\rho} = \frac{1}{3} (\delta^{\mu\lambda} \delta^{\nu\rho} - \delta^{\mu\rho} \delta^{\nu\lambda}) \equiv \frac{1}{3} \Delta^{\mu\nu\lambda\rho},$$

$$\int_{SO(4)} dM \ \theta_M^{\mu\nu} \theta_M^{\lambda\rho} \theta_M^{\alpha\beta} \theta_M^{\gamma\delta} = -\frac{1}{27} (\Delta^{\mu\nu\lambda\rho} \Delta^{\alpha\beta\gamma\delta} + \Delta^{\mu\nu\alpha\beta} \Delta^{\lambda\rho\gamma\delta} + \Delta^{\mu\nu\gamma\delta} \Delta^{\lambda\rho\alpha\beta})$$

$$+ \frac{1}{9} \Big\{ (\delta^{\nu\lambda} \delta^{\rho\alpha} \delta^{\beta\gamma} \delta^{\delta\mu} + \delta^{\nu\alpha} \delta^{\beta\gamma} \delta^{\delta\lambda} \delta^{\rho\mu} + \delta^{\nu\gamma} \delta^{\delta\lambda} \delta^{\rho\alpha} \delta^{\beta\mu})$$

$$+ [\mu\nu] [\lambda\rho] [\alpha\beta] [\gamma\delta] \Big\}, \quad (3.32)$$

where dM denotes the Haar measure of SO(4), and $[\mu\nu][\lambda\rho][\alpha\beta][\gamma\delta]$ represents the antisymmetrized terms of the first one with respect to the superscripts in each of the brackets. Here the coefficients in the right hand side (RHS) of Eq.(3.32) are determined so that they are consistent with Eq.(3.30). After taking such an average, Eq.(3.23) now becomes

$$\mathcal{M}_{A} = \frac{2}{\Lambda^{2}} \left[\frac{52}{27} \frac{1}{k^{2}} 2(p \cdot q)^{2} + \frac{14}{27} p \cdot q + \frac{1}{36} k^{2} \right] + (s \leftrightarrow t) + (t \leftrightarrow u)$$

$$= -\frac{52}{27\Lambda^{2}} \left(\frac{su}{t} + \frac{tu}{s} + \frac{st}{u} \right), \qquad (3.33)$$

which correctly reproduces Eq.(3.29).

Therefore, if $\theta^{\mu\nu}$ is appropriately averaged as Eq.(3.32), the scattering amplitude in the induced gravity scenario coincides with that of the ordinary gravity. The question is the meaning and validity of averaging $\theta^{\mu\nu}$. In the analysis above, we first calculated the amplitude with a fixed $\theta^{\mu\nu}$ and then averaged over its direction. On the other hand, turning back to the matrix model, $\theta^{\mu\nu}$ is determined by the commutator of matrices, and the path integral over them naturally includes the integration over $\theta^{\mu\nu}$ as a fundamental variable. In the language of NC field theory, this implies that $\theta^{\mu\nu}$ could have independent fluctuation, and that the average over $\theta^{\mu\nu}$ corresponds to the integral over $\theta^{\mu\nu}$:

$$Z = \int d\theta f(\theta) \int \mathcal{D}A \int \mathcal{D}\Phi \exp\left(-S\right), \qquad (3.34)$$

where $f(\theta)$ is some weight function. In order to justify this picture, it is necessary to check whether the average as Eq.(3.32) can actually produce the correct results in other scattering processes of gravity.

Further investigation is needed on the treatment of $\theta^{\mu\nu}$, with the emphasis on its degrees of freedom. For example, lifting $\theta^{\mu\nu}$ as an independent field is attractive in the aspect of degrees of freedom. It is possible to compensate the discrepancy between the off-shell degrees of freedom of the NC U(1) gauge field and the ordinary gravity, because $\theta^{\mu\nu}$ has six independent components. We can also consider a new matrix model or novel interpretation of matrix variables in which gravity does not necessarily come from the noncommutativity [27] [61].

3.3 Summarizing remark

We have investigated the emergent gravity scenario by examining the two-body scattering of the scalar particles exchanging the NC U(1) gauge field. As long as we take Eq.(3.23) literally, the fundamental force acting between test particles looks somewhat different from that of the ordinary gravity, although the NC U(1) gauge field can be viewed as the metric fluctuation. However, once we take the average over the direction of $\theta^{\mu\nu}$, the resulting amplitude matches that of ordinary gravity. The origin of the averaging procedure can be attributed to the path integral of the matrices in the matrix model.

On the other hand, the mechanism discussed in this chapter is somewhat awkward because we do not have manifest diffeomorphism invariance in the model. What we have done is that we break that invariance and Lorentz invariance by introducing the noncommutativity, and then integrate them out. If such a procedure correctly described gravity, it is natural to expect the existence of more straightforward description, where the system is seen at least isotropic from the beginning. In the next Chapter, we will introduce a promising treatment of the matrix model which meets such a insight.

Chapter 4

The Operator Interpretation of the Matrix Model

From this chapter, we focus on a special interpretation of the matrix model, namely the operator interpretation. Among the many interpretations, the distinctive feature of the operator interpretation is that the unitary symmetry of the matrix model is translated into the symmetries including the diffeomorphism and local Lorentz symmetries. Because there is a belief that the two symmetries is closely associated to the off-shell DoF of graviton and is essential to describing gravity, we can expect that the matrix model with the operator interpretation and briefly review the advantages of it. Note that, in this and subsequent two chapters, we mainly focus on the bosonic sector of the IIB matrix model.

4.1 The operator interpretation

In the momentum interpretation, we assume that matrices contains the information of the spacetime in the form of momenta. Roughly speaking, there we interpret matrices as first-order derivative operators:

$$A_a \sim i\partial_a + a_a(x). \tag{4.1}$$

These act on some functional space, the elements of which is defined on the flat spacetime. One can make such identification in the large-N limit. We shall generalize this treatment. Consider the general linear operator acting on the smooth functional space $C^{\infty}(\mathbb{R}^d, \mathbb{C})$, which consists of functions living in the d- dimensional flat spacetime. When we regard a matrices as such a operator, it is represented as a integral kernel or a bi-linear field. Furthermore, it is formally expanded in terms of local quantity and is written as a infinite-order derivative operator:

$$A_a: f(x) \mapsto (A_a \cdot f)(x) = \int d^d y K_a(x, y) f(y)$$

$$\sim a_a(x) f(x) + a_a^{\mu}(x) \partial_{\mu} f(x) + a_a^{\mu\nu}(x) \partial_{\mu} \partial_{\nu} f(x) + \cdots$$
(4.2)

Intuitively, the first term in Eq.(4.2) corresponds to the diagonal element of the matrix, and the terms with higher derivatives can be read as the matrix elements farther from the diagonal ones. Each coefficient field should be regarded as the field living on the d-dimensional flat space.

Before lifting the functional space to the one defined on curved spacetimes, we see how the diffeo-invariance and spacetime emerge in the matrix model. The fundamental DoF in the IIB matrix model is hermitian operators, a matrix are expanded as

$$A_{a} = a_{a}(x) + \frac{i}{2} \{a_{a}^{\mu}(x), \partial_{\mu}\} + \frac{i^{2}}{2} \{a_{a}^{(\mu\nu)}(x), \partial_{\mu}\partial_{\nu}\} + \cdots$$
(4.3)

Here, $\{\cdot, \cdot\}$ is the anticommutators and $a_a^{\mu\nu\cdots}$ is a tensor field with the Greek indices symmetric. The original U(N) symmetry in Eq.(2.25) is written as

$$\delta A_a = [\Lambda, A_a], \quad \Lambda : \text{ hermitian matrix.}$$

$$(4.4)$$

In the operator interpretation, Λ is expanded as a derivative operator as well as A_a :

$$\Lambda = \lambda(x) + \frac{i}{2} \{\lambda^{\mu}(x), \partial_{\mu}\} + \frac{i^2}{2} \{\lambda^{\mu\nu}(x), \partial_{\mu}\partial_{\nu}\} + \cdots$$
(4.5)

In the special case where we take the gauge parameter as $\Lambda = \frac{i}{2} \{\lambda^{\mu}, \partial_{\mu}\}$, the transformation law for each coefficient field in Eq.((4.4)) is

$$\delta a_a(x) = -\lambda^{\mu} \partial_{\mu} a_a(x),$$

$$\delta a_a^{\mu}(x) = -\lambda^{\nu} \partial_{\nu} a_a^{\mu}(x) + a_a^{\nu}(x) \partial_{\nu} \lambda^{\mu},$$

$$\vdots$$
(4.6)

This is nothing but the diffeomorphism transformation. On the other hand, such unitary transformation has no influence on the ten-dimensional index a, that the matrix originally has. One can re-treat it as the index associated to the local Lorentz transformation, which emerges from the Lorentz symmetry of Eq.(2.25). These symmetries imply that $a_a^{\mu}(x)$ is the vielbein.

While we have assumed that the functional space is defined on the flat spacetime, the spacetime is in turn identified as a classical solution of the model. Consider the following

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solution of the EoM:

$$A_{a} = \begin{cases} i\delta_{a}^{\mu}\partial_{\mu} & (a = 1, \cdots, d), \\ 0 & (a = d + 1, \cdots, 10) \end{cases}, \quad \Psi = 0.$$
(4.7)

This means that $a_a^{\mu}(x) = \delta_a^{\mu}$ $(a = 1, \dots, d)$, and that the fluctuation around it is equivalent to the large-*N* reduction of the Yang-Mills theory. Thus the solution represents the *d*-dimensional flat spacetime. This situation is similar to the general relativity. the EH action is defined the integral of the curvature over some curved spacetime, which is determined (at least when one consider the on-shell structure) through its EoM.

In order to reproduce curved spacetimes in from the matrix model, it might be a natural generalization to replace the derivatives with the covariant ones in the matrices:

$$A_{a} = a_{a}(x) + \frac{i}{2} \{ a_{a}^{\mu}(x), \nabla_{\mu} \} + \frac{i^{2}}{2} \{ a_{a}^{\mu\nu}(x), \nabla_{\mu}\nabla_{\nu} \} + \cdots, \qquad (4.8)$$

$$\nabla_{\mu} = \partial_{\mu} + \omega_{\mu}^{bc} \mathcal{O}_{bc}. \tag{4.9}$$

Here ω_{μ}^{bc} is the spin connection and \mathcal{O}_{bc} is the Lorentz generator, whose representation depends on that of the function it acts on.

There are two difficulties in the above naive generalization. One is involved with the definition of tensor operators, including vectors. A tensor living in the curved manifold is defined on each local patch, in a way that the locally-defined quantities are glued by the transition function in the overlap regions of the patches. Since the gluing mixes all the components of the tensor, we can state that they are simultaneously defined consistently to form a single object, a tensor. On the other hand, each component of the matrices in the matrix model is independently well-defined. Thus there is the gap between an operator and a set of matrices. Related to this problem, the second problem is that the discrepancy of the multiplication rule for matrices and covariant derivatives. For example, the product of the first and second components of the covariant derivatives gives

$$\nabla_1 \nabla_2 = \partial_1 \nabla_2 + \omega_1^{2c} \nabla_c. \tag{4.10}$$

It contains all the components. As for the matrices, $A_1 \cdot A_2$ does not contains A_3 , A_4 and so on, of course. Eq.(4.10) comes from the fact that ∇_a is a vector operators and hence is not closed on a functional space of a specific representation. This point is discussed in the next section. These difficulties tell us that we cannot regard a matrices naively as a derivative operator, $A_a \sim a_a^{\mu} \nabla_{\mu}$ for example.

4.2 Dressed derivative operators on curved spacetimes

In fact, the obstacles to generalization mentioned above are resolved by the extending the functional space.

In general, a field living on the d-dimensional spacetime belongs to the irreducible representation of the Lorentz group, Spin(d) or $Spin^{c}(d)$.¹ Consider the space that consists of the whole configurations of fields in the r-representation. This is the set of sections of a fiber bundle, whose bottom is the spacetime \mathcal{M} and whose fiber is the r-representation space $V_{\rm r}$ of the Lorentz group. The structure group of the fiber is Spin(d). We represent the fiber bundle and its section as $E_{\rm r}$ and $\Gamma(E_{\rm r})$, respectively. An action of the covariant derivative on that space changes the representation:

$$\nabla_a: \Gamma(E_{\mathbf{r}}) \longrightarrow \Gamma(T \otimes E_{\mathbf{r}}), \tag{4.11}$$

where T is the tangent bundle on the spacetime. Apparently, the action of the covariant derivative (more generally, any tensor operator) cannot be closed in the space of a specific representation.

However, this is not the case when we take the regular representation for r.

The regular representation of a group G is defined as action of G over G itself. Its representation space is described as the functional space living in the group:

$$V_{\text{reg}} \equiv \{ f(g) | f : G \longrightarrow \mathbb{C} \}, \tag{4.12}$$

$$h \in G$$
, then $(h \cdot f)(g) \equiv f(h^{-1}g)$. (4.13)

This representation contains the entire information of the group. The interesting feature of it is that its tensor product with an arbitrary irreducible representation is decomposed into the direct summation of regular representations:

$$V_{\rm r} \otimes V_{\rm reg} \simeq V_{\rm reg} \oplus \dots \oplus V_{\rm reg}.$$
 (4.14)

Here the number of V_{reg} in the RHS is equal to $\dim(V_r)$. This relation means that an element of $V_r \otimes V_{\text{reg}}$ can be write with $\dim(V_r)$ elements of V_{reg} by some diagonalization. More explicitly, suppose $f_i(g) \in V_r \otimes V_{\text{reg}}$ and i is the index for r-representation. This element transforms by the action of $h \in G$ as

$$f_i(g) \to (h \cdot f)_i(g) = R_i^{\langle r \rangle j}(h) f_j(h^{-1}g), \qquad (4.15)$$

where $R_{i}^{(\mathbf{r})j}(h)$ is the matrix element of h in r-representation. The isomorphism Eq.(4.14) is

¹We have to consider $Spin^{c}(d)$ in the case the spacetime does not have the spin structure.

obtained by considering the following element:

$$f_{(i)}(g) \equiv R^{\langle \mathbf{r} \rangle j}_{(i)}(g^{-1}) f_j(g).$$
(4.16)

This indeed has one-to-one correspondence to $f_i(g)$, and transforms as

$$f_{(i)}(g) \to (h \cdot f)_{(i)}(g) = R^{\langle r \rangle j}_{(i)}(g^{-1})(h \cdot f)_{j}(g)$$

$$= R^{\langle r \rangle j}_{(i)}(g^{-1})R^{\langle r \rangle k}_{j}(h)f_{k}(h^{-1}g)$$

$$= R^{\langle r \rangle j}_{(i)}((h^{-1}g)^{-1})f_{j}(h^{-1}g)$$

$$= f_{(i)}(h^{-1}g). \qquad (4.17)$$

Although the index (i) is for the representation matrix, it gets no influence from the above transformation. It is merely a label for the diagonalized elements. In the following the parenthesis of an index shows that the index is a mere label and does not mix under the Lorentz transformation.

Such a exceptional feature of the regular representation is equipped to $\Gamma(E_{\text{reg}})$ as well. Assuming that the spacetime \mathcal{M} is covered by the local patches $\{U_i\}$, elements of $\Gamma(E_{\text{reg}})$ is first defined locally as a map $U_i \times G \to \mathbb{C}$. Next they are consistently glued together in the overlapping region:

$$f^{[i]}: U_i \times G \longrightarrow \mathbb{C} \quad f^{[j]}: U_i \times G \longrightarrow \mathbb{C},$$

$$x \in U_i \cap U_j \quad \text{then} \quad f^{[i]}(x, g) = f^{[j]}(x, t_{ij}g). \tag{4.18}$$

Here $t_{ij}(x)$ is the transition function. Therefore, There is an isomorphism involving $\Gamma(E_{reg})$ and an arbitrary fiber bundle E_r , corresponding to Eq.(4.14):

$$\Gamma(E_{\rm r} \otimes E_{\rm reg}) \simeq \Gamma(E_{\rm reg}) \oplus \dots \oplus \Gamma(E_{\rm reg}).$$
 (4.19)

In terms of elements, This isomorphism is given by

$$f_{(k)}^{[i]}(x,g) = R_{(k)}^{\langle \mathbf{r} \rangle \ l} f_l^{[i]}(x,g).$$
(4.20)

By combining Eqs.(4.11), (4.19) and (4.20), one can see that the action of the following operator is closed (at least locally) on $\Gamma(E_{reg})$:

$$\nabla_{(a)} \equiv R_{(a)}^{\langle \mathbf{V} \rangle \ b}(g^{-1}) \nabla_b \in \operatorname{End}(\Gamma(E_{\operatorname{reg}})).$$
(4.21)

Here, $R_{(a)}^{\langle V \rangle b}(g^{-1})$ is the matrix element of g^{-1} in the vector representation. $\nabla_{(a)}$ is a scalar operator in a sense. It is seen by checking its definition. First, the ordinary covariant derivative

operator ∇_a is glued on the overlap of the patches $x \in U_i \cap U_j$ as

$$\nabla_a^{[i]} = R_a^{\langle V \rangle \ b}(t_{ij}(x)) \nabla_b^{[j]}. \tag{4.22}$$

Then the gluing of $\nabla_{(a)}$ is traced as

$$\begin{aligned}
\nabla_{(a)}^{[i]} &= R_{(a)}^{\langle V \rangle \ b}(g^{-1}) \nabla_{b}^{[i]} \\
&= R_{(a)}^{\langle V \rangle \ b}(g^{-1}) R_{b}^{\langle V \rangle \ c}(t_{ij}(x)) \nabla_{c}^{[j]} \\
&= R_{(a)}^{\langle V \rangle \ b}((t_{ij}(x)g)^{-1}) \nabla_{b}^{[j]} \\
&= \nabla_{(a)}^{[j]}.
\end{aligned}$$
(4.23)

This tells that each component of $\nabla_{(a)}$ is independently and globally defined.

As above, we have defined the "dressed" operator $\nabla_{(a)} = R^{\langle V \rangle \ b}_{(a)}(g^{-1})\nabla_b$, which circumvent the two difficulty mentioned in the previous section. Note that it is defined on $\Gamma(E_{\text{reg}})$. In fact, there is a isomorphism between this and the principal fiber bundle E_{prin} :

$$\Gamma(E_{\rm reg}) \simeq C^{\infty}(E_{\rm prin}). \tag{4.24}$$

This relation is shown from the fact that E_{prin} is locally equivalent to $U_i \times G$, and that functions from E_{prin} to \mathbb{C} obey the same gluing rule Eq.(4.18) as elements of $\Gamma(E_{\text{reg}})$. Moreover, one can show the hermicity of $i\nabla_{(a)}$ on this space. While we have constructed $\nabla_{(a)}$ on $\Gamma(E_{\text{reg}})$, it is naturally lifted to a dressed operator acting on $\Gamma(E_{\text{reg}})$ by Eq.(4.19):

$$\hat{\nabla}_{(a)}: \Gamma(E_{\mathbf{r}} \otimes E_{\mathrm{reg}}) \ni f_i(x,g) \longmapsto R_i^{\langle \mathbf{r} \rangle(j)}(g) \nabla_{(a)} f_{(j)}(x,g) \in \Gamma(E_{\mathbf{r}} \otimes E_{\mathrm{reg}}).$$
(4.25)

The above discussion is applicable to any tensor operator. That is, we can always obtain from a tensor operator the corresponding scalar operator on $C^{\infty}(E_{\text{prin}})$, by dressing its indices with the matrix elements. We can interpret the matrices A_a in the IIB matrix model as the dressed operators $A_{(a)}$ on $C^{\infty}(E_{\text{prin}})$ with G = Spin(d). It is expanded as a seriese of $\nabla_{(a)}$.

The explicit components of $\nabla_{(a)}$ are

$$\nabla_{(a)}f(x,g) = R^{\langle V \rangle \ b}_{\ (a)}(g^{-1})e^{\mu}_{b}(x) \Big(\partial_{\mu} + \omega^{cd}_{\mu}(x)\mathcal{O}_{cd} - i\tilde{a}_{\mu}(x)\Big)f(x,g).$$
(4.26)

Here, we can identify $e_b^{\mu}(x)$ to the vielbein, ω_{μ}^{cd} to the spin connection, and $\tilde{a}_{\mu}(x)$ to the U(1) gauge field. The Lorentz gengerator \mathcal{O}_{cd} is defined on $C^{\infty}(E_{\text{prin}})$ by the following operation:

$$i\epsilon^{ab}\mathcal{O}_{ab}f(x,g) \equiv f(x,(1+i\epsilon^{ab}M_{ab})^{-1}g) - f(x,g),$$
(4.27)

wehre M_{ab} is the Lorentz generator in the fundamental representation.

4.3 Emergence of curved spacetime

From the standpoint that the matrices is interpreted as the operator on $C^{\infty}(E_{\text{prin}}) \simeq \Gamma(E_{\text{reg}})$, The action of the IIB matrix model should be written as below:

$$S = \frac{1}{g^2} \operatorname{Tr} \left(-\frac{1}{4} [A_{(a)}, A_{(b)}] [A_{(c)}, A_{(d)}] \delta^{(a)(c)} \delta^{(b)(d)} - \frac{i}{2} \bar{\Psi}^{(\alpha)} (\Gamma^{(a)})^{(\beta)}_{(\alpha)} [A_{(a)}, \Psi_{(\beta)}] \right),$$
(4.28)

Here $(a), (b) = 1, \dots, 10$ and $(\alpha), (\beta) = 1, \dots, 16$. By using the matrix elements of Spin(10) in the vector and spinor representation, $R^{\langle V \rangle}$ and $R^{\langle S \rangle}$, two operators

$$A_{a} = R_{a}^{\langle \mathbf{V} \rangle (b)}(g) A_{(b)},$$

$$\Psi_{\alpha} = R_{\alpha}^{\langle \mathbf{S} \rangle (\beta)}(g) \Psi_{(\beta)}$$
(4.29)

are maps from $\Gamma(E_{\text{reg}})$ to $\Gamma(T \otimes E_{\text{reg}})$ and from $\Gamma(E_{\text{reg}})$ to $\Gamma(S \otimes E_{\text{reg}})$ (S is the spin bundle), respectively.

Since the operation of $A_{(a)}$ and $\Psi_{(\alpha)}$ is closed, we can define their products. In particular, the dressed operators are lifted to elements of $\operatorname{End}(\Gamma(E_{\rm r} \otimes E_{\rm reg}))$ in a way that it commutes the (un)dressing. In other words, the following equation holds:

$$A_{(a)}A_{(b)}A_{(c)}\cdots = R_{(a)}^{\langle V \rangle \ a'}(g^{-1})R_{(b)}^{\langle V \rangle \ b'}(g^{-1})R_{(c)}^{\langle V \rangle \ c'}(g^{-1})A_{a'}A_{b'}A_{c'}\cdots$$
(4.30)

 $A_{(a)}$ in the LHS is the lifted one. $A_{a'}$ in the RHS have influence on the indices b', c', \cdots of the function, similarly to the ordinary covariant derivative. The same discussion holds on $\Psi_{(\alpha)}$ and $\Psi_{\alpha'}$. With this lift and the orthogonality of the representation matrix $R^{\langle r \rangle T} R^{\langle r \rangle} = 1$, Eq.(4.28) is written as the followings:

$$S = \frac{1}{g^2} \operatorname{Tr} \left(-\frac{1}{4} [A_a, A_b] [A_c, A_d] \delta^{ac} \delta^{bd} + \frac{1}{2} \bar{\Psi}^{\alpha} (\Gamma^a)_{\alpha}^{\ \beta} [A_a, \Psi_{\beta}] \right).$$
(4.31)

Apparently, its form is completely the same as Eq.(4.28). However, A_a and Ψ_{α} in Eq.(4.31) is no longer matrices; they and fields from their expansion do not obey the ordinary multiplication rule.

As a first step to study Eq.(4.31), we analyze its EoM. For simplicity, we drop the fermionic DoF or set the classical solution with $\Psi = 0$. The EoM is then given by

$$[A^{(a)}, [A_{(a)}, A_{(b)}]] = 0. (4.32)$$

This is equivalent to

$$[A^a, [A_a, A_b]] = 0 (4.33)$$
due to the definition of the lift. We solve it with the following ansatz:

$$A_a = i\nabla_a = ie^{\mu}_a(x) \left(\partial_{\mu} + \omega^{bc}_{\mu}(x)\mathcal{O}_{bc} - i\tilde{a}_{\mu}(x)\right).$$
(4.34)

When we substitute this form to the LHS of Eq.(4.33), it is calculated as

$$[\nabla^{a}, [\nabla_{a}, \nabla_{b}]] = [\nabla^{a}, R_{ab}^{\ cd} \mathcal{O}_{cd} - if_{ab}]$$

$$= [\nabla^{a}, R_{ab}^{\ cd}] \mathcal{O}_{cd} + R_{ab}^{\ cd} [\nabla^{a}, \mathcal{O}_{cd}] - i[\nabla^{a}, f_{ab}]$$

$$= (\nabla^{a} R_{ab}^{\ cd}) \mathcal{O}_{cd} - R_{ab}^{\ cd} \frac{1}{2} (\delta_{ca} \nabla_{d} - \delta_{da} \nabla_{c}) - i(\nabla^{a} f_{ab})$$

$$= -2 (\nabla^{d} R^{c}_{\ b}) \mathcal{O}_{cd} - R_{b}^{\ a} \nabla_{a} - i(\nabla^{a} f_{ab})]. \qquad (4.35)$$

Here, we have used the Bianchi identity $\nabla^{(a} R_{bc}{}^{de)} = 0$ so as to move from the third line to the forth line.² The condition for Eq.(4.35) to vanish is

$$R_{ab} = 0, \quad \nabla^a f_{ab} = 0. \tag{4.36}$$

Therefore, the EoM includes the Einstein equation with vanishing energy-momentum tensor, and the Maxwell equation for the U(1) gauge field. Note that the solutions of the first equation of Eqs.(4.36) is the vacuum spacetimes. They contains no back-reaction from the gauge field. The first equation should have contain the energy-momentum tensor of the gauge field. This mismatch is understood as below. While we have consider the classical solution, the dynamical variables e_a^{μ} , ω_{μ}^{bc} and a_{μ} acquire the loop correction in their behavior. The exact EoM is Eqs.(4.36) with such corrections added, and the additional terms are expected to be proportional to powers of some dimensionfull parameter. For example, If it is the string scale, then the EoM is expected to take the form of

$$R_{ab} - \frac{1}{2}\delta_{ab}R = \alpha' \left(f_a^{\ c} f_{bc} - \frac{1}{4}\delta_{ab}f_{cd}f^{cd} \right), \quad \nabla^a f_{ab} = \alpha' \left(\cdots \right). \tag{4.37}$$

Recalling that in string theory α' -expansion is loop expansion in the world-sheet theory. The above observation might perhaps be a clue to investigate the connection between the matrix model and string theory.

The treatment we have constructed so far is applicable to general zero-dimensional matrix model. For example, consider the model with the following action:

$$S = \frac{1}{g^2} \operatorname{Tr} \left(-\frac{1}{4} [A_{(a)}, A_{(b)}] [A^{(a)}, A^{(b)}] + \frac{m^2}{2} A_{(a)} A^{(a)} \right).$$
(4.38)

²Parenthesis for several indices stands for the symmetrization: $X^{(a_1a_2\cdots a_n)} \equiv (X^{a_1\cdots a_n} + all \text{ permutation terms})/n!$. On the other hand, square bracket represents the antismmetrization.

The EoM from it is

$$[A^{(a)}, [A_{(a)}, A_{(b)}]] + m^2 A_{(b)} = 0.$$
(4.39)

By substituting Eq.(4.34), it turns into

$$R_{ab} = -2m^2 \delta_{ab}, \quad \nabla^a f_{ab} = 0. \tag{4.40}$$

As a solution of these eauations, we obtain the maximally-symmetric spacetime with the cosmological constant $\Lambda = -(d-2)m^2$. Even though the indices originally run from 0 to 9, we can obtain d-dimensional spacetime with $1 \leq d \leq 9$ by considering the ansatz $A_{(a)} = 0$ $((a) = d + 1, \dots, 10)$.

When we consider the fluctuation around a spacetime of classical solution, it can be seen as local fields living on that background. Here we have to be careful in expanding a matrix. Since it is an operator acting on $C^{\infty}(E_{\text{prin}})$, we have to take into account the expansion with respect to the derivatives in the fiber-directions, in addition to the spacetime-directions. A derivative with respect to the fiber coordinates $g \in G$ is a Lorentz generator \mathcal{O}_{ab} . The expansion of the matrix is shown by the following equations: $\nabla_{(a)}$

$$A_{a} = \hat{a}_{a}(x,g) + \frac{i}{2} \{ \hat{a}_{a}^{\ b}(x,g), \nabla_{b} \} + \frac{i^{2}}{2} \{ \hat{a}_{a}^{\ bc}(x,g), \nabla_{b} \nabla_{c} \} + \cdots, \qquad (4.41)$$

where

$$\hat{a}_{a}(x,g) = a_{(0)a}(x,g) + \frac{i}{2} \{a_{(1)a}{}^{bc}(x,g), \mathcal{O}_{bc}\} + \frac{i^{2}}{2} \{a_{(2)a}{}^{bc,de}(x,g), \mathcal{O}_{bc}\mathcal{O}_{de}\} + \cdots,$$

$$\hat{a}_{a}{}^{b}(x,g) = a_{(0)a}{}^{b}(x,g) + \frac{i}{2} \{a_{(1)a}{}^{b,cd}(x,g), \mathcal{O}_{cd}\} + \frac{i^{2}}{2} \{a_{(2)a}{}^{b,cd,ef}(x,g)\mathcal{O}_{cd}\mathcal{O}_{ef}\} + \cdots.$$

$$\vdots \qquad (4.42)$$

As for the indices of each field, those contracted with ∇_a are symmetric. As well, pairs of indices contracted with \mathcal{O}_{ab} are symmetric. It is easy to see that the expansion Eq.(4.41) is background-independent, *i.e.* it is well-defined without specification of the spacetime where the covariant derivative is defined. When we write the operator as³

$$A_a = \hat{a}_a + \hat{a}_a^{\ b} \nabla_b + \cdots \tag{4.43}$$

Rewriting the second term on another background, $(e'^b_\mu, \omega'^{bc}_\mu)$ gives⁴

$$\hat{a}_a^{\ b} \nabla_b = \hat{a}_a^{\ b} e_b^\mu \left(\partial_\mu + \omega_\mu^{bc} \mathcal{O}_{bc} \right)$$

³we neglect hermicity for simplicity.

⁴Now we take G = Spin(d), and set to zero the U(1) gauge field.

$$= \hat{a}_{a}^{\ \mu} \left(\partial_{\mu} + \omega_{\mu}^{bc} \mathcal{O}_{bc} \right)$$

$$= \hat{a}_{a}^{\prime \mu} \left(\partial_{\mu} + \omega_{\mu}^{\prime bc} \mathcal{O}_{bc} \right) + \hat{a}_{a}^{\prime \mu} \delta \omega_{\mu}^{bc} \mathcal{O}_{bc}$$

$$= \hat{a}_{a}^{\prime \mu} \nabla_{\mu}^{\prime} + \hat{a}_{a}^{\prime b} e_{b}^{\prime \mu} \delta \omega_{\mu}^{\prime bc} \mathcal{O}_{bc}, \qquad (4.44)$$

with $\delta \omega_{\mu}^{bc} = \omega_{\mu}^{bc} - \omega_{\mu}^{\prime bc}$. Although there appear some extra term, it can be absorbed into redefinition of $a'_{(1)a}{}^{bc}$, $a'_{\prime(2)a}{}^{bc,de}$, \cdots , that consists in the expansion with respect to \mathcal{O}_{ab} Eq.(4.42). The operator on the new background written as

$$A_a = \hat{a}'_a + \hat{a}'^b_a \nabla'_b + \cdots$$

$$\tag{4.45}$$

This represents the background-independence of Eq.(4.41).

In the above expansion Eq.(4.41) and Eq.(4.42), the coefficient fields belong to the regular representation of the Lorentz group. They are further expanded into all the irreducible representations according to Peter-Weyl theorem:⁵

To summarize, an operator A_a is triply expanded into infinite series, and it contains numerous DoF of local fields. Since this fact makes analysis difficult, it is required to reduce those DoF under some assumption. Below we briefly discuss the symmetries of several fields by restricting them to zero modes in the expansion Eq.(4.46).⁶

The unitary symmetry of the model is

$$\delta A_{(a)} = i[\Lambda, A_{(a)}], \quad \delta \Psi_{(\alpha)} = i[\Lambda, \Psi_{(\alpha)}], \tag{4.47}$$

or equivalently,

$$\delta A_a = i[\Lambda, A_a], \quad \delta \Psi_\alpha = i[\Lambda, \Psi_\alpha]. \tag{4.48}$$

We can also expand Λ as Eqs.(4.41) and (4.42). In terms of the coefficient fields, it exhibits very large symmetry. As a part of it, with the choice of the gauge parameter as $\Lambda \lambda(x)$, the

⁵Peter-Weyl theorem itself is applicable to the case of compact groups. However, its generalization to locallycompact group is possible.

⁶In [27], the authors required that (a), (α) transform as vector and spinor indices under the right action of Lorentz group. The left and right actions of a group commute. Using this fact and the definition of dressed operators Eq.(4.29) implies that the undressed coefficient fields must be invariant under the right action, hence is independent of fiber coordinate $g \in G$.

lowest spin field transforms as

$$\delta a_a(x) = \partial_a \lambda(x). \tag{4.49}$$

This is just U(1) gauge transform, with the identification of $a_a(x)$ to the gauge field. Note that $a_a^{\mu}(x)$ is interpreted as the total vielbein: $a_a^{\mu}(x) = e_a^{\mu}(x) + \delta e_a^{\mu}(x)$.

As another choice, when we take the gauge parameter as $\Lambda = \frac{i}{2} \{\lambda^{\mu}(x), \partial_{\mu}\}$, this is diffeomorphism, the same one as Eq.(4.6):

$$\delta a_a(x) = -\lambda^{\mu}(x)\partial_{\mu}a(x),$$

$$\delta a_a^{\mu}(x) = -\lambda^{\nu}(x)\partial_{\nu}a_a^{\mu}(x) + a_a^{\nu}(x)\partial_{\nu}\lambda^{\mu}(x),$$

$$\delta a_a^{\mu\nu}(x) = -\lambda^{\rho}(x)\partial_{\rho}a_a^{\mu\nu}(x) + 2a_a^{\rho(\mu}(x)\partial_{\rho}\lambda^{\nu)}(x),$$

$$\vdots$$
(4.50)

A general field $a_a^{\mu_1\cdots\mu_{s-1}}(x)$ transforms as a rank-(s-1) symmetric tensor. It indicates that the indices contracted with the covariant derivatives are the coordinate indices.

Moreover, the symmetry Eq.(4.48) includes the local Lorentz symmetry. It is realized by choosing the parameter as $\Lambda = \frac{i}{2} \{\lambda_{(1)}^{ab}(x), \mathcal{O}_{ab}\}$. The background vielbein and e_a^{μ} and spin connection ω_{μ}^{ab} transform as

$$\delta e_a^{\mu}(x) = -\lambda_{(1)a}{}^b(x) e_b^{\mu}(x),$$

$$\delta \omega_{\mu}^{ab}(x) = \partial_{\mu} \lambda_{(1)}{}^{ab}(x) + 2\lambda_{(1)c}{}^{[a}(x) \omega_{\mu}^{b]c}(x).$$
(4.51)

In the same manner, the fluctuating fields around them, the gauge field a_a , the total vielbein $a_a^b(x)$ and spin connection $a_{(1)a}^{bc}$, transform according to the following law:

$$\delta a_{a}(x) = -\lambda_{(1)a}{}^{b}(x)a_{b}(x),$$

$$\delta a_{a}^{b}(x) = -\lambda_{(1)a}{}^{c}(x)a_{c}^{b}(x) - \lambda_{(1)c}{}^{b}(x)a_{a}^{c}(x),$$

$$\delta a_{(1)a}{}^{bc} = -\lambda_{(1)a}{}^{d}(x)a_{(1)d}{}^{bc} + 2\lambda_{(1)}{}^{[b}{}_{d}(x)a_{(1)a}{}^{c]d}(x).$$
(4.52)

We have dealt with the expansion and symmetry of the bosonic matrices A_a . We can make the same discussion on the fermionic matrices Ψ_{α} . For example, its lowest spin field $\chi_{\alpha}(x)$ transforms under the above local Lorantz transformation as

$$\delta\chi_{\alpha}(x) = -\frac{1}{4}\lambda_{(1)} {}^{bc}(x) (\Gamma_{bc})_{\alpha\beta} \chi_{\beta}(x).$$
(4.53)

Chapter 5

Stability of the Matrix Model with the Operator Interpretation

While the matrix model has the advantages mentioned in the last chapter, It has some subtly or difficulty. The most problematic feature is that the model contains infinitely many fields. It is a obstacle against making concrete analysis. Furthermore, all of those numerous fields are massless at the classical level. This might cause instability in the model, especially through the radiative correction. Existence of gauge symmetries to remove their longitudinal component is in question as well.

Therefore, in this chapter, we study the matrix model with the operator interpretation with the focus on such aspects. One desirable prescription is to truncate DoF so that one can control the theory. We attempt to make it, and find that naive truncation leads to a inconsistency of the model. Then we analyze the gauge symmetries and radiative corrections. We find evidence partially for positive-definiteness of the matrix model.

5.1 Minimality of the operator interpretation

As we have mentioned in the previous chapter, general elements of $End(C^{\infty}(\mathcal{M}))$ cannot be understood as matrices because they are not generally closed as an algebra under the multiplication nor the commutator [,]. However, we can actually construct a set of operators which are closed under those operations by restricting $End(C^{\infty}(\mathcal{M}))$. In this section, we attempt to construct the minimal consistent operator interpretation. As a result, we will find the necessity of the principal bundle, and the original operator interpretation is indeed the minimal.

5.1.1 An attempt to construct a further minimal model

Among various possibilities, the simplest one is the set of the first-order differential operators:

$$A_a \in \mathcal{A} \equiv \left\{ f^{\mu}(\hat{x})\hat{p}_{\mu} + g(\hat{x}) \mid \hat{x}^{\mu}h(x) = x^{\mu}h(x), \ \hat{p}_{\mu}h(x) = -i\frac{\partial h}{\partial x^{\mu}}, \ h \in C^{\infty}(\mathcal{M}) \right\}.$$
(5.1)

One can easily check that \mathcal{A} is closed under the commutator. \mathcal{A} can also be understood as a set of quantum mechanical operators constructed by \hat{x}^{μ} and $\hat{p}_{\mu} = -i\partial/\partial \hat{x}^{\mu}$. In the following discussion, we consider the "semi-classical" limit of those operators:

$$(\hat{x}^{\mu}, \hat{p}_{\mu}) \rightarrow (x^{\mu}, p_{\mu})$$

$$(5.2)$$

$$[\hat{f}, \hat{g}] \rightarrow -i\{f, g\} = -i\sum_{\mu=1}^{D} \left(\frac{\partial f}{\partial p_{\mu}} \frac{\partial g}{\partial x^{\mu}} - \frac{\partial f}{\partial x^{\mu}} \frac{\partial g}{\partial p_{\mu}} \right),$$
(5.3)

$$\operatorname{Tr}(\cdots) \rightarrow \int d^D x \int d^D p(\cdots).$$
 (5.4)

In this limit, the bosonic EoM $[A^b, [A_b, A_a]] = 0$ in the matrix model becomes 1

$$\{A^b, \{A_b, A_a\}\} = 0.$$
(5.6)

The simplest solution of this equation is $A_a = \delta^{\mu}_a p_{\mu}$, and this corresponds to the flat spacetime as does in other interpretations. Now, let us consider the fluctuation around this solution:

$$A_{a} = \delta^{\mu}_{a} p_{\mu} + \widetilde{A}_{a}(x, p)$$

= $\delta^{\mu}_{a} p_{\mu} + f_{a}(x) + f_{a}^{\ \mu}(x) p_{\mu}$
= $e_{a}^{\ \mu}(x) p_{\mu} + f_{a}(x), \ e_{a}^{\ \mu}(x) = \delta^{\mu}_{a} + f_{a}^{\ \mu}(x).$ (5.7)

In the semi-classical limit, the bosonic part of the original IIB action becomes

$$S_{\rm IIB} = -\frac{1}{4g^2} \text{Tr} \left([\hat{p}_a + \tilde{A}_a(\hat{x}, \hat{p}), \hat{p}_b + \tilde{A}_b(\hat{x}, \hat{p})]^2 \right) \to \frac{1}{4g^2} \int d^D x \int d^D p \ G_{ab}(x, p) G^{ab}(x, p),$$
(5.8)

$$\int d^{D}x \int d^{D}p \ f\{g,h\} = \int d^{D}x \int d^{D}p \ g\{h,f\},$$
(5.5)

we have

$$\delta S = \frac{1}{g^2} \int d^D x \int d^D p \, \{A^a, A^b\} \{\delta A_a, A_b\} = \frac{1}{g^2} \int d^D x \int d^D p \, \delta A_a \{A_b, \{A^a, A^b\}\}$$

which coincides with the semi-classical limit of Eq.(4.33). We can easily check Eq.(5.5) by assuming that the integral of the total derivative terms vanish.

 $^{^{1}}$ As a consistency check, we derive it from the action in the semi-classical limit. As long as the Poisson bracket satisfies the cyclicity condition

where

$$G_{ab}(x,p) = \{A_a, A_b\} = \delta^{\mu}_a \partial_{\mu} \widetilde{A}_b(x,p) - \delta^{\mu}_b \partial_{\mu} \widetilde{A}_a(x,p) + \{\widetilde{A}_a, \widetilde{A}_b\}.$$
(5.9)

Furthermore, the original U(N) transformation Eq.(4.48) now becomes ²

$$\begin{split} \delta \widetilde{A}_a(x,p) &= \{ \delta^{\mu}_a p_{\mu} + \widetilde{A}_a(x,p), \Lambda(x,p) \} \\ &= -\delta^{\mu}_a \partial_{\mu} \Lambda(x,p) + \{ \widetilde{A}_a(x,p), \Lambda(x,p) \}, \end{split} \tag{5.11}$$

which leads to the transformation of each fields as

$$\delta f_a(x) = e_a{}^{\mu}(x)\partial_{\mu}\lambda(x) - (\partial_{\mu}f_a(x))\lambda^{\mu}(x), \qquad (5.12)$$

$$\delta e_a{}^\mu(x) = e_a{}^\nu(x)\partial_\nu\lambda^\mu - (\partial_\nu e_a{}^\mu(x))\lambda^\nu(x), \tag{5.13}$$

where we have also expanded $\Lambda(x, p)$ as $\lambda(x) + p_{\mu}\lambda^{\mu}(x)$. Here, the first term in Eq.(5.13) corresponds to the gauge transformation, and the other terms correspond to the diffeomorphism. Note that the action Eq.(5.8) is, of course, invariant under Eq.(5.11), or Eqs.(5.13) and (5.12) up to a total derivative term:

$$\delta S_{\rm IIB} = \frac{1}{8g^2} \int d^D x \int d^D p \ \{G_{ab}, \Lambda\} G^{ab}.$$
(5.14)

Therefore, the EoM Eq.(5.6) is also invariant under the transformation.

Let us now consider the dynamics of the fluctuations at the classical level. By substituting Eq.(5.7) to Eq.(5.6), we obtain

$$\Box_{ab}f^b + \partial_{\mu}f^b \left(\bar{\partial}_a e_b{}^{\mu} - \bar{\partial}_b e_a{}^{\mu}\right) + p_{\mu} \left[\Box_{ab}e^{b\mu} + (\partial_{\nu}e^{b\mu}) \left(\bar{\partial}_a e_b{}^{\nu} - \bar{\partial}_b e_a{}^{\nu}\right)\right] = 0, \tag{5.15}$$

where

$$\Box_{ab} = \bar{\partial}_c \cdot \bar{\partial}^c \delta_{ab} - \bar{\partial}_a \cdot \bar{\partial}_b, \tag{5.16}$$

$$\bar{\partial}_a = e_a{}^\mu \partial_\mu. \tag{5.17}$$

Note that $\bar{\partial}_a$'s do not commute each other because of $e_a{}^{\mu}$. Eq.(5.15) holds for arbitrary p_{μ} , so it is equivalent to the following two equations

$$\Box_{ab}f^b + \partial_{\mu}f^b \left(\bar{\partial}_a e_b{}^{\mu} - \bar{\partial}_b e_a{}^{\mu}\right) = 0, \qquad (5.18)$$

$$\Box_{ab}e^{b\mu} + (\partial_{\nu}e^{b\mu})\left(\bar{\partial}_{a}e_{b}^{\ \nu} - \bar{\partial}_{b}e_{a}^{\ \nu}\right) = 0.$$
(5.19)

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$
(5.10)

²This gauge symmety does exist as long as the Poisson bracket satisfies the Jacobi identity

The Poisson bracket in this section (Eq.(5.3)) satisfies it.

42 Chapter 5 Stability of the Matrix Model with the Operator Interpretation

Now let us study how many DoF remain at the liberalized (free) level:

$$\begin{cases} \left(\Box\delta_{ab} - \partial_a\partial_b\right)f^b = 0, \ \delta f_a = \partial_a\lambda - (\partial_\mu f_a)\lambda^\mu, \\ \left(\Box\delta_{ab} - \partial_a\partial_b\right)f^{b\mu} = 0, \ \delta f_a^{\ \mu} = (\delta_a^\nu + f_a^\nu)\partial_\nu\lambda^\mu - (\partial_\nu f_a^{\ \mu})\lambda^\nu, \end{cases}$$
(5.20)

where $\partial_a = \delta^{\mu}_a \partial_{\mu}$ and $\Box = \partial^{\mu} \partial_{\mu}$ are the ordinary differential and d'Alembert operator, respectively. As for f_a , this is completely the same as the ordinary gauge field. Thus, by choosing the Lorentz gauge, its EoM, gauge condition, and residual symmetry are given by

$$\Box f_a = 0, \ \partial_a f^a = 0, \ \delta f_a = \partial_a \lambda, \ \Box \lambda = 0, \tag{5.21}$$

from which one can see that only two physical DoF remain. Next, as for $f_a{}^{\mu}$, we can also choose a Lorentz-like gauge

$$\partial_a f^{a\mu} = 0 \tag{5.22}$$

by using the diffeomorphism. As a result, we obtain the following EoM, gauge condition and residual symmetry:

$$\Box f_a{}^{\mu} = 0, \ \partial_a f^{a\mu} = 0, \ \delta f_a{}^{\mu} = \partial_a \lambda^{\mu}, \ \Box \lambda^{\mu} = 0$$
(5.23)

The above gauge conditions for f^a and $f^{a\mu}$ are summarized with a condition for A_a :

$$\{p_a, A^a(x, p)\} = 0, \tag{5.24}$$

which fixes the original U(N) gauge symmetry.

In order to discuss physical DoF, let us move to the Fourier component $\tilde{f}_a^{\mu}(k)$. In the light cone coordinate, we can choose $k = (k^+, 0, 0, 0)$ without loss of generality. Then, the gauge fixing condition and the residual symmetry give

$$\tilde{f}^{-\mu} = 0, \ \tilde{f}^{+\mu} = 0,$$
 (5.25)

so the remaining DoF are

$$\tilde{f}^{i+}, \ \tilde{f}^{i-}, \ \tilde{f}^{ij} \ (i=1,2).$$
 (5.26)

The first two are vectors, and the third one coincides with the massless states of bosonic closed string theory: graviton, Kalb-Ramond field and dilaton. In the presence of both \tilde{f}^{i+} and \tilde{f}^{i-} , the theory violates positivity and is unstable.³ However, we can eliminate \tilde{f}^{i-} by assuming the following additional condition:

$$\partial_{\mu}f^{a\mu} = 0, \tag{5.27}$$

³Although we are treating the Euclidean matrix model, the above analysis also applies to the Lorentzian model straightforwardly. In this sense we refers to the stability here.

which leads to a condition for the diffeomorphism:

$$\partial_{\mu}\lambda^{\mu}(x) = 0 \,\, \Leftrightarrow \delta f_a{}^a = 0. \tag{5.28}$$

The second equation means that the metric fluctuation is traceless, so the above transformation is the volume-preserving diffeomorphism. Note that the condition Eq.(5.27) is consistent with the commutator. Consider

$$\left\{f_{a}{}^{\mu}p_{\mu},g_{b}{}^{\mu}p_{\mu}\right\} = p_{\mu}\left[(\partial_{\nu}f_{a}{}^{\mu})g_{b}{}^{\nu} - (\partial_{\nu}g_{b}{}^{\mu})f_{a}{}^{\nu}\right] \equiv p_{\mu}F_{ab}{}^{\mu},\tag{5.29}$$

then $F_{ab}^{\ \mu}$ also satisfies

$$\partial_{\mu}F_{ab}{}^{\mu} = 0 \tag{5.30}$$

if $f_a^{\ \mu}$ and $g_b^{\ \mu}$ satisfy $\partial_{\mu} f_a^{\ \mu} = \partial_{\mu} g_b^{\ \mu} = 0$. The symmetry in this case is the volume-preserving diffeomorphism, not the general one There are a number of works to discuss the gravitational system that possesses only the volume-preserving diffeomorphism. It is sometimes called the unimodular gravity [63]. That theory is equivalent to the general relativity in many aspects and is a reasonable gravitational system.

With the discussion above, it seems to be the case that the present would-be minimal choice of operator space Eq.(5.1). However, it has some difficulty. From EoM, Eqs.(5.18), (5.19), one can see that there is no suitable *D*-dimensional action that produces those EoMs because the second one has no f_a dependence. This fact means that we should not expand $\tilde{A}_a(x,p)$ by p_{μ} at the action level, and that we should treat the stationary point of the action Eq.(5.8) with respect to the matrices $\tilde{A}_a(x,p)$. Apart from this problem, it is seen as a fault that there seems to be no way to define the restriction Eq.(5.27) in terms of matrices. Therefore we have to consider the result of the previous section unsatisfactory in the viewpoint of the matrix model. Furthermore, the local Lorentz transformation that act on the indices a, b, \cdots is not given explicitly. All of the problem suggest that the treatment in the previous section lacks some piece in order to contain gravitational DoF. The interpretation discussed so far is thus too-small operator interpretation.

5.1.2 The necessity of the principal bundle

Contrary to the above description, the original one proposed in [27,29], which we have introduced in the previous section, includes the local Lorentz and general diffeo-invariance as a part of the original U(N) symmetry Eq.(4.48). When one reduce DoF of the model, we can consider a slightly restricted version of it. That is, it is possible to interpret matrices as operators on E_{prin} which are first-order in spacetime derivative. In the following, we repeat the same analysis as above taking the original interpretation, and then discuss the restriction of operators to first-order ones. Although the result itself is trivial, it demonstrates the advantage of the principal bundle, that plays an essential role in the description.

In the original operator interpretation, we take as the operator space as

$$A_{a} \in \mathcal{A} \equiv \left\{ F(\hat{x}, \hat{p}, \hat{g}, \mathcal{O}) \mid \hat{x}^{\mu} f(x, g) = x^{\mu} f(x, g), \ \hat{p}_{\mu} f(x, g) = -i \frac{\partial f}{\partial x^{\mu}}, \\ \hat{g}_{ij} f(x, g) = g_{ij} f, \mathcal{O}_{ab} f(x, g) = -(M_{ab} \cdot g)_{ij} \frac{\partial f}{\partial g_{ij}}, \\ f \in C^{\infty}(E_{\text{prin}}) \right\},$$
(5.31)

where M_{ab} is the fundamental representation of $\hat{\mathcal{O}}_{ab}$. In order to consider the semi-classical limit, we need to define the Poisson bracket on the phase space of the principal bundle. Note that we also have the derivative operator \mathcal{O}_{ab} in addition to \hat{p}_{μ} , and they are replaced with c-numbers in the semi-classical limit:

$$\hat{p}_{\mu} \to p_{\mu}, \quad \mathcal{O}_{ab} \to t_{ab}.$$
 (5.32)

For the base space directions (x, p), it is natural that we employ the same Poisson bracket as Eq.(5.3). Furthermore, it seems reasonable to assume that there is no nontrivial Poisson structure between the (x, p) and (g, t) directions. For the fiber directions (g, t), we shall define the Poisson bracket naturally from the algebraic structure of Spin(D). As a result, the nonzero components of the Poisson bracket are given by

$$\{p_{\mu}, x^{\nu}\} = \delta_{\mu}^{\ \nu}, \{t_{s}, g_{ij}\} = i(\mathcal{M}_{s}g)_{ij}, \ \{t_{s}, t_{t}\} = if_{stu}t_{u}.$$
(5.33)

where g_{ij} is an element of Spin(D) in fundamental representation⁴ and \mathcal{M}_s is the fundamental representation of \mathcal{O}_s . In these expressions, the subscript *s* represents antisymmetric double local Lorentz indices [*ab*]. For example, the last equation of Eq.(5.33) actually means $\{t_{ab}, t_{cd}\} =$ $i[(\mathcal{M}_{ab})_{ce}t_{ed} + (\mathcal{M}_{ab})_{de}t_{ce}]$. Using Eq.(5.33), we can write the Poisson bracket of general functions on the principal bundle as

$$\{f,h\} \equiv \frac{\partial f}{\partial p_{\mu}} \frac{\partial h}{\partial x^{\mu}} - \frac{\partial f}{\partial x^{\mu}} \frac{\partial h}{\partial p_{\mu}} + i(\mathcal{M}_{s}g)_{ij} \left(\frac{\partial f}{\partial t_{s}} \frac{\partial h}{\partial g_{ij}} - \frac{\partial f}{\partial g_{ij}} \frac{\partial h}{\partial t_{s}}\right) + if_{stu}t_{s} \frac{\partial f}{\partial t_{t}} \frac{\partial h}{\partial t_{u}}.$$
 (5.34)

This Poisson bracket satisfies the cyclicity condition Eq.(5.5) and the Jacobi identity Eq.(5.10).

The important point of the analysis is to take into account the factor $R_{(a)}^{\ b}$. For general functions $F_{(a)} = R_{(a)}^{\ c}F_c$ and $H_{(b)} = R_{(b)}^{\ d}H_d$, their Poisson bracket becomes,

$$\{F_{(a)}, H_{(b)}\} = R_{(a)}^{\ c} R_{(b)}^{\ d} \left[\{F_c, H_d\} + i(M_s F)_c \frac{\partial H_d}{\partial t_s} - i \frac{\partial F_c}{\partial t_s} (M_s H)_d\right].$$
(5.35)

⁴Even though some constraint is posed on $\{g_{ij}\}$ so that $g \in Spin(D)$, the Poisson bracket (5.33) is well-defined.

In particular,

$$\{p_{(a)}, H_{(b)}\} = R_{(a)}^{\ c} R_{(b)}^{\ d} \left[\partial_c H_d + i(M_s p)_c \frac{\partial H_d}{\partial t_s}\right]$$
$$\equiv R_{(a)}^{\ c} R_{(b)}^{\ d} D_c H_d.$$
(5.36)

We have introduced the twisted derivative D_c by the above equation.

Let us now consider the fluctuation around the flat background $p_{(a)} = R_{(a)}^{\ \ b}(g^{-1})\delta^{\mu}_{b}p_{\mu}$:

$$A_{(a)} = p_{(a)} + \widetilde{A}_{(a)}$$

= $R_{(a)}^{\ b}(g^{-1}) \left[\delta_b^{\mu} p_{\mu} + \widetilde{A}_b(x, p, g, t) \right]$
= $R_{(a)}^{\ b}(g^{-1}) \left[\delta_b^{\mu} p_{\mu} + f_b(x, g) + f_b^{\mu}(x, g) p_{\mu} + \omega_b^s(x, g) t_s \right]$
= $R_{(a)}^{\ b}(g^{-1}) \left[e_b^{\ \mu}(x, g) p_{\mu} + f_b(x, g) + \omega_b^{\ s}(x, g) t_s \right].$ (5.37)

Here, in the third and firth lines, we have restricted $\tilde{A}_b(x, p, g, t)$ to be first order in (p, t). This restriction is algebraically consistent because the Poisson bracket is closed among the first-order operators in (p, t). The semi-classical limit of the original action then becomes

$$S_{\text{IIB}} = \frac{1}{4g^2} \int dq \ \{A_{(a)}, A_{(b)}\} \{A^{(a)}, A^{(b)}\}$$

= $\frac{1}{4g^2} \int dq \ \tilde{G}_{ab} \tilde{G}^{ab},$ (5.38)

where

$$\int dq \equiv \int d^D x \int d^D p \int_{Spin(D)} dg \int_{\mathbb{R}^{d(d-1)/2}} dt,$$
(5.39)

$$\tilde{G}_{ab} = D_a \tilde{A}_b - D_b \tilde{A}_a + \{\tilde{A}_a, \tilde{A}_b\} + i(M_s \tilde{A})_a \frac{\partial A_b}{\partial t_s} - i \frac{\partial A_a}{\partial t_s} (M_s A)_b.$$
(5.40)

Note that indices contracted outside the Poisson bracket can be replaced with ones without parentheses due to the orthogonality of $R_{(a)}{}^{b}(g^{-1})$. Of course this action is invariant under the gauge transformation written as

$$\delta \widetilde{A}_{(a)} = \{\Lambda(x, p, g, t), p_{(a)} + \widetilde{A}_{(a)}\},$$
(5.41)

which is equivalent to the transformation

$$\delta \widetilde{A}_a = -\partial_a \Lambda - i \frac{\partial \Lambda}{\partial t_s} (M_s p)_a + \{\Lambda, \widetilde{A}_a\} - i \frac{\partial \Lambda}{\partial t_s} (M_s \widetilde{A})_a.$$
(5.42)

In terms of expanded fields in Eq.(5.37), and the expanded gauge parameters

$$\Lambda(x, p, g, t) = \lambda(x, g) + \lambda^{\mu}(x, g)p_{\mu} + \lambda^{s}(x, g)t_{s}, \qquad (5.43)$$

the transformation lows are summarized as follows:

• "U(1)" gauge transformation

$$\begin{cases} \delta f_a = -e_a^{\ \mu} \partial_\mu \lambda - i \omega_a^{\ s} (\hat{M}_s \cdot \lambda), \\ \delta e_a^{\ \mu} = 0, \\ \delta \omega_a^{\ s} = 0. \end{cases}$$
(5.44)

• "Diffeomorphism" transformation

$$\begin{cases} \delta f_a = \lambda^{\mu} \partial_{\mu} f_a, \\ \delta e_a^{\ \mu} = -e_a^{\ \nu} \partial_{\nu} \lambda^{\mu} + \lambda^{\nu} \partial_{\nu} e_a^{\ \mu} - i \omega_a^{\ s} (\hat{M}_s \cdot \lambda^{\mu}), \\ \delta \omega_a^{\ s} = \lambda^{\mu} \partial_{\mu} \omega_a^{\ s}. \end{cases}$$
(5.45)

• "Local Lorentz" transformation

$$\begin{cases} \delta f_a = i\lambda^s (M_s f)_a + i\lambda^s (\hat{M}_s \cdot f_a), \\ \delta e_a^{\ \mu} = i\lambda^s (M_s e^{\ \mu})_a + i\lambda^s (\hat{M}_s \cdot e_a^{\ \mu}), \\ \delta \omega_a^{\ s} = -e_a^{\ \mu} \partial_\mu \lambda^s + i\lambda^t f_{\ tu}^s \omega_a^{\ u} - i\lambda^t (M_t \omega^{\ s})_a + i\lambda^t (\hat{M}_t \cdot \omega_a^{\ s}) - i\omega_a^{\ t} (\hat{M}_t \cdot \lambda^s). \end{cases}$$

$$(5.46)$$

Here, we have introduced an operator \hat{M}_s , which is defined as

$$i(\hat{M}_s \cdot f)(x, p, g, t) \equiv \frac{\partial}{\partial \epsilon^s} f(x, p, e^{i\epsilon^t \mathcal{M}_t} g, t) \Big|_{\epsilon=0}.$$
(5.47)

Choosing the specific gauge parameters, which is independent from g_{ij} , these transformations are the exact U(1) gauge, diffeomorphism and local Lorentz transformations, respectively.

If one expands each field according to Peter-Weyl theorem:

$$f(x,g) = \sum_{\text{r:irr.rep.}} R_i^{\langle \mathbf{r} \rangle j}(g) f_j^{\langle \mathbf{r} \rangle i}(x), \qquad (5.48)$$

the operation of \hat{M}_s is equivalent to infinite simal transformation for each representation:

$$(\hat{M}_s \cdot f)(x, p, g, t) = \sum_{\text{r:irr.rep.}} M_{s\,i}^{\langle \mathbf{r} \rangle k} R_k^{\langle \mathbf{r} \rangle j}(g) f_j^{\langle \mathbf{r} \rangle i}(x),$$
(5.49)

where i, j, k are identified to be local Lorentz indices.

Now we focus on the dynamics of the system which has no g-dependence. Here we identify $\omega_a^{\ s}$ to the spin connection. The EoM are given by

$$\{A^{(b)}, \{A_{(b)}, A_{(a)}\}\} = 0.$$
(5.50)

We shall restrict the space of the operators as much as possible, posing on e_a^{μ} and ω_a^{s} the metricity condition:

$$\nabla_{\mu}e_{a}^{\ \nu}=0, \tag{5.51}$$

and assume that $\omega_a{}^s$ is torsion-free. In other words, we consider the first-order differential operators which contains only f_a and $e_a{}^{\mu}$ as the independent DoF. The space of such operators is closed with respect to the ordinary commutator. From these two condition one can deduce the following formula:

$$\partial_a e_b^{\ \mu} - \partial_b e_a^{\ \mu} + \omega_{ab}^{\ c} e_c^{\ \mu} - \omega_{ba}^{\ c} e_c^{\ \mu} = e_a^{\mu} e_b^{\nu} (\Gamma_{\mu\nu}^{\lambda} - \Gamma_{\nu\mu}^{\lambda}) = 0.$$
(5.52)

The linearized EoM are then

$$\begin{cases} \partial_b F^{ab} = 0, \\ \Box f_a^{\mu} - \partial_a \partial^b f_b^{\mu} + \partial^b \omega_b{}^s (iM_s)_a{}^{\mu} - \partial^b \omega_a{}^s (iM_s)_b{}^{\mu} - (\partial^b \omega_a{}^s - \partial_a \omega^{b,s})(iM_s)_b{}^{\mu} = 0, \\ \Box \omega_a{}^s - \partial_a \partial^b \omega_b{}^s = 0. \end{cases}$$
(5.53)

However, by using Eq.(5.52) and the explicit form of the vector representation matrices $(iM_s)_c^d = (iM_{ab})_c^d = \delta_{ac}\delta_b^d - \delta_a^d\delta_{bc}$, one can derive an equation:

$$\Box f_a^{\mu} - \partial_a \partial^b f_b^{\mu} + \partial^b \omega_b^{\ s} (iM_s)_a^{\ \mu} - \partial^b \omega_a^{\ s} (iM_s)_b^{\ \mu} = 0.$$
(5.54)

Therefore the second equation of Eq.(5.53) falls into a simple equation

$$\partial^b \omega_{ab}^{\ \mu}(e) - \partial_a \omega_b^{b \ \mu}(e) = 0, \qquad (5.55)$$

and the third equation of Eq.(5.53) is not independent from the second. Consequently, the EoM for the spacetime fluctuation is given by

$$\frac{1}{2}\Box h_a^{\ \mu} - \frac{1}{2}(\partial_a \partial^b h_b^{\mu} + \partial^{\mu} \partial^b h_{ba}) + \frac{1}{2}\partial_a \partial^{\mu} \delta^b_{\ \nu} h_b^{\ \nu} = 0,$$
(5.56)

$$h_a^{\ \mu} = f_a^{\ \mu} + f_a^{\ \mu}. \tag{5.57}$$

One can easily see that Eq.(5.56) is equivalent to the usual linearized Einstein equation, through combining it with its own trace part.

Looking over the above equations, we find that the crucial point is that the dynamics of the vielbein $f_a{}^{\mu}$ emerges only through the spin connection $\omega_a{}^s$ in contrast to the previous section, where there is no cancellation in the explicit kinetic terms for $f_a{}^{\mu}$.

Furthermore, Eq.(5.56) shows no dynamics of the antisymmetric part of f_a^{μ} . This is not a

problem since the local Lorentz transformation should be used to make the local Lorentz frame parallel to the coordinate system, which means f_a^{μ} is made symmetric.

Once we set $e_a^{\mu} = e_a^{\mu}$, we need not pose further extra conditions, because the condition to eliminate the unstable mode Eq.(5.27) is automatically satisfied through gauge-fixing condition for the diffeomorphism Eq.(5.22). In this case, the theory is stable and the dynamical variables independent of the fiber coordinates g are the U(1) gauge field and the pure vielbein, only. Therefore we have showed that the theory is stable without any additional condition, as expected in the previous work. This feature suggest that the principal bundle is essential for equipping general diffeomorphism in the operator interpretation.

Another essential point is that we have set the metricity condition. We will encounter in spin connection a part of longitudinal components propagating. They are the torsion and socalled non-metricity. While the dynamics of the torsion was discussed in [28], it remains to be settled whether all of those propagating components observe the positivity. On the other hand, it is welcome that the metricity condition can be imposed with no conflict with dynamics. We will push this structure forward in order to analyze symmetries for fields that are coefficients of higher derivatives, in the next section.

5.2 Higher spin gauge symmetries in the IIB matrix model

We have analyzed the symmetries and dynamics of lower spin fields. On the other hand, we must treat more physics than that of gravity and U(1) gauge field in the matrix model. Since string theory or its strong tension limit contains higher spin fields, the matrix model should be. Therefore, we do not truncate operators to be first-order. However, it is still meaningful that we focus only on a coefficient field of specific order derivatives. In this section, we take such an approach.

U(N) symmetry of the matrices are translated into a lot of symmetries of local fields, in addition to diffeomorphism and local Lorentz symmetry. On the other hand, one has to introduce many DoF, which, written in terms of local fields, formally appears to be massless higher spin fields. It is not clear whether these fields are actually physical DoF, and whether there are gauge symmetries which eliminate their potentially dangerous components, such as the longitudinal components of a vector field. In this section, we investigate the symmetry of higher spin fields in some class, and see that the auxiliary fields need to be introduced in order to close the gauge transformation.⁵ There are gauge symmetries to remove the longitudinal components of the would-be spin *s* field and parts of the auxiliary fields. In addition, we pose some generalization of torsion-free conditions, which enable us to rewrite the rest parts of the auxiliary fields in terms of the physical field. As a result, we see that when we focus on the spin

 $^{^{5}}$ With another interpretation, where one regards matrices as noncommutative coordinates, a higher spin structure has recently been found as well [64, 65]. It has similarity to that discussed in this section.

s fields, the gauge symmetries and the torsion-free conditions leave the transverse components of the fields in the totally symmetric representation.

5.2.1 Higher spin symmetries in U(N)

In this subsection, we investigate such aspect of the matrix model. We focus on a restricted class of the fields, namely the bosonic fields that are independent of group coordinates g. Thus they are not in the product representation of the tensor one and the regular one.

In the ordinary field theory, a massless spin s field is described by a rank-s symmetric double-traceless tensor field [66] :

$$a_{\mu(s)}(x)$$
 s.t. $a^{\nu_1\nu_2}_{\ \nu_1\nu_2\mu(s-4)} = 0,$ (5.58)

where $\mu(s)$ denotes the symmetrized indices $(\mu_1 \cdots \mu_s)$.⁶ The gauge transformation of it is written as

$$\delta a_{\mu(s)} = \partial_{\mu} \lambda_{\mu(s-1)} \tag{5.59}$$

with $\lambda_{\mu(s-1)}$ is a rank-(s-1) symmetric traceless tensor parameter. We formally express the symmetrized indices the same letter.

Turning back the matrix model, again we take the sami-classical limit. This limit enables us to ignore the order of the derivatives and coordinates in the expansion of $A_a(x,g)$, and simplifies the analysis. Naively, it seems natural that a spin s field in the flat spacetime background is described in the operator interpretation as

$$A^{a} = p^{a} + a^{a\mu(s-1)}(x)p_{\mu(s-1)},$$
(5.60)

where $p_{\mu(s-1)} := p_{(\mu_1} \cdots p_{\mu_{s-1}})$. The first term in Eq.(5.60) is for the background. We attempt to find the appropriate gauge transformation for the field. First, the most simple gauge parameter we have is the following form:

$$\Lambda = \lambda^{\mu(s-1)}(x)p_{\mu(s-1)},$$
(5.61)

which realizes the transformation

$$\delta A^{(a)} = \{A^{(a)}, \Lambda\}$$

$$\Leftrightarrow \quad \delta a^{a\mu(s-1)} = \partial^a \lambda^{\mu(s-1)} + O(a \times \lambda). \tag{5.62}$$

In the analysis, we will ignore the second term in the RHS of Eq. (5.62). Although the validity

⁶In the spin-3 case, any tracelessness is not imposed.

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Figure 5.1: The representational structure of the field. The bar between the indices in the field denotes the tensor product, as mentioned below. The field is of a tensor product of vector representation and rank-(s-1) symmetric one. It is decomposed into rank-s symmetric representation and the rest "hook-type" one. Note that all the representations contain the trace part, and they are reducible. (Source: [45], doi:10.1016/j.nuclphysb.2019.114801)

of it needs to be analyzed, in this section we assume that the discussion around the elimination of the DoF can be held focusing only on the inhomogeneous term. Of course, Eq.(5.62) is not sufficient for the elimination of the longitudinal components of $a^{a\mu(s-1)}$, because it includes non-totally symmetric tensor components. It comes from the fact that $a^{a\mu(s-1)}$ behave as the product representation of the vector one (having the index *a*) and rank-(*s* - 1) symmetric tensor one ($\mu(s - 1)$). The representation is decomposed into two representations and their traces (Fig.(5.1)). The extra components are the two-row representation tensor, characterized by the second tableaux in Fig.(5.1).

In terms of the field, we rewrite $a^{a\mu(s-1)}$ as $a^{a|\mu(s-1)}$ and the decomposition as

$$a^{a|\mu(s-1)} = h^{a\mu(s-1)} + b^{a,\mu(s-1)}.$$
(5.63)

From now on, we separate the indices for tensor products by bars, and for different rows in the Young tableaux by commas. The sequence of indices without commas or bars are symmetric.

The problem is whether there is any gauge transformation to remove $b^{a,\mu(s-1)}$. We take a new gauge parameter in the following form:

$$\Lambda = \lambda^{\mu(s-1)} p_{\mu(s-1)} + \lambda^{c,d\mu(s-2)} t_{cd} \, p_{\mu(s-2)}.$$
(5.64)

With this parameter we get the transformation law as below:

$$\delta A^{a} = \partial^{a} \lambda^{\mu(s-1)} p_{\mu(s-1)} - \frac{1}{2} (\lambda^{a,\mu(s-1)} - \lambda^{\mu,a\mu(s-2)}) p_{\mu(s-1)} + \partial^{a} \lambda^{c,d\mu(s-2)} t_{cd} p_{\mu(s-2)}.$$
(5.65)

In terms of the fields this is written as

$$\delta h^{a\mu(s-1)} = \partial^{(a}\lambda^{\mu(s-1))},\tag{5.66}$$

$$\delta b^{a,\mu(s-1)} = -\frac{s}{2(s-1)} \lambda^{a,\mu(s-1)} + (\partial^a \lambda^{\mu(s-1)})_{\mathbb{P}(s-1,1)}.$$
(5.67)

In the above equations, $\mathbb{P}(m, n)$ represents the projection into the representation for the Young tableaux which consists of an *m*-boxes row and an *n*-boxes row. The coefficient s/(s-1) appears

from the normalization of the projection, $(\lambda^{a,\mu(s-1)})_{\mathbb{P}(s-1,1)} = \lambda^{a,\mu(s-1)}$. Eq.(5.67) indicates that we can remove all the components of $b^{a,\mu(s-1)}$ by this transformation. Furthermore, we can remove the longitudinal components of totally symmetric tensor $a^{a\mu(s-1)}$ with $b^{a,\mu(s-1)}$ kept zero by choosing $\lambda^{a,\mu(s-1)}$ appropriately.

However, Eq.(5.65) includes the extra change of A^a , *i.e.* the third term in the RHS. In order to close the transformation law, it is necessary to introduce new DoF. Therefore we are forced to consider the operator of the following form:

$$A^{a} = p_{a} + a^{a|\mu(s-1)}(x)p_{\mu(s-1)} + \omega^{a|c,d\mu(s-2)}t_{cd}p_{\mu(s-2)},$$
(5.68)

where $\omega^{a|c,d\mu(s-2)}$ is an additional field. Then we have again the problem of whether $\omega^{a|c,d\mu(s-2)}$ can be removed by any gauge transformation.

Before discussing the gauge transformation, note that $\omega^{a|c,\mu(s-1)}$ is seen as a higher spin counterpart for the spin connection. In the spin-2 case, the spin connection $\omega^{a|b,c}$ is written in terms of vielbein through the torsion-free condition

$$T^{a|bc} = \partial^{b} e^{c|a} - \partial^{c} e^{b|a} + \omega^{c|d,a} e_{d|}{}^{b} - \omega^{b|d,a} e_{d|}{}^{c} = 0.$$
(5.69)

Keeping this fact in mind, we shall pose the generalized torsion-free condition:

$$\frac{2(s-1)}{s}(\partial^b a^{c|\mu(s-1)} - \partial^c a^{b|\mu(s-1)}) + \omega^{b|c,\mu(s-1)} - \omega^{c|b,\mu(s-1)} = 0.$$
(5.70)

In spin 2 case, this coincides with Eq.(5.69) with the vielbein being small fluctuation around the flat space. The general solution of Eq.(5.70) is written as

$$\omega^{a|b,\mu(s-1)} = \frac{s-1}{s} \Big(\partial^b a^{a|\mu(s-1)} - \partial^a a^{b|\mu(s-1)} + \partial^b a^{\mu|a\mu(s-2)} - \partial^\mu a^{b|a\mu(s-2)} + \partial^a a^{\mu|b\mu(s-2)} - \partial^\mu a^{a|b\mu(s-2)} \Big) + \zeta^{ab,\mu(s-1)},$$
(5.71)

where $\zeta^{ab,\mu(s-1)}$ is an arbitrary tensor corresponding to the Young tableaux whose two rows consist of (s-1) and 2 boxes, respectively. Therefore the additional field $\omega^{a|b,\mu(s-1)}$ is written with $a^{a|\mu(s-1)}$ through the above equation, except for components of $\zeta^{ab,\mu(s-1)}$.

Fortunately, it is possible to eliminate $\zeta^{ab,\mu(s-1)}$ by another gauge transformation. we choose a gauge parameter of the form below:

$$\Lambda = \lambda^{\mu(s-1)} p_{\mu(s-1)} + \lambda^{c,d\mu(s-2)} t_{cd} p_{\mu(s-2)} + \lambda^{c(2),d(2)\mu(s-3)} t_{cd}^2 p_{\mu(s-3)},$$
(5.72)

with the notations are defined as

$$t_{cd}^n := t_{c_1 d_1} \cdots t_{c_n d_n}. \tag{5.73}$$

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The gauge transformation of A^a is then

$$\begin{split} \delta A^{a} = &\partial^{a} \lambda^{\mu(s-1)} p_{\mu(s-1)} - \frac{1}{2} (\lambda^{a,\mu(s-1)} - \lambda^{\mu,a\mu(s-2)}) p_{\mu(s-1)} + \partial^{a} \lambda^{c,d\mu(s-1)} t_{cd} \, p_{\mu(s-2)} \\ &- \frac{1}{2} (\lambda^{ac,d\mu(s-2)} - \lambda^{\mu c,ad\mu(s-3)} + \lambda^{ca,d\mu(s-2)} - \lambda^{c\mu,da\mu(s-3)}) t_{cd} \, p_{\mu(s-2)} \\ &+ \partial^{a} \lambda^{c(2),d(2)\mu(s-3)} t_{cd}^{2} \, p_{\mu(s-3)}, \end{split}$$
(5.74)

hence

$$\delta a^{a|\mu(s-1)} = \partial^a \lambda^{\mu(s-1)} - \frac{s}{2(s-1)} \lambda^{a,\mu(s-1)}, \qquad (5.75)$$

$$\delta\omega^{a|c,d\mu(s-2)} = \partial^a \lambda^{c,d\mu(s-2)} - \frac{s}{4(s-2)} \lambda^{ac,d\mu(s-2)}.$$
(5.76)

Eq.(5.75) is equivalent to Eqs.(5.67), while Eq.(5.76) is consistent with the imposed condition Eq.(5.70). As a result, a part of $\omega^{a|c,d\mu(s-2)}$ can be removed by the second term in Eq.(5.76), and the rest part is written in terms of $a^{a|\mu(s-1)}$. Therefore, there is no independent DoF in $\omega^{a|c,d\mu(s-2)}$.

Due to the last term in Eq.(5.74), we have to introduce further additional field in order to close the gauge transformation. Remarkably, the present discussion is somewhat similar to that of the higher spin gauge theory in form language [67].⁷ In the viewpoint of the gauge transformation, we find that the present analysis can be done almost in parallel with the study in [69], although the generalized torsion-free conditions are different. Therefore, we state the discussion briefly. In order to close gauge transformation completely, we have to consider the operator of the following form:

$$A^{a} = p^{a} + a^{a|\mu(s-1)}p_{\mu(s-1)} + \sum_{n=1}^{s-1} \omega^{a|c(n),d(n)\mu(s-1-n)} t^{n}_{cd} p_{\mu(s-1-n)}$$
(5.77)

Appropriate gauge parameter is given by

$$\Lambda = \lambda^{\mu(s-1)} p_{\mu(s-1)} + \sum_{n=1}^{s-1} \lambda^{c(n),d(n)\mu(s-1-n)} t_{cd}^{n} p_{\mu(s-1-n)}, \qquad (5.78)$$

which leads to the transformation laws

$$\delta a^{a|\mu(s-1)} = \partial^a \lambda^{\mu(s-1)} - \frac{s}{2(s-1)} \lambda^{a,\mu(s-1)}, \tag{5.79}$$

 $\delta\omega^{a|c(n),d(n)\mu(s-1-n)} = \partial^a \lambda^{c(n),d(n)\mu(s-1-n)}$

$$-\frac{s}{2^n(s-1-n)}\lambda^{ac(n),d(n)\mu(s-1-n)}. \quad (1 \le n \le s-2),$$
(5.80)

$$\delta\omega^{a|c(s-1),d(s-1)} = \partial^a \lambda^{c(n),d(n)}.$$
(5.81)

⁷For a review see [68].

Now we impose a set of generalized torsion-free conditions

$$\frac{2^{n}(s-1-n)}{s} (\partial^{a} \omega^{b|c(n),d(n)\mu(s-1-n)} - \partial^{b} \omega^{a|c(n),d(n)\mu(s-1-n)}) + \omega^{a|bc(n),d(n)\mu(s-1-n)} - \omega^{b|ac(n),d(n)\mu(s-1-n)} = 0. \quad (1 \le n \le s-2)$$
(5.82)

Due to this equations, a part of each extra fields $\omega^{a|c(n),d(n)\mu(s-1-n)}$ is written in terms of the "lower" extra fields recursively. At the same time, the rest part of $\omega^{a|c(n),d(n)\mu(s-1-n)}$ can be removed with the gauge transformation, in particular with the second term in Eq.(5.80). As for the highest extra field $\omega^{a|c(s-1),d(s-1)}$, there is no gauge parameter with which we can eliminate the DoF of the field. However, the generalized torsion-free condition for it can be solved and the whole part of it is expressed with $\omega^{a|c(s-2),d(s-2)\mu}$ without ambiguity:

$$\omega^{a|bc(s-2),d(s-1)} = -\frac{1}{2} \left[\partial^{a} \omega^{b|c(s-2),d(s-1)} - \partial^{b} \omega^{a|c(s-2),d(s-1)} - (s-1) \left(\partial^{d} \omega^{a|c(s-2),bd(s-2)} - \partial^{a} \omega^{d|c(s-2),bd(s-2)} - \partial^{b} \omega^{d|c(s-2),ad(s-2)} - \partial^{b} \omega^{d|c(s-2),ad(s-2)} - \partial^{d} \omega^{b|c(s-2),ad(s-2)} + \partial^{d} \omega^{a|bc(s-3),cd(s-2)} - \partial^{a} \omega^{d|bc(s-3),cd(s-2)} \right) \right]$$
(5.83)

In the derivation of the above equation, we have made use of the Bianchi identity

$$\omega^{a|c(s-1),cd(s-2)} = \omega^{a|dc(s-2),d(s-1)} = 0, \tag{5.84}$$

and a relation which is derived from it,

$$\omega^{a|dc(s-2),bd(s-2)} = -\frac{1}{s-1}\omega^{a|bc(s-2),d(s-1)}.$$
(5.85)

The fact that the $\omega^{a|c(s-1),d(s-1)}$ can be solved is on the same foot as that the spin connection can be solved in terms of the vielbein.

According to these discussion, we can conclude that $b^{a,\mu(s-1)}$ and all the extra fields $\omega^{a|c(n),d(n)\mu(s-1-n)}$ are eliminated either with gauge transformation or with generalized torsion-free condition. In this sense, the extra fields are auxiliary fields. Furthermore, we can still remove the longitudinal component of $h^{a\mu(s-1)}$ by an appropriate gauge transformation. It is driven both by the parameter $\lambda^{\mu(s-1)}$ and the higher rank parameters $\lambda^{c(n),d(n)\mu(s-1-n)}$. The former removes the longitudinal components directly, while the latter compensate the change in $\omega^{a|c(n),d(n)\mu(s-1-n)}$ and keep them zero. Therefore, we are left the transverse component of $h^{a\mu(s-1)}$ as the only physical DoF.

After gauge-fixing and eliminating fields except $h^{a\mu(s-1)}$, the matrices take the following

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form:

$$A^{a} = p^{a} + \sum_{n=0}^{s-1} \frac{1}{n!} \partial^{c(n)} h^{d(n)a\mu(s-1-n)} t^{n}_{cd} \ p_{\mu(s-1-n)}.$$
(5.86)

On the other hand, the explicit form of the residual gauge degrees of freedom which remove the longitudinal components of $h^{a\mu(s-1)}$ is written as

$$\Lambda = \sum_{n=0}^{s-1} \frac{s-n}{n!} \partial^{c(n)} \lambda^{d(n)\mu(s-1-n)} t_{cd}^n \ p_{\mu(s-1-n)}.$$
(5.87)

Then we find that the unitary transformation of matrices is equivalent of a higher spin gauge transformation:

$$\delta A^a = \{A^a, \Lambda\} \quad \Leftrightarrow \quad \delta h^{a\mu(s-1)} = \partial^{(a}\lambda^{\mu(s-1))}. \tag{5.88}$$

Here $h^{a\mu(s-1)}$ does not belong to an irreducible representation, since it contains the trace part. In this respect, there is some difference between the field and the ordinary higher spin fields, which satisfies the double-traceless condition Eq.(5.58). In the ordinary case, the condition is required to make the theory gauge-invariant, with the gauge parameter being traceless. As for our case, we already have gauge invariance with the traceful field $h^{a\mu(s-1)}$ and the parameter $\lambda^{\mu(s-1)}$. Thus we need no further condition. The longitudinal traceless part is removed by gauge transformation, since $\lambda^{\mu(s-1)}$ is traceful. Therefore, we have no positivityviolating component, though it is unclear whether the lower spin fields as the trace parts can be eliminated.

5.2.2 Equations of motion for higher spin fields

In the previous section, we have discussed the higher spin symmetry in the kinematical aspect. In other words, what we have shown is that the unitary transformation of the matrices, when translated into terms of derivative operators, includes gauge transformations, and that they remove components of fields except the transverse ones of totally symmetric part.

However, the transformation law for the totally symmetric field is somewhat different from the Fronsdal theory, due to absence of traceless conditions both for the field and for gauge parameter. Therefore there emerges one question: in what form the equations of motion are. Even in the free part, we do not expect it to be the Fronsdal operator. Apparently it conflicts with the existence of higher spin symmetry. In this section we explicitly write down the equations of motions for the field and discuss their structure.

In this part we truncate the interaction part. It is still worth analyzing since the ordinary higher spin field theory is established rigorously as free field theory. We shall expand the equations of motion for matrices by substituting Eq.(5.86):

$$0 = \{p_b + A_b, \{p^b + A^b, p^a + A^a\}\}$$

$$\sim \{p_b, \{p^b, A^a\}\}$$

$$= \left[\partial^{c(s-2)} \left(\Box h^{a\mu d(s-2)} - 2\partial^b \partial^{(a} h_b^{\mu)d(s-2)} + \partial^a \partial^\mu \bar{h}^{d(s-2)}\right)\right] t_{cd}^{s-2} p_\mu$$

$$+ \left[\partial^{c(s-1)} \left(\partial^b \partial^a h_b^{d(s-1)} - \Box h^{ad(s-1)}\right)\right] t_{cd}^{s-1},$$
(5.89)

where $\bar{h}^{d(s-2)} = h_b^{bs(s-2)}$. It is remarkable that the coefficients of $t_{cd}^n p_{\mu(s-1-n)}$ $(0 \le n \le s-3)$ vanish, leaving the two equations (neglecting the interaction):

$$\partial^{c(s-2)} \left(\Box h^{a\mu d(s-2)} - 2\partial^b \partial^{(a} h_b^{\ \mu)d(s-2)} + \partial^a \partial^\mu \bar{h}^{d(s-2)} \right) = 0, \tag{5.90}$$

$$\partial^{c(s-1)} \left(\partial^b \partial^a h_b^{\ d(s-1)} - \Box h^{ad(s-1)} \right) = 0.$$
(5.91)

Here we emphasize that the indices of c's and d's are symmetrized respectively, while the two types of indices are antisymmetrized. Note that Eq.(5.91) is obtained from Eq.(5.90) by taking a derivative ∂^{ν} and antisymmetrizing μ and ν . Therefore, we have derived a single equation of motion for the higher spion field.

Eq.(5.90) is different from the Fronsdal equation, or equivalently, from the vanishing condition of the Fronsdal operator:

$$\Box h^{a\mu d(s-2)} - 2\partial^b \partial^{(a} h_b^{\ \mu d(s-2))} + \partial^{(a} \partial^\mu \bar{h}^{d(s-2))} = 0.$$
(5.92)

Rather, Eq.(5.90) can be understood as a vanishing condition of a kind of curvature. the equation can be written as

$$\eta_{cc} R^{c(s),d(s)} = 0, \quad R^{c(s),d(s)} = \partial^{c(s)} h^{d(s)}.$$
(5.93)

In the viewpoint of symmetry, $R^{c(s),d(s)}$ corresponds to the Young tableaux of two rows, both of which consist of s boxes. This quantity is the generalization of the (linearized) Riemann curvature, that was discussed in [70]. It is the only gauge-invariant quantity without the traceless conditions. Thus the appearance of the generalized curvature in the equations of motion is consistent, since we have higher spin symmetry without traceless conditions. Moreover, in s = 2 case, the above equation is nothing but the Rich-flat condition obtained in Eq.(4.33). This coincidence is reasonable because we need neither double-tracelessness for the field, nor the tracelessness for the gauge parameter. in the higher spin case, we conclude that the higher spin field is not the Fronsdal field, but the generalized curvature field.

On the other hand, once we take the interaction into account, the analysis will get far complicated. In the free part of the equations of motion, we obtained the vanishing condition of a derivative operator of degree-(s - 1). However, in the presence of the interaction terms coming from products of the second term in Eq.(5.86), the degree of the derivative operator increases, to 2s - 3 at most. In that case, we have many independent equations since all the coefficients of a degree-(2s - 3) derivative operator must vanish. Moreover, as long as we consider a single field of spin s only, most of those equations should be regarded as some constraints on the interaction terms. One way to avoid it is to introduce new fields and to make each equation contain free kinetic terms for the field. It is likely that a true consistent description is obtained only when we take into account fields of all spin at the same time. it is equivalent to considering the most general derivative operators of infinite degree, without any truncation.

Such formulation is too complicated to study by directly expanding matrices as derivative operators. A new method to investigate needs to be established. Related to this issue, it is remarkable that the higher spin gauge transformation in the matrix model includes both homogeneous and inhomogeneous terms. Our study has focused on the inhomogeneous term only, since we have examined whether there are sufficient gauge parameters to eliminate unwanted components. The exact gauge symmetries are far complicated, and it enables the model to include interaction terms. The relationship to the various no-go theorems that prohibit the existence of interacting higher-spin particles needs to be analyzed as well. Further investigation is required. The analysis of higher spin symmetries for a general class of fields is another open question. As reviewed in the previous section, the essential part of the operator interpretation is actually the introduction of the principal bundle. Although we have dealt with the zero modes in the fiber direction, $a^{a|\mu(s-1)}(x, y)$, the study on symmetries of general fields are a future work. The stability, which seems to be put in danger by higher derivative term in the equations of motions (5.93), also requires further investigation. The study will be reported elsewhere.

5.3 One-loop corrections and induced mass terms

We have confirmed that the original description with the principal bundle is minimal possibility to contain gravity in the operator interpretation. Then it is necessary to study quantum correction to that model. In this section, we investigate some mass term induced by loop diagrams and see that the theory is still stable. We compute one-loop corrections to the action Eq.(5.38) and its supersymmetrized version, and read off the mass term for each field.

5.3.1 One-loop computation for the bosonic action

In order to compute one-loop corrections, one should confront a problem of constructing the propagator. It is unclear whether one can define the propagator $1/D_a^2$ with Eq.(5.36), because D_a has the explicit dependence on the coordinates (x, p). Instead of directly define $1/D_a^2$, we

transform the coordinates and redefine functions as

$$t_s \to \tilde{t}_s = t_s - ix^{\mu} \delta^a_{\ \mu} (M_s)^{\ b}_a \delta^{\ \nu}_b p_{\nu}, \tag{5.94}$$

$$X(x, p, g, t) \to X(x, p, g, \tilde{t}).$$
(5.95)

For the redefined functions with indices, the Poisson bracket Eq.(5.34) changes to the following form:

$$\{F_{(a)}, H_{(b)}\} \longrightarrow \{F_{(a)}, H_{(b)}\}'$$

$$= R_{(a)}^{\ c} R_{(b)}^{\ d} \left[\frac{\partial F_c}{\partial p_{\mu}} \frac{\partial H_d}{\partial x^{\mu}} - \frac{\partial F_c}{\partial x^{\mu}} \frac{\partial H_d}{\partial p_{\mu}} + i(\mathcal{M}_s g)_{ij} \left(\frac{\partial F_c}{\partial \tilde{t}_s} \frac{\partial H_d}{\partial g_{ij}} - \frac{\partial F_c}{\partial g_{ij}} \frac{\partial H_d}{\partial \tilde{t}_s} \right)$$

$$+ i f_{stu} \tilde{t}_s \frac{\partial F_c}{\partial \tilde{t}_t} \frac{\partial H_d}{\partial \tilde{t}_u}$$

$$- i(M_s p)_{\mu} \frac{\partial F_c}{\partial p_{\mu}} \frac{\partial H_d}{\partial \tilde{t}_s} - i(xM_s)^{\mu} \frac{\partial F_c}{\partial \tilde{t}_s} \frac{\partial H_d}{\partial x^{\mu}}$$

$$+ i(M_s p)_{\mu} \frac{\partial F_c}{\partial \tilde{t}_s} \frac{\partial H_d}{\partial p_{\mu}} + i(xM_s)^{\mu} \frac{\partial F_c}{\partial x^{\mu}} \frac{\partial H_d}{\partial \tilde{t}_s}$$

$$+ i(M_s F)_c \frac{\partial H_d}{\partial \tilde{t}_s} - i \frac{\partial F_c}{\partial \tilde{t}_s} (M_s H)_d \right].$$
(5.96)

In particular, $D_c H_d \rightarrow \partial_c H_d$, and thus we can define the propagator for $A_a(x, p, g, \tilde{t})$. For convenience, we will write the new coordinates \tilde{t}_s as t_s in the following.

Now let us compute the loop corrections to the action Eq.(5.38) with the background field method. We consider the quantum fluctuation of A_a . Expanding Eq.(5.38) as

$$A_a \to A_a + \phi_a,\tag{5.97}$$

and adding the gauge-fixing terms, we obtain

$$S = \int dq \left[\frac{1}{2} \partial_a A_b \partial^a A^b - \frac{1}{2} \partial^a A_a \partial^b A_b + R_{(a)}^{\ c} R_{(b)}^{\ d} \partial_c A_d \{A^{(a)}, A^{(b)}\}' + \frac{1}{4} \{A_{(a)}, A_{(b)}\}' \{A^{(a)}, A^{(b)}\}' + \frac{1}{2} \partial_a \phi_b \partial^a \phi^b + R_{(a)}^{\ c} R_{(b)}^{\ d} (\partial_c \phi_d - \partial_d \phi_c) \{\phi^{(a)}, A^{(b)}\}' + \frac{1}{2} R_{(a)}^{\ c} R_{(b)}^{\ d} (\partial_c A_d - \partial_d A_c) \{\phi^{(a)}, \phi^{(b)}\}' + \frac{1}{2} \{\phi_{(a)}, A_{(b)}\}' \{\phi^{(a)}, A^{(b)}\}' - \frac{1}{2} \{\phi_{(a)}, A_{(b)}\}' \{\phi^{(b)}, A^{(a)}\}' + \frac{1}{2} \{\phi_{(a)}, \phi_{(b)}\}' \{A^{(a)}, A^{(b)}\}' - b \Box c + R_{(a)}^{\ c} \partial_c b \{A^{(a)}, c\}' \Big].$$
(5.98)

Here we have taken the Feynman gauge, and introduced the Faddeev-Popov ghost c and antighost b. Because we are interested in the induced mass terms for A_a , we calculate loop corrections with a condition

$$\partial_a A_b = 0. \tag{5.99}$$

It is convenient to introduce the "momentum variables" (k, r, h, u), which are dual to (x, p, g, t), and an operator $(\mathbb{M}_s)_a^b$, which is defined as

$$(i\mathbb{M}_s \cdot A)_a(p,g,t) = \frac{\partial}{\partial \epsilon^s} \left[(e^{-i\epsilon^t M_t} A)_a (e^{i\epsilon^v M_v} p, e^{i\epsilon^w \mathcal{M}_w} g, e^{i\frac{1}{2}\epsilon^{w'} \mathcal{M}_{w'}^{\mathrm{adj}}} t) \right] \Big|_{\epsilon=0},$$
(5.100)

where $\mathcal{M}_{f}^{\mathrm{adj}}$ is the adjoint representation of \mathcal{O}_{s} . With these preparation, one can read off the propagators and the vertices from Eq.(5.98). The factors needed for the calculation are bellows:

$$\langle \phi_a(k,r,h,u)\phi_b(-k,-r,-h,-u)\rangle = \frac{\delta_{ab}}{k^2},\tag{5.101}$$

$$\langle b(k,r,h,u)c(-k,-r,-h,-u)\rangle = \frac{1}{k^2},$$
(5.102)



$$= i \left[(k \cdot \bar{r}) \{ k^b \delta^{ac} + k^a \delta^{bc} - 2k^c \delta^{ab} \} - \{ k^b u_s (\mathbb{M}_s)_e^c \delta^{ae} + k^a u_s (\mathbb{M}_s)_e^c \delta^{be} - 2(k \cdot \mathbb{M}_s)^c \delta^{ab} \} \right],$$
(5.103)



$$= -\left[(k \cdot \bar{r})^2 (2\delta^{ab} \delta^{cd} - \delta^{ad} \delta^{bc} - \delta^{ac} \delta^{bd}) - u_s u_t \{ 2\delta^{ab} \delta^{ef}(\mathbb{M}_s)_e^{\ c}(\mathbb{M}_t)_f^{\ d} - \delta^{ae} \delta^{bf}(\mathbb{M}_t)_e^{\ d}(\mathbb{M}_s)_f^{\ c} - \delta^{ae} \delta^{bf}(\mathbb{M}_t)_e^{\ c}(\mathbb{M}_s)_f^{\ d} \} \right], \quad (5.104)$$



Figure 5.2: One-loop diagrams needed for the calculation of induced mass terms. (Source: [44], doi:10.1016/j.nuclphysb.2017.10.011)



$$= i \left[-(k \cdot \bar{r})k^{c} + u_{s}(k \cdot \mathbb{M}_{s})^{c} \right].$$
(5.105)

By calculating the one-loop diagrams (Fig.5.2), we obtain the mass terms induced in the effective action:

$$\Gamma \Big|_{\text{mass}} = -\frac{1}{g^2} \frac{d-2}{d+2} \int dq \left[\alpha \left(\frac{\partial A_b}{\partial p_a} \right)^2 - \alpha \frac{4}{d} \left(\frac{\partial A_a}{\partial p_a} \right)^2 - \beta \frac{2(d-2)}{d^2} ((\mathbb{M}_s \cdot A)_a)^2 + \gamma \left(\frac{\partial A_a}{\partial t_s} \right)^2 \right],$$

$$= -\frac{1}{g^2} \frac{d-2}{d+2} \int dq \left[\alpha \{ x^{(a)}, p_{(b)} + A_{(b)} \}^2 - \alpha \frac{4}{d} \{ x^{(a)}, p_{(a)} + A_{(a)} \}^2 - \beta \frac{2(d-2)}{d^2} \{ t_s, p_{(a)} + A_{(a)} \}^2 + \gamma \{ g_{ij}, p_{(a)} + A_{(a)} \}^2 \right],$$

$$-\beta \frac{2(d-2)}{d^2} \{ t_s, p_{(a)} + A_{(a)} \}^2 + \gamma \{ g_{ij}, p_{(a)} + A_{(a)} \}^2 \right],$$

$$(5.106)$$

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$$\alpha = \int \frac{d^d k d^d r dh d^{d(d-1/2)} u}{(2\pi)^{d+d+d(d-1)/2}}, \quad \beta = \int \frac{d^d k d^d r dh d^{d(d-1/2)} u}{(2\pi)^{d+d+d(d-1)/2}} \frac{u^2}{k^2}, \quad \gamma \propto \int d^d k \frac{1}{k^2} \operatorname{Tr}(\mathbb{M}_s \mathbb{M}_s).$$
(5.107)

The second equality holds up to unimportant constant. Since α , β and γ are divergent, we need to take the cutoff regularization. Also note that the meaning of γ is somewhat ambiguous and its numerical factor is not determined. However, \mathbb{M}_s 's can be regarded as a sort of angular momentum operators. It is then natural to consider γ as the sum of the eigenvalues of their Casimir operator, along with the momentum integral.

The EoM with gauge condition is now changed as

$$\Box A_{a} + \alpha \frac{2(d-2)}{d+2} \left(\frac{\partial}{\partial p_{b}}\right)^{2} A_{a} - \alpha \frac{8(d-2)}{d(d+2)} \frac{\partial}{\partial p_{a}} \left(\frac{\partial A_{b}}{\partial p_{b}}\right) - \gamma \left(\frac{\partial}{\partial t_{s}}\right)^{2} A_{a} -\beta \frac{4(d-2)^{2}}{d^{2}(d+2)} \left(\mathbb{M}_{s}\mathbb{M}_{s} \cdot A\right)_{a} = 0, \quad (5.108)$$
$$\partial_{a} A^{a} = 0. \quad (5.109)$$

To derive the above equation, one has to be careful to take variation of $((\mathbb{M}_s \cdot A)_a)^2$. As mentioned above, $(\mathbb{M}_s)_a^b$ is equivalent to angular momentum operator, and its operation is multiplying the corresponding generator to all the indices of the field. Therefore it should be identified to be the derivative with respect to fiber direction, and one obtains in the action $((\mathbb{M}_s \cdot A)_a)^2 = -A^a(\mathbb{M}_s\mathbb{M}_s \cdot A)_a$ by partial integration.

One can analyze Eq.(5.108) by expanding A_a . Note that the three terms in the first line in Eq.(5.108) vanish since we have restricted A_a to be first order in p and t_s . From the last term in Eq.(5.108), we get mass terms for each field component. While \mathbb{M}_a can be interpreted as the derivative with respect to the fiber coordinates, it is the generators of Spin(D) and $(\mathbb{M}_s)^2$ is the Casimir operator. Consequently, each field arising from P-W expansion Eq.(5.48) gets the positive mass squared, the value of which is the eigenvalue of the Casimir operator according to the representation. The important point here is that a field of any nontrivial representation of Lorentz group, which has implicit g-dependence, acquires a mass term. This means that vielbein fluctuation $f_a^{\mu}(x)$ get massive as well, even though it has no explicit g-dependence.

5.3.2 Inclusion of the fermionic sector

The above result seems to be quite bad news for us, since there is no gravitational field when one take into account the quantum correction. However, the original IIB matrix model has the fermionic sector as well, and it is possible that its quantum correction drastically changes the result, as is the case in most supersymmetric theories. Therefore, we shall repeat the same analysis as above on the action obtained from the full IIB matrix model. In this subsection, we write the essence of the analysis briefly.



Figure 5.3: The additional loop diagram. The loop is of Majorana fermion. (Source: [44], doi:10.1016/j.nuclphysb.2017.10.011)

The action corresponding Eq.(5.98) is now

$$S = S|_{\text{bos}} + \frac{1}{2}\bar{\psi}^T C\Gamma^a \partial_a \psi + \frac{1}{2}(\bar{\psi}^T C\Gamma^{(a)})_{(\alpha)} \{A_{(a)}, \psi_{(\alpha)}\}',$$
(5.110)

where $S|_{\text{bos}}$ is the bosonic part Eq.(5.98), Γ^a is the *d*-dimensional gamma matrix and *C* is the charge conjugation matrix. One can easily check that the Eq. (5.110) is obtained as the semiclassical limit of the IIB matrix model. The new parts needed for computing loop diagrams is bellow:

$$\langle \psi_{\alpha}(k,r,h,u)\psi_{\beta}(-k,-r,-h,-u)\rangle = \frac{i(kC^{-1})_{\alpha\beta}}{k^2},$$
 (5.111)



$$= i \left[(k \cdot \bar{r}) \{ k^b \delta^{ac} + k^a \delta^{bc} - 2k^c \delta^{ab} \} - \{ k^b u_s (\mathbb{M}_s)_e^c \delta^{ae} + k^a u_s (\mathbb{M}_s)_e^c \delta^{be} - 2(k \cdot \mathbb{M}_s)^c \delta^{ab} \} \right],$$
(5.112)

Using them, we compute the loop diagrams containing a fermion loop (Fig.5.3), and obtain the induced mass terms in this case:

$$\Gamma\Big|_{\text{mass}} = -\frac{1}{g^2} \frac{1}{d+2} \left((d-2) - 2^{[d/2]-\kappa} \right) \\ \times \int dq \left[\alpha \left(\frac{\partial A_b}{\partial p_a} \right)^2 - \alpha \frac{4}{d} \left(\frac{\partial A_a}{\partial p_a} \right)^2 - \beta \frac{2(d-2)}{d^2} ((\mathbb{M}_s \cdot A)_a)^2 \right], \quad (5.113)$$

$$\kappa = \begin{cases}
1 & \text{(for the Majorana fermion)} \\
2 & \text{(for the Majorana-Weyl fermion)}
\end{cases}$$
(5.114)

From this induced mass term, we conclude that all of the fields, including the vielbein, remain massless in the IIB matrix model (d=10). On the other hand, if a matrix model contains more DoF in the fermionic sector than in the bosonic one, then the the effective action has tachyonic field and violates stability.

5.4 Summarizing remarks

In this section, we have analyzed the stability of the matrix model in the operator interpretation, which is originally proposed in [27,29]. We have shown that the principal bundle is essential for both the general diffeo-invariance and stability. Therefore, the original interpretation is indeed the minimal description of the operator interpretation. We have also seen how the metricity condition works to describe the curvature in EoM.

Next, we have confirmed that coefficient fields in higher derivative operators do transform as higher spin fields under the appropriate gauge transformation in the U(N) symmetry. The coefficient fields include as well redundant components that are eliminated either by gauge transformations or by generalized torsion-free conditions. Note that, however, the existence of the higher spin symmetries have been addressed only for simple sector, namely bosonic fiber-zero modes in the flat spacetime background.

We have computed the mass terms induced by loop corrections. While the IIB matrix model is protected from the corrections and remains massless theory, its bosonic version acquires mass terms with the appropriate sign. On the other hand, if we consider a matrix model which has more fermionic DoF than bosonic ones, fields in the effective action get tachyonic and therefore violate the stability. The important observation is that the vielbein and spin connection are massless only in the IIB matrix model. This implies the uniqueness of it as a model describing gravity. In the original description of the IIB matrix model [1], the correct gravitational interaction is realized due to supersymmetry. Our result has the same feature as that work, from the viewpoint that supersymmetry realizes gravity.

There still remain some open questions, parts of which we have already mentioned. Amongst them the most critical and central is the definition of the trace in the action and the interaction terms.

We have studied the higher spin symmetries at the level of transformation laws and of EoM. It should be that the symmetries can be seen explicitly in the action as well. However, we have no way so far to write down the action directly in terms of derivative operators. FOr example, consider the following action:

$$S = \frac{1}{g^2} \operatorname{Tr} \left(-\frac{1}{4} [A_a, A_b]^2 - \frac{m^2}{2} A_a^2 \right).$$
 (5.115)

In the operator interpretation, it possesses a U(1) gauge field and the gauge symmetry for it:

$$A_a = i\partial_a + a_a(x) + \cdots, \qquad (5.116)$$

$$\delta A_a = i[\Lambda, A_a], \quad \Lambda = \lambda(x) \quad \Leftrightarrow \quad \delta a_a(x) = -\partial_a \lambda(x). \tag{5.117}$$

It suggests the existence of U(1) gauge symmetry for massive vector field without the Stueckelberg field in Eq.(5.115), if we can convert the action in terms of operators. It appears to conflict with our understanding on ordinary field theory. Although the question will be settled once we make the well-defined trace in the action, it is a very difficult issue. At least, the truncation of operators to first-order should not be performed. It is because the following trace diverges:

$$\operatorname{Tr}(A_a^2) = \operatorname{Tr}(\partial^2) + \cdots$$
$$\sim \int \frac{d^d p}{(2\pi)^d} p^2 + \cdots .$$
(5.118)

Therefore we must regard matrices, at the starting point, as infinite-order derivative operators without the truncation, as was claimed repeatedly. Still, the definition of the trace remains to be determined. While the EoM for higher spin fields contain higher derivative, it is in fact unclear whether it really shows Ostrogradsky instability of the theory, because we do not have the action and hence the Hamiltonian for the fields. Related to this problem, the structure of interaction terms in terms of the local fields need to be analyzed.

If we can define an appropriate trace for infinite-order derivative operators, then we will eventually obtain a higher spin field theory which includes infinitely many fields with infinitely many interactions. It seems to have a deep relationship to a high energy description of string theory, presenting its importance as a nonperturbative formulation of string theory. Therefore, we will tackle this issue as the most crucial future work.

Chapter 6

Hill-climbing Saddle Point Inflation —— a Suggested Low-energy Model from the Matrix Model

In the previous chapters, we have studied the IIB matrix model as a candidate for the theory of quantum gravity. Since it is supposed to be a non-perturbative formulation of string theory, it describes physics at very high energy region, namely Planck scale (or the string scale). The reason for our interest to it was its simple structure in the action and its rich dynamics implying its inclusion of gravitational physics.

On the other hand, the connection between the matrix model and low-energy physics has yet to be revealed. The latter is described in terms of local field theory. As we have mentioned in Chapter 1, our eventual objective is not only to construct a quantum theory including gravity but also to theoretically connect it to low-energy effective theory which describes our current universe. More concretely, we should make an effort to reproduce SM from the IIB matrix model.

For this direction of study, A possible key is the inflation. There is a robust belief that an inflaton physics is responsible the exponential expansion of early universe, and that the origin of it consists in high-energy physics, typically at Planck scale. At the same time, the inflaton decays to SM particle after the inflation explain the reheating. It can be a bridge between the theories for the two different energy scale physics.

In the IIB matrix model, as has been stated in the previous section, bosonic DoF acquire mass terms with supersymmetries broken. Therefore, it should have at least one massive scalar at low energy region where the supersymmetries are broken, and the scalar behaves as an inflaton.

The crucial questions are the following two: what the true color of an inflaton is, and what the structure of its potential comes from. While string theory often suggests an axion playing an role of the inflaton, it is another reasonable philosophy that the inflaton is identified to a particle discovered already. Therefore, we are attracted to identify Higgs particle as inflation [37].

Assuming it, an inflaton potential is of course that of Higgs field. For its structure, there is some suggestion from the IIB matrix model with the operator interpretation. It was reported in [31] that its low-energy effective field theory should be with an action of an unusual form. ¹ It could be an answer for fine-tuning problems for various coupling constants, including the Higgs coupling constant. In particular, it favors the Higgs effective potential to have a critical structure, which has two degenerate vacua at electro-weak and near-Planck scales. Related to this suggestion, there has been a proposed principle that coupling constants are naturally tuned so that the universe can have degenerate vacua. This is called the Multiple Point criticality Principle (MPP). The low-energy effective theory of IIB matrix model, thus, gives some validity for MPP.

Furthermore, when one focuses on Higgs potential from phenomenological point of view, there is a good basis to believe it has degenerate two minima. The observed Higgs mass ~ 125GeV indicates that the SM can be safely interpolated up to the Planck scale without any divergence or instability. The observed Higgs quartic coupling $\lambda \sim 0.12$ also shows an interesting behavior of the Higgs potential around the Planck scale M_{pl} ; The potential can have another degenerate minimum around that scale. The origin of this behavior comes from the fact that λ and its beta function β_{λ} can simultaneously vanish around M_{pl} . This is the very suggested structure from MPP or the IIB matrix model. It is surprising that the Higgs mass was predicted to be around 130GeV about 20 years ago based on MPP [34].

Various phenomenological and theoretical studies of such a degenerate vacuum have been done so far [71–75]. One of them is the Higgs inflation with a non minimal coupling $\xi \phi^2 R/M_{pl}^2$ [37]. When this scenario was proposed, it was argued that we need large $\xi \sim 10^5$ in order to obtain the successful inflationary predictions of the cosmic microwave background (CMB). However, the criticality of the Higgs potential makes it possible to realize the inflation even if ξ is relatively small $\sim \mathcal{O}(10)$ by using small but nonzero $\lambda \sim 10^{-6}$ around M_{pl} . See [76] for the detailed analyses.

Although the SM criticality can help the realization of the Higgs inflation, it is difficult to realize the MPP simultaneously because the latter requires $\lambda = 0$ around the Planck scale and we can no longer maintain the monotonicity of the Higgs potential above the scale $\sim M_{pl}/\sqrt{\xi}$. Recently, a new inflationary scenario was proposed in [35] which enables an inflation even if the inflaton potential has multiple degenerate vacua. Then, the authors applied it to the SM Higgs and showed that it is actually possible to obtain a successful inflation while satisfying the MPP [36]. In those papers, the authors studied a few cases such that the inflaton potential

¹See Appendix C for the detail.

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behaves as a quadratic potential around another potential minimum. Although the inflationary predictions of this scenario does not strongly depend on the details of the inflaton potential such as the coefficients of the Taylor expansion, they can depend on the leading exponent of the (Jordan-frame) potential and the choice of the conformal factor. In this part, we generalize their works to the cases where the inflaton potential has a saddle point around the Planck scale. Our study is meaningful from the point of view of the MPP because this situation can be understood as a natural generalization of this principle, and therefore it can associated to the high-energy physics as the IIB matrix model, in principle. Although some fine-tunings are needed in order to realize a saddle point, some theoretical studies [33, 77–79] suggest that we can naturally archive such fine-tunings by considering physics beyond ordinary field theory.

6.1 Brief review of hill-climbing inflation

Let us briefly review the hill-climbing inflation. We consider the following action of an inflaton ϕ_J in the Jordan-frame:

$$S = \int d^4x \sqrt{-g_J} \left(\frac{M_{pl}}{2}^2 \Omega R_J - \frac{K_J}{2} (\partial \phi_J)^2 - V_J(\phi_J) \right), \tag{6.1}$$

where $(\partial \phi_J)^2 = g_J^{\mu\nu} \partial_\mu \phi_J \partial_\nu \phi_J$. If we identify ϕ_J as the Higgs, the usual Higgs potential corresponds to $V_J(\phi_J)$ in this framework. Then, by doing the Weyl transformation

$$g_{\mu\nu} = \Omega g_{J\mu\nu},\tag{6.2}$$

we have

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{pl}^2}{2} R - \frac{1}{2} \left(\frac{K_J}{\Omega} + \frac{3}{2} \left(M_{pl} \frac{\partial \ln \Omega}{\partial \phi_J} \right)^2 \right) (\partial \phi_J)^2 - \frac{V_J(\phi_J)}{\Omega^2} \right], \tag{6.3}$$

where R is the Ricci scalar in the Einstein-frame and we have neglected the total derivative term. Let us now assume that the second term of the kinetic terms dominates. In this case, we can regard $\chi \equiv M_{pl}\sqrt{3/2}\ln\Omega$ or $-M_{pl}\sqrt{3/2}\ln\Omega$ as a fundamental field instead of ϕ_J .² For example, in the case of the ordinary Higgs inflation, we have

$$\Omega(\phi_J) = 1 + \xi \frac{\phi_J^2}{M_{pl}^2}, \ V_J(\phi_J) = \frac{\lambda \phi_J^4}{4}, \tag{6.4}$$

which leads to the following potential in the Einstein-frame:

$$V_E(\chi) = \frac{\lambda \phi_J^4}{4\Omega^2} = \frac{\lambda M_{pl}^4}{4\xi^2} (1 - \Omega^{-1})^2$$

²The choice of the sign depends on the region we consider; When we consider $\Omega \geq 1 \ (\leq 1)$, we take $\chi = (-)M_{pl}\sqrt{3/2}\ln\Omega$.

$$\simeq \frac{\lambda M_{pl}^4}{4\xi^2} \left(1 - \exp\left(-\sqrt{\frac{2}{3}}\frac{\chi}{M_{pl}}\right) \right)^2, \tag{6.5}$$

from which we can see that $V_E(\chi)$ becomes exponentially flat when $\chi \gg M_{pl} \Leftrightarrow \Omega \gg 1$. See also Ref. [76] for more detailed analyses.

On the other hand, a new possibility has been proposed in Ref. [35], where it is shown that we can also consider the $\Omega \ll 1$ region instead of $\Omega \gg 1$. In this case, because V_E is given by $V_E = V_J/\Omega^2$, V_J needs to behave as

$$V_J = V_0 \Omega^2 \left(1 + \dots \right) \tag{6.6}$$

around $\Omega = 0$ in order to realize the inflationary era, i.e. $H = \dot{a}/a = const$. Because the conformal factor Ω should approaches one after inflation, the inflaton *climbs up* the Jordan-frame potential. This is the reason why the authors of Ref. [35] call this scenario "*Hill-climbing* (*Higgs*) inflation". Let us briefly summarize the inflationary predictions of this scenario. By expanding the Jordan-frame potential V_J as a function of Ω

$$V_J = V_0 \Omega^2 (1 + \sum_{m \ge n} \eta_m \Omega^m), \tag{6.7}$$

we obtain

$$\epsilon = \frac{M_{pl}^2}{2} \left(\frac{V'}{V}\right)^2 \simeq \frac{1}{3} \left(\sum_m \eta_m m \Omega^m\right)^2,\tag{6.8}$$

$$\eta = M_{pl}^2 \frac{V''}{V} \simeq -\frac{2}{3} \sum_m \eta_m m^2 \Omega^m,$$
(6.9)

where the prime represents the derivative with respect to χ and we have used the relation $\chi = \sqrt{3/2} \ln \Omega$. Furthermore, we can relate the initial value of Ω to the *e*-folding number N:

$$N = \int dt H = \frac{1}{M_{pl}^2} \int d\chi \frac{V}{\frac{\partial V}{\partial \chi}} \simeq \frac{3}{2\eta_n n^2} \frac{1}{\Omega_{ini}^n}.$$
(6.10)

From those equations, we obtain the following inflationary predictions:

$$n_s = 1 - 6\epsilon + 2\eta \simeq 1 - \frac{2}{N}, \ r = 16\epsilon = \frac{12}{n^2 N^2}. \tag{6.11}$$

Note that both of them do not depend on the details of the inflaton potential such as its coefficients η_n 's. This is the similar behavior of the ξ or α attractor [80–82]. However, the leading exponent n depends on a specific model we consider and the choice of the conformal factor. In the following, we consider the hill-climbing inflation around a (UV) saddle point of an inflaton potential.

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Figure 6.1: Upper (Lower): A schematic behavior of the Jordan (Einstein)-frame potential around the saddle point ϕ_0 ($\chi = \infty$). Here, the solid (dashed) contour corresponds to k =odd (even). (Source: [46], doi:10.1016/j.physletb.2018.01.007)

6.2 Hill-climbing saddle point inflation

Let us now consider a general situation where the Jordan-frame potential has a saddle point ϕ_0 around the Planck scale:

$$V_J(\phi_0) = 0, \ V_J^{(1)}(\phi_0) = 0, \ V_J^{(2)}(\phi_0) = 0, \ \cdots, \ V_J^{(k)}(\phi_0) = 0$$
 (6.12)

with $V_J^{(i)}$ denoting the *i*-th derivative of V_J . In the following, we assume

$$\begin{cases} V_J^{(k+1)}(\phi_0) > 0 & \text{for odd } k, \\ V_J^{(k+1)}(\phi_0) < 0 & \text{for even } k, \\ V_J^{(k+2)}(\phi_0) \neq 0 \end{cases}$$
(6.13)

in order to realize a positive vacuum energy in $\phi_J \leq \phi_0$.³ This is schematically shown in the upper panel of Fig.6.1. In this case, we can expand V_J around ϕ_0 as

$$V_J(\phi_J) = \frac{V_J^{(k+1)}}{(k+1)!} (\phi_J - \phi_0)^{k+1} + \frac{V_J^{(k+2)}}{(k+2)!} (\phi_J - \phi_0)^{k+2}$$
$$= \frac{|V_J^{(k+1)}| \phi_0^{k+1}}{(k+1)!} \left(1 - \frac{\phi_J}{\phi_0}\right)^{k+1}$$

³The third assumption is not necessary for our present set up. We can also consider a more general situation such that $V_J^{(k+1)}(\phi_0) \neq 0$, $V_J^{(k+2)}(\phi_0) = 0$, \cdots , $V_J^{(k+m)}(\phi_0) = 0$, $V_J^{(k+m+1)}(\phi_0) \neq 0$.

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$$\times \left(1 + v_1^{(k+2)} \left(\frac{\phi_J}{\phi_0} - 1\right) + v_2^{(k+3)} \left(\frac{\phi_J}{\phi_0} - 1\right)^2\right),\tag{6.14}$$

where

$$v_1^{(k+2)} = \frac{\phi_0 V_J^{(k+2)}}{(k+2)V_J^{(k+1)}}, \ v_2^{(k+3)} = \frac{\phi_0 V_J^{(k+3)}}{(k+2)(k+3)V_J^{(k+1)}}.$$
(6.15)

As for the conformal factor Ω , we can consider various possibilities:⁴

$$\Omega(\phi_J)^2 = \left(1 - \frac{\phi_J}{\phi_0}\right)^{k+1} \left(1 + \sum_{i \ge 0} \omega_i \left(1 - \frac{\phi_J}{\phi_0}\right)^i\right), \tag{6.16}$$

$$\sum_{i\geq 0}\omega_i = 0,\tag{6.17}$$

where the second equation guarantees $\Omega(0) = 1$. In this letter, in order to give some concrete inflationary predictions, we consider the following two models:

$$\Omega = \begin{cases} \left(1 - \frac{\phi_J^2}{\phi_0^2}\right)^{\frac{k+1}{2}} & \text{(Model 1)}, \\ \left(1 - \frac{\phi_J^4}{\phi_0^4}\right)^{\frac{k+1}{2}} & \text{(Model 2)}, \end{cases}$$
(6.18)

which correspond to Model 1 and Model 2 presented in Ref. [36], respectively. In the case of Model 1, the Einstein-frame potential becomes

$$V_E \simeq \frac{|V_J^{(k+1)}|\phi_0^{k+1}}{(k+1)!2^{k+1}} \left(1 + \left(\frac{k+1}{2} - v_1^{(k+2)}\right) \left(1 - \frac{\phi_J}{\phi_0}\right) + \left(v_2^{(k+3)} - \frac{k+1}{2}v_1^{(k+2)} + \frac{(k+1)(k+2)}{8}\right) \left(1 - \frac{\phi_J}{\phi_0}\right)^2 \right) \\ \simeq V_0 \left(1 + \eta_{\frac{2}{k+1}}\Omega^{\frac{2}{k+1}} + \eta_{\frac{4}{k+1}}\Omega^{\frac{4}{k+1}} \right),$$
(6.19)

where

$$V_{0} = \frac{|V_{J}^{(k+1)}|\phi_{0}^{k+1}}{(k+1)!2^{k+1}}, \ \eta_{\frac{2}{k+1}} = \frac{1}{2} \left(\frac{k+1}{2} - v_{1}^{(k+2)} \right),$$

$$\eta_{\frac{4}{k+1}} = \frac{1}{2^{2}} \left(v_{2}^{(k+3)} - \frac{k+1}{2} v_{1}^{(k+2)} + \frac{(k+1)(k+2)}{8} \right),$$

(Model 1) (6.20)

⁴In this letter, we assume that the conformal factor Ω also becomes zero at a saddle point of V_J . This fine-tuning might also be explained by some new physics [33, 77–79].

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from which we can see that the resultant leading exponent depends on the coefficients of the Jordan-frame potential. ⁵ In the lower panel of Fig.6.1, we schematically show the Einstein-frame potential V_E . Here note that the saddle point ϕ_0 corresponds to $\chi = \infty$ because of the relation $\chi = -M_{pl}\sqrt{3/2}\ln\Omega$. Here, the solid (dashed) contour corresponds to k = odd (even). In the case of Model 2, we have

$$V_{0} = \frac{|V_{J}^{(k+1)}|\phi_{0}^{k+1}}{(k+1)!2^{2(k+1)}}, \ \eta_{\frac{2}{k+1}} = \frac{1}{4} \left(\frac{3(k+1)}{2} - v_{1}^{(k+2)} \right),$$

$$\eta_{\frac{4}{k+1}} = \frac{1}{4^{2}} \left(v_{2}^{(k+3)} - \frac{3(k+1)}{2} v_{1}^{(k+2)} + \frac{(k+1)(9k+10)}{8} \right),$$

(Model 2) (6.21)

Thus, both of the models typically give the leading exponent $n = \frac{2}{(k+1)}$ as long as we do not require a fine-tuning of the coefficients.⁶ As a result, the tensor-to-scalar ratio becomes larger when we increase k. Note that, in this framework, the coefficient of the leading term in the potential must be negative, $\eta_{\frac{2}{k+1}} < 0$, which enables χ to roll down it. Furthermore, the potential height V_0 is also constrained by the curvature perturbation

$$A_s = \frac{V_0}{24\pi^2 \epsilon M_{pl}^4} = 2.2 \times 10^{-9} \propto \frac{V_J^{(k+1)}(\phi_0)\phi_0^{k+1}}{M_{pl}^4}.$$
(6.22)

In Fig.6.2, we plot the parameter regions obtained from Eq.(6.22). Here, the (k+1)-th derivative $V_J^{(k+1)}(\phi_0)$ is normalized by ϕ_0^{k-3} , and each bands corresponds to each k's. The solid (dashed) contours represent N = 50 (60).

In Fig.6.3, we also show the inflationary predictions obtained from the analytic formulas Eq.(6.11). Here, the different color lines represent different k's and the small (large) dots correspond to N = 50 (60). Note that n_s does not change within this analytic formula because it only depends on the *e*-folding N. As is already mentioned in Ref. [36], the higher order terms of the inflaton potential can have slightly large contributions to the inflationary dynamics, and numerical studies are necessary in order to give more precise predictions. This is left for future investigations.

we have applied the idea of the hill-climbing inflation to the models where the inflaton potential has a saddle point around the Planck scale and shown that it is possible to archive a successful inflation. A notable feature of this class of models is that the leading exponent of the Jordan-frame potential as a function of the conformal factor is typically given by 2/(k+1),

⁵For example, in the case of the Higgs potential, we have $k = 1, v_1^{(k+2)} = 3$, which lead to $\eta_1 = -1$. This agrees with the previous study Ref. [36].

⁶If we consider general V_J and Ω , the coefficients $\eta_{2i/(k+1)}$'s are simple polynomials of $(v_i^{(k+i+1)}, \omega_i)$, and it is possible to eliminate some of the first $\eta_{2i/(k+1)}$'s by choosing specific values of those parameters. Then, the leading exponent can be $n = \frac{2l}{k+1}$ with arbitrary l. The Model 2 of the hill-climbing Higgs inflation Ref. [36] is such a case.


Figure 6.2: The parameter regions that produce the observed value of the scalar perturbation $A_s = 2.2 \times 10^{-9}$. The upper (lower) panel corresponds to Model 1 (2). Here, the different color bands represent different k's respectively, and the solid (dashed) lines corresponds to N = 50 (60). (Source: [46], doi:10.1016/j.physletb.2018.01.007)

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which leads to a large tensor-to-scalar ratio. Although we have just concentrated on a saddle point of the inflaton potential, we can also consider various realizations of the hill-climbing inflation by using a variety of V_J and Ω . So it might be interesting to investigate such possibilities and construct a phenomenological model that can realize a successful inflation. From the point that such a inflation model should be induced from high-energy physics, it is a tempting future work to study the effective theory of the IIB matrix model in comparison with these generalized hill-climbing structure.



Figure 6.3: The inflationary predictions of the hill-climbing saddle point inflation. Here, the different color lines represent different k's and the small (large) dots correspond to N = 50 (60). (Source: [46], doi:10.1016/j.physletb.2018.01.007)

Chapter 7

Massive Higher Spin Fields in Curved Spacetime

Another implication of the IIB matrix model is that many massive higher spin fields (MHSF) should exist in our universe. It is a resulting observation of Chapter 5. It is now no doubt that the IIB matrix model contains higher spin fields, and with the supersymmetries broken at some energy scale, they get massive through radiative corrections. Although the mechanism of breaking supersymmetries is yet to be studied, the breaking did occur in our universe since the experiments tell us the absence of low-energy supersymmetry. Therefore, even if real MHSF are so heavy that the current experiments and observations do not confirm its existence, it is important to describe them in terms of field theory. One may think that it is sufficient to treat them within the framework of string theory. However, it is still a reasonable expectation as well that we can describe them when we do not take into account their UV behavior, including UV divergence.¹

Attempts to construct MHSF theories showed up with the papers written by Fierz and Pauli, who formulated a free field theory of massive spin 2 particles in the Minkowski space [83,84]. In general, the natural object to describe a spin s particle is a rank-s traceless symmetric tensor field, but this has more independent components than necessary, because a spin s particle has only 2s + 1 DoF. Therefore, the Lagrangian should give the EoM that yield necessary and sufficient constraints to eliminate the redundant DoF. In fact, for the s = 2 case, Fierz and Pauli showed that an appropriate Lagrangian can be obtained if one introduces an auxiliary scalar field in addition to a rank-2 traceless tensor. These fields can actually be combined to form a single traceful symmetric tensor $h_{\mu\nu}$, which we call the Fierz-Pauli (FP) field.² For the case s > 2, the Lagrangian with the desired property was given by Singh-Hagen [38,39], which

¹Apart from the connection of string theory or the IIB matrix model with low-energy theories, MHSF is of interest from the phenomenological viewpoint since the excited hadrons are indeed such objects. The analysis of this chapter can also be seen as an attempt to describe them coupled with gravity.

 $^{^{2}}$ In [85] it was shown that the FP theory is the unique formulation of a spin 2 particle without ghosts or tachyons.

consist of traceless symmetric tensors of ranks $s, s-2, s-3, s-4, \ldots, 0$. These fields can be combined to form two traceful symmetric tensors of ranks s and s-3.³

All the works above only consider the case where the background spacetime is flat. However, for curved backgrounds, it is non-trivial to formulate MHSF theories.⁴ In fact, as we will see in section 7.1, the mechanism to derive the constraints from the EoM breaks down because covariant derivatives do not commute with each other. There was also an argument that the transverse condition is not compatible with the wave equation for arbitrary backgrounds [92]. Although verious works have been made in this direction [93–107], It seems that currently there are no consistent massive higher spin theories for general backgrounds that reduces to the flat case smoothly.

In this chapter, as a first step to investigate higher spin theories, we give the quadratic Lagrangian for spin 2 particles in general gravitational backgrounds, and discuss a fundamental problem occurs with particles of spin larger than 3.

The analysis of spin 2 particles is essentially related to [108,109]. First, we have an analysis in Hamiltonian formalism which is motivated by the canonical analysis of ordinary gravity with ADM decomposition. This makes the points clear. Then we give the Lagrangian and the direct analysis. The connection to massive gravity theory [110–113] is presented as well.

On the other hand, we show that a spin 3 particle fails to couple to gravity in the subsequent section. The EoM for it do not contain enough constraints in the general background. The same breakdown is likely to occur with arbitrary spin higher than 2. This result implies that we cannot describe single MHSF in curved spacetime. The way to overcome this problem is probably to introduce infinitely many fields of different spin. The situation is consistent to the operator interpretation of the IIB matrix model, where we have to consider infinity-order derivative operators with higher spin fields being its coefficients.

7.1 Breakdown of the transverse condition for curved backgrounds

In this section, we demonstrate that FP's original mechanism to eliminate the redundant DoF of a massive higher rank tensor field does not work for generic curved backgrounds.

We start by arguing that there is no such issue for massive spin 1 field A^{μ} (Proca field). The action of the Proca field in the flat Minkowski spacetime is given by

$$S = \int d^4x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} m^2 A^{\mu} A_{\mu} \right], \qquad (7.1)$$

where $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ and the metric is chosen to be $\eta_{\mu\nu} = \text{diag}[-1, +1, +1, +1]$. Its EoM

³The massless limit of that Lagrangian was studied by Fronsdal [66] [86].

⁴For specific types of background, consistent EoM are obtained for massless fields by using the spacetime symmetry [87-90]. An attempt to generalize the theory to the massive case was made in [91].

are given by

$$\partial_{\nu}F^{\mu\nu} + m^2 A^{\mu} = 0.$$
 (7.2)

The divergence of Eq.(7.2) gives the transverse condition $\partial_{\mu}A^{\mu} = 0$, and the substitution of this to the EoM in turn gives the wave equation, $(\Box - m^2) A^{\mu} = 0$. Thus, the action (Eq.(7.1)) gives the EoM which automatically include the constraint that eliminates the redundant DoF correctly. It is easy to see that this mechanism also works in general curved backgrounds. In fact, if we covariantize the action as

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} m^2 A^{\mu} A_{\mu} \right]$$
(7.3)

with $F_{\mu\nu} \equiv \nabla_{\mu}A_{\nu} - \nabla_{\nu}A_{\mu}$, then the EoM are given by

$$\nabla_{\nu}F^{\mu\nu} + m^2 A^{\mu} = 0, \qquad (7.4)$$

whose divergence again gives the transverse condition, $\nabla_{\mu}A^{\mu} = 0$, because $\nabla_{\mu}\nabla_{\nu}F^{\mu\nu} = [\nabla_{\mu}, \nabla_{\nu}]F^{\mu\nu} = R_{\mu\nu}{}^{\mu}{}_{\alpha}F^{\alpha\nu} + R_{\mu\nu}{}^{\nu}{}_{\alpha}F^{\mu\alpha} = -2R_{\mu\nu}F^{\mu\nu} = 0$. Note that one could have added curvature terms to the action of the form $\int d^4x \sqrt{-g} \left[a R_{\mu\nu}A^{\mu}A^{\nu} + b R A^{\mu}A_{\mu}\right]$, where the coupling constants a and b are not determined only by requiring the action to become Eq.(7.1) in the flat limit. Such non-minimal couplings can be used to absorb the discrepancy that may arise when kinematic terms are covariantized in a different manner [e.g., a kinetic term $\partial_{\mu}A^{\mu}\partial_{\nu}A^{\nu}$ (up to total derivatives) can be covariantized in two ways: $\nabla_{\mu}A^{\mu}\nabla_{\nu}A^{\nu}$ or $\nabla_{\mu}A^{\nu}\nabla_{\nu}A^{\mu}$].

Now we discuss the FP field. The Lagrangian in the flat spacetime is given by

$$\mathcal{L} = h_{\mu\nu} \mathcal{E}_0^{\mu\nu\rho\sigma} h_{\rho\sigma} - \frac{m^2}{2} \left(h_{\mu\nu} h^{\mu\nu} - h^2 \right), \qquad (7.5)$$

where $\mathcal{E}_0^{\mu\nu\rho\sigma}$ is the Lichnerowicz operator for the flat spacetime:⁵

$$\mathcal{E}_{0}^{\mu\nu\rho\sigma}h_{\rho\sigma} \equiv \frac{1}{2}(\Box h^{\mu\nu} - \eta^{\mu\nu}\Box h) + \frac{1}{2}(\partial^{\mu}\partial^{\nu}h + \eta^{\mu\nu}\partial^{\rho}\partial^{\sigma}h_{\rho\sigma}) - \partial^{(\mu}\partial_{\lambda}h^{\nu)\lambda}.$$
 (7.6)

The kinetic term $\mathcal{L}_{\mathcal{E}_0} = h_{\mu\nu} \mathcal{E}_0^{\mu\nu\rho\sigma} h_{\rho\sigma}$ can be formally obtained from the EH action⁶

$$S_{\rm EH}[\hat{g}] = \frac{1}{2} \int d^4x \sqrt{-\hat{g}}\hat{R}$$
(7.7)

by setting $\hat{g}_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu}$ and taking quadratic terms in $h_{\mu\nu}$. The EoM take the form

$$0 = 2 \mathcal{E}_0^{\mu\nu\rho\sigma} h_{\rho\sigma} - m^2 (h^{\mu\nu} - \eta^{\mu\nu} h)$$

⁵We normalize the symmetrization as $X^{(\mu\nu)} \equiv (1/2) (X^{\mu\nu} + X^{\nu\mu}).$

⁶Throughout this chapter quantities with turret should be understood to represent those associated with $\hat{g}_{\mu\nu}$.

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$$= (\Box - m^2)(h^{\mu\nu} - \eta^{\mu\nu}h) + \eta^{\mu\nu}\partial^{\rho}\partial^{\sigma}h_{\rho\sigma} - 2\partial^{(\mu}\partial_{\lambda}h^{\nu)\lambda} + \partial^{\mu}\partial^{\nu}h.$$
(7.8)

A rank-2 symmetric tensor $h_{\mu\nu}$ has ten independent components, while a massive spin 2 particle has five DoF. In the flat background, the extra DoF are actually eliminated from the EoM as the Proca field. In fact, the divergence, double divergence, and trace of Eq.(7.8) respectively give

$$-m^2(\partial_\nu h^{\mu\nu} - \partial^\mu h) = 0, \qquad (7.9)$$

$$-m^2(\partial_\mu\partial_\nu h^{\mu\nu} - \Box h) = 0, \qquad (7.10)$$

$$2(\partial_{\mu}\partial_{\nu}h^{\mu\nu} - \Box h) + 3m^{2}h = 0.$$
(7.11)

Thus, when $m \neq 0$, we obtain the traceless condition, h = 0, from Eqs.(7.10) and (7.11). Then, substituting it to Eq.(7.9), we get the transverse condition, $\partial_{\nu}h^{\mu\nu} = 0$. Consequently, $h_{\mu\nu}$ is a rank-2 traceless symmetric, divergence-free tensor, which has five independent components. Note that the EoM (Eq.(7.8)) are then reduced to the Klein Gordon equations:

$$(\Box - m^2)h^{\mu\nu} = 0. (7.12)$$

We thus see that the reduction mechanism works for a massive spin 2 field as long as the background is flat.

Next we show the breakdown of the reduction mechanism when the flat theory is naïvely lifted to curved backgrounds. A natural extension of Eq.(7.5) is obtained (a) by replacing the derivatives with covariant derivatives, or (b) by substituting $\hat{g}_{\mu\nu} = g_{\mu\nu} + 2h_{\mu\nu}$ to Eq.(7.7) and taking only quadratic terms in $h_{\mu\nu}$. The discrepancy between (a) and (b) appears as the difference of non-minimal couplings (e.g., the difference of the coefficient of $R h^{\mu\nu} h_{\mu\nu}$). In this section we adopt the prescription (b).

The Lagrangian now takes the form

$$\mathcal{L} = \sqrt{-g} \Big[h_{\mu\nu} \mathcal{E}^{\mu\nu\rho\sigma} h_{\rho\sigma} - \frac{m^2}{2} (h_{\mu\nu} h^{\mu\nu} - h^2) \Big] \,. \tag{7.13}$$

Here, $h = g^{\mu\nu}h_{\mu\nu}$, and $\mathcal{E}^{\mu\nu\rho\sigma}$ is the Lichnerowicz operator acting on symmetric tensors in a curved spacetime:

$$\mathcal{E}^{\mu\nu\rho\sigma}h_{\rho\sigma} = \frac{1}{2}(\Box h^{\mu\nu} - g^{\mu\nu}\Box h) + \frac{1}{2}(\nabla^{\mu}\nabla^{\nu}h + g^{\mu\nu}\nabla^{\rho}\nabla^{\sigma}h_{\rho\sigma}) - \nabla^{(\mu}\nabla_{\lambda}h^{\nu)\lambda} + R^{\mu\rho\nu\sigma}h_{\rho\sigma} + R^{\rho(\mu}h_{\rho}^{\nu)} - \frac{1}{2}(g^{\mu\nu}R^{\rho\sigma}h_{\rho\sigma} + R^{\mu\nu}h) - \frac{1}{2}Rh^{\mu\nu} + \frac{1}{4}Rg^{\mu\nu}h, \quad (7.14)$$

which reduces to Eq.(7.6) in the flat limit and enjoys the following properties:

$$\frac{1}{2}\sqrt{-\hat{g}}\,\hat{R} = \sqrt{-g}\left[\frac{1}{2}\,R - G^{\mu\nu}h_{\mu\nu} + h_{\mu\nu}\,\mathcal{E}^{\mu\nu\rho\sigma}h_{\rho\sigma} + O(h^3)\right] \quad \left(\hat{g}_{\mu\nu} = g_{\mu\nu} + 2h_{\mu\nu}\right),\tag{7.15}$$

$$\nabla_{\nu} \left(\mathcal{E}^{\mu\nu\rho\sigma} h_{\rho\sigma} \right) = \frac{1}{2} G^{\rho\sigma} \left(2\nabla_{\rho} h^{\mu}_{\sigma} - \nabla^{\mu} h_{\rho\sigma} \right) , \qquad (7.16)$$

$$g_{\mu\nu}\left(\mathcal{E}^{\mu\nu\rho\sigma}h_{\rho\sigma}\right) = \nabla_{\mu}\nabla_{\nu}h^{\mu\nu} - \Box h, \qquad (7.17)$$

where $G^{\mu\nu} = R^{\mu\nu} - (R/2) g^{\mu\nu}$ is the Einstein tensor. The EoM are given by

$$2\mathcal{E}^{\mu\nu\rho\sigma}h_{\rho\sigma} - m^2(h^{\mu\nu} - g^{\mu\nu}h) = 0.$$
(7.18)

The divergence, double divergence, and trace of Eq.(7.18) respectively give

$$G^{\rho\sigma} \left(2\nabla_{\rho} h^{\mu}_{\sigma} - \nabla^{\mu} h_{\rho\sigma} \right) - m^2 (\nabla_{\nu} h^{\mu\nu} - \nabla^{\mu} h) = 0, \qquad (7.19)$$

$$\nabla_{\mu} \left[G^{\rho\sigma} \left(2 \nabla_{\rho} h^{\mu}_{\sigma} - \nabla^{\mu} h_{\rho\sigma} \right) \right] - m^2 \left(\nabla_{\mu} \nabla_{\nu} h^{\mu\nu} - \Box h \right) = 0 , \qquad (7.20)$$

$$2(\nabla_{\mu}\nabla_{\nu}h^{\mu\nu} - \Box h) + 3m^{2}h = 0.$$
 (7.21)

Thus, if h vanished or at least could be expressed as a function of the traceless part of $h_{\mu\nu}$, Eq.(7.19) would give four constraints on the transverse component. However, Eqs.(7.20) and (7.21) lead to

$$h = -\frac{2}{3m^4} \nabla_\mu \left[G^{\rho\sigma} \left(2\nabla_\rho h^\mu_\sigma - \nabla^\mu h_{\rho\sigma} \right) \right].$$
(7.22)

This is, except for the vacuum case $(G_{\mu\nu} = 0)$, a second-order differential equation for the trace h and the traceless part of $h_{\mu\nu}$, which cannot be regarded as a constraint eliminating unnecessary DoF. The situations are the same also for the cases of other spins, except for spin 1 (Proca field). In the spin 1 case, the divergence of the EoM always results in a first-order differential equation corresponding to the transverse condition, irrespective of how non-minimal couplings are introduced. For the case of higher spins, however, there is no choice of non-minimal couplings so as to cancel the RHS of Eq.(7.22). Another problem will emerge when formally substituting Eq.(7.22) to Eq.(7.18), since it results in fourth-order differential equations with respect to time. It is a singular perturbation, and yields an exponential growth of the amplitudes because the perturbation becomes much larger than the original kinetic term at short time scales. These facts seem to indicate that the Lagrangian above fails to describe a consistent FP field in a general background. In the following, we resolve this issue by giving up the attempt to express the constraint in a form that is directly related to the transverse condition and also by paying the cost of breaking the manifest covariance in the analysis.

7.2 Fierz-Pauli field in general curved backgrounds

In this section, we construct a consistent, linear field theory of a massive spin 2 field in a general curved spacetime.

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We start with the Lagrangian (Eq.(7.13)) with non-minimal couplings to the curvature:

$$S = \int d^4 x \mathcal{L}, \quad \mathcal{L} = \mathcal{L}_{\mathcal{E}} + \mathcal{L}_m + \mathcal{L}_R \tag{7.23}$$

with

$$\mathcal{L}_{\mathcal{E}} = \sqrt{-g} h_{\mu\nu} \mathcal{E}^{\mu\nu\rho\sigma} h_{\rho\sigma}, \quad \mathcal{L}_m = -\sqrt{-g} \frac{m^2}{2} (h_{\mu\nu} h^{\mu\nu} - h^2), \qquad (7.24)$$

$$\mathcal{L}_{R} = \sqrt{-g} \left[\frac{a_{1}}{2} R_{\mu\nu\rho\sigma} h^{\mu\rho} h^{\nu\sigma} + \frac{a_{2}}{2} R_{\mu\nu} h^{\mu\rho} h^{\nu}_{\rho} + \frac{a_{3}}{2} R h_{\mu\nu} h^{\mu\nu} + \frac{b_{1}}{2} R h^{2} + b_{2} R_{\mu\nu} h^{\mu\nu} h \right].$$
(7.25)

Here \mathcal{L}_R expresses the non-minimal couplings, and the coupling constants a_1 , a_2 , a_3 , b_1 , b_2 cannot be determined *a priori* only by requiring the action to become the FP action in the flat limit. Note that such terms also exist in $\mathcal{L}_{\mathcal{E}}$. In the remaining of this section, we show that Eqs.(7.23)–(7.25) describe a massive spin 2 field with correct DoF if and only if the constants in \mathcal{L}_R satisfy the two conditions⁷

$$a_2 + 2b_2 = -1, (7.26)$$

$$a_3 + b_1 = \frac{1}{2} \,. \tag{7.27}$$

The counting of DoF is usually easiest in the Hamiltonian formalism, and for this purpose we introduce the ADM decomposition of the metric:

$$(\hat{g}_{\mu\nu}) = \begin{pmatrix} -\hat{N}^2 + \hat{g}_{ij}\hat{N}^i\hat{N}^j & \hat{g}_{ij}\hat{N}^i \\ \hat{g}_{ij}\hat{N}^j & \hat{g}_{ij} \end{pmatrix}.$$
 (7.28)

The functions \hat{N} and $\vec{\hat{N}} = (\hat{N}^i)$ (i = 1, 2, 3) are called the lapse and the shift, respectively, and \hat{g}_{ij} describes the induced metric on a timeslice. The EH action then takes the following form up to surface integrals:

$$S_{EH} = \int d^4x \frac{1}{2} \hat{N} \sqrt{\hat{g}} \left[{}^{(3)}\hat{R} + \hat{K}_{ij}\hat{K}^{ij} - \hat{K}^2 \right] \quad (\hat{K} \equiv \hat{g}^{ij}\,\hat{K}_{ij}) \,. \tag{7.29}$$

Here, ${}^{(3)}\hat{R}$ is the Ricci scalar associated with \hat{g}_{ij} , and $\hat{K}_{ij} \equiv (1/2\hat{N}) \left[\dot{\hat{g}}_{ij} - \delta_{\vec{N}} \hat{g}_{ij}\right]$ is the extrinsic curvature of the timeslice $(\delta_{\vec{N}}$ is the Lie derivative with respect to the shift \vec{N}). We now expand the action around a classical background metric. By using the diffeo-invariance of the EH action, we can set the background to the following form without loss of generality:

$$(g_{\mu\nu}) = \begin{pmatrix} -1 & 0 \\ 0 & g_{ij} \end{pmatrix}.$$
 (7.30)

⁷The relations were first obtained in [109].

We then replace the metric in the action as

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + \mathsf{h}_{\mu\nu} \quad \left(\mathsf{h}_{\mu\nu} \equiv 2h_{\mu\nu}\right),\tag{7.31}$$

or equivalently, rewrite the lapse and shifts in Eq.(7.29) as

$$\hat{N}^2 = 1 - \mathbf{h}_{00} + \hat{g}^{ij} \mathbf{h}_{0i} \mathbf{h}_{0j} \,, \tag{7.32}$$

$$\hat{g}_{ij}\hat{N}^j = \mathsf{h}_{0i} \,, \tag{7.33}$$

$$\hat{g}_{ij} = g_{ij} + \mathsf{h}_{ij} \,. \tag{7.34}$$

The quadratic terms in $h_{\mu\nu}$ give $\mathcal{L}_{\mathcal{E}}$, whose explicit form is given by

$$\begin{aligned} \mathcal{L}_{\mathcal{E}} &= \left[\frac{1}{2} \hat{N} \sqrt{\hat{g}} \left[{}^{(3)} \hat{R} + \hat{K}_{ij} \hat{K}^{ij} - \hat{K}^{2} \right] \right]_{(2)} \\ &= \left[\frac{\hat{N} \sqrt{\hat{g}}}{2} {}^{(3)} \hat{R} + \frac{1}{2} \hat{C}^{ijkl} \left(\dot{\hat{g}}_{ij} - \delta_{\vec{N}} \hat{g}_{ij} \right) \left(\dot{\hat{g}}_{kl} - \delta_{\vec{N}} \hat{g}_{kl} \right) \right]_{(2)} \\ &= \frac{1}{2} \hat{C}^{ijkl}_{(0)} \dot{\mathsf{h}}_{ij} \dot{\mathsf{h}}_{kl} + \hat{C}^{ijkl}_{(1)} \dot{\mathsf{h}}_{ij} \dot{g}_{kl} - \hat{C}^{ijkl}_{(0)} \dot{\mathsf{h}}_{ij} (\delta_{\vec{N}} g_{kl})_{(1)} \\ &+ \left[\frac{\hat{N} \sqrt{\hat{g}}}{2} {}^{(3)} \hat{R} + \frac{1}{2} \hat{C}^{ijkl} \left(\dot{g}_{ij} - \delta_{\vec{N}} g_{ij} - \delta_{\vec{N}} \mathsf{h}_{ij} \right) \left(\dot{g}_{kl} - \delta_{\vec{N}} g_{kl} - \delta_{\vec{N}} \mathsf{h}_{kl} \right) \right]_{(2)}, \end{aligned}$$
(7.35)

where

$$\hat{C}^{ijkl} \equiv \frac{\sqrt{\hat{g}}}{4\hat{N}} \left[\frac{1}{2} \left(\hat{g}^{ik} \hat{g}^{jl} + \hat{g}^{il} \hat{g}^{jk} \right) - \hat{g}^{ij} \hat{g}^{kl} \right],$$
(7.36)

and a subscript in parenthesis denotes the order in $\mathsf{h}_{\mu\nu}.$

We now move on to the Hamiltonian formalism by making the Legendre transformation with respect to \dot{h}_{ij} . Since \dot{h}_{ij} is contained only in $\mathcal{L}_{\mathcal{E}}$, the conjugate variable to h_{ij} is given by

$$\pi^{ij} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{h}}_{ij}} = \frac{\partial \mathcal{L}_{\mathcal{E}}}{\partial \dot{\mathbf{h}}_{ij}}$$
$$= \hat{C}^{ijkl}_{(0)} \dot{\mathbf{h}}_{kl} + \hat{C}^{ijkl}_{(1)} \dot{g}_{kl} - \hat{C}^{ijkl}_{(0)} (\delta_{\vec{N}} g_{kl})_{(1)}, \qquad (7.37)$$

which can be solved for $\dot{\mathsf{h}}_{ij}$ as

$$\dot{\mathsf{h}}_{ij} = (\hat{C}_{(0)}^{-1})_{ijkl} \left(\pi^{kl} - \hat{C}_{(1)}^{klmn} \dot{g}_{mn} + \hat{C}_{(0)}^{klmn} (\delta_{\vec{N}} g_{mn})_{(1)} \right).$$
(7.38)

The Hamiltonian is then obtained as

$$\begin{aligned} \mathcal{H} &= \pi^{ij} \dot{\mathbf{h}}_{ij} - \mathcal{L}_{\mathcal{E}} - \mathcal{L}_m - \mathcal{L}_R \\ &= \frac{1}{2} (\hat{C}_{(0)}^{-1})_{ijkl} \big(\pi^{ij} - \hat{C}_{(1)}^{ijmn} \dot{g}_{mn} + \hat{C}_{(0)}^{ijmn} (\delta_{\vec{N}} g_{mn})_{(1)} \big) \big(\pi^{kl} - \hat{C}_{(1)}^{klpq} \dot{g}_{pq} + \hat{C}_{(0)}^{klpq} (\delta_{\vec{N}} g_{pq})_{(1)} \big) \end{aligned}$$

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$$-\left[\frac{\hat{N}\sqrt{\hat{g}}}{2}\,^{(3)}\hat{R} + \frac{1}{2}\hat{C}^{ijkl}(\dot{g}_{ij} - \delta_{\vec{N}}g_{ij} - \delta_{\vec{N}}h_{ij})(\dot{g}_{kl} - \delta_{\vec{N}}g_{kl} - \delta_{\vec{N}}h_{kl})\right]_{(2)} - \mathcal{L}_m - \mathcal{L}_R.$$
(7.39)

Since h_{0i} is generically quadratic and has no kinetic terms, the corresponding DoF will drop out from the system by solving the EoM for h_{0i} and by substituting the obtained solution to the action. Then, if the resulting Hamiltonian has only linear terms in h_{00} , there will arise the primary constraint, from which will follow the secondary constraint as a condition for the primary constraint to be consistent under the time evolution. Furthermore, a further consistency condition will arise for the secondary constraint, which in turn will determine the form of h_{00} . Thus, if the Hamiltonian has only linear terms in h_{00} after the elimination of h_{0i} , the variables h_{00} and h_{0i} will disappear from the system, leaving two constraints. This means that the system has ten (= 6 + 6 - 2) DoF, which agree with those of a massive spin 2 field. We are going to show that this is the case if and only if the conditions (Eqs.(7.26) and (7.27)) are met.

There are actually two sources of h_{00}^2 terms. One is the h_{00}^2 terms that already exist in the Hamiltonian before solving the EoM for h_{0i} . The other is the h_{00}^2 terms that come out after h_{0i} is eliminated from the Hamiltonian.

First we point out that the latter source is absent, noticing that the mass term \mathcal{L}_m ,

$$\mathcal{L}_m = -\sqrt{-g} \, \frac{m^2}{8} \left[-2g^{ij} \mathsf{h}_{0i} \mathsf{h}_{0j} + g^{ik} g^{jl} \mathsf{h}_{ij} \mathsf{h}_{kl} + 2\mathsf{h}_{00} g^{ij} \mathsf{h}_{ij} - (g^{ij} \mathsf{h}_{ij})^2 \right], \tag{7.40}$$

contains quadratic terms in h_{0i} when $m \neq 0$. If the Lagrangian contains the terms of the form $h_{00}h_{0i}$, the EoM for h_{0i} take the form $h_{0i} = h_{00} \times A_{0i} + \cdots$ and give h_{00}^2 terms when substituted back to the Lagrangian. However, as we will see below, there are no such terms in the Lagrangian. Since there are no $h_{00}h_{0i}$ terms in \mathcal{L}_m , we only need to confirm the absence of such terms in the rest of the Hamiltonian Eq.(7.39). As for \mathcal{L}_R , we see that $a_2 R_{\mu\nu} h^{\mu\rho} h_{\rho}^{\nu}$ and $b_2 R_{\mu\nu} h^{\mu\nu} h$ actually give dangerous terms $-a_2 R_{0i} h_{00} h_0^i$ and $-2b_2 R_{0i} h_0^i h_{00}$. However, they can be ignored in our present approximation, because their contributions to the coefficients of h_{00}^2 will be $O(R^2/m^2)$ and can be neglected to the first order in the curvature. As for the remaining part of Eq.(7.39), we see from Eqs.(7.32)–(7.34) that terms linear in h_{0i} appear only through $\delta_{\vec{x}}g_{ij}$. Thus, the possible terms containing $h_{00}h_{0i}$ are

$$-(\hat{C}_{(0)}^{-1})_{ijkl}\hat{C}_{(1)}^{ijmn}\dot{g}_{mn}\hat{C}_{(0)}^{klpq}(\delta_{\vec{N}}g_{pq})_{(1)} - \left(-\hat{C}^{ijkl}\dot{g}_{ij}\delta_{\vec{N}}g_{kl}\right)_{(2)}.$$
(7.41)

However, the $h_{00} h_{0i}$ terms cancel out in Eq.(7.41), because it can be rewritten as

$$- \hat{C}_{(1)}^{ijmn} \dot{g}_{mn} (\delta_{\vec{N}} g_{ij})_{(1)} + \hat{C}_{(0)}^{ijkl} \dot{g}_{ij} (\delta_{\vec{N}} g_{kl})_{(2)} + \hat{C}_{(1)}^{ijkl} \dot{g}_{ij} (\delta_{\vec{N}} g_{kl})_{(1)} + \hat{C}_{(2)}^{ijkl} \dot{g}_{ij} (\delta_{\vec{N}} g_{kl})_{(0)} = \hat{C}_{(0)}^{ijkl} \dot{g}_{ij} (\delta_{\vec{N}} g_{kl})_{(2)} + \hat{C}_{(2)}^{ijkl} \dot{g}_{ij} (\delta_{\vec{N}} g_{kl})_{(0)},$$
(7.42)

which does not contain $h_{00} h_{0i}$.

We thus find that h_{0i} do not play any role in investigating the possible appearance of h_{00}^2 terms, so that we can safely set $h_{0i} = 0$ for further arguments. Since h_{00}^2 terms can appear only through \hat{N} in \hat{C}^{ijkl} , we only need to look at the h_{00}^2 terms in the reduced Hamiltonian

$$\mathcal{H} \sim \frac{1}{2} \left[(\hat{C}_{(0)}^{-1})_{ijkl} \hat{C}_{(1)}^{ijmn} \hat{C}_{(1)}^{klpq} \dot{g}_{mn} \dot{g}_{pq} - \hat{C}_{(2)}^{ijkl} \dot{g}_{ij} \dot{g}_{kl} \right] - \left[\frac{\hat{N} \sqrt{g}}{2} \, {}^{(3)}R \right]_{(2)} - \mathcal{L}_m - \mathcal{L}_R \,. \tag{7.43}$$

Here, the symbol ~ stands for an equality that holds when \hat{N}^i and h_{ij} are set to 0. \hat{C}^{ijkl} now takes the form

$$\hat{C}^{ijkl} = \hat{C}^{ijkl}_{(0)} + \hat{C}^{ijkl}_{(1)} + \hat{C}^{ijkl}_{(2)} + \cdots \\
\sim \frac{\sqrt{g}}{4\hat{N}} \left[\frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk}) - g^{ij} g^{kl} \right] \\
= \frac{1}{4} \sqrt{g} \left(1 + \frac{1}{2} \mathsf{h}_{00} + \frac{3}{8} \mathsf{h}_{00}^2 + \cdots \right) \left[\frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk}) - g^{ij} g^{kl} \right]$$
(7.44)

with

$$\hat{C}_{(0)}^{ijkl} \sim \frac{\sqrt{g}}{4} \left[\frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk}) - g^{ij} g^{kl} \right], \tag{7.45}$$

$$\hat{C}_{(1)}^{ijkl} \sim \frac{1}{2} \,\mathsf{h}_{00} \,\hat{C}_{(0)}^{ijkl} \,, \tag{7.46}$$

$$\hat{C}_{(2)}^{ijkl} \sim \frac{3}{8} \,\mathsf{h}_{00}^2 \,\hat{C}_{(0)}^{ijkl} \,. \tag{7.47}$$

Because \mathcal{L}_m does not include h_{00}^2 terms, we thus get

$$\mathcal{H} \sim \frac{1}{2} \left(\frac{1}{4} - \frac{3}{8} \right) \mathsf{h}_{00}^2 \, \hat{C}_{(0)}^{ijkl} \dot{g}_{ij} \dot{g}_{kl} + \frac{\sqrt{g}}{16} \, {}^{(3)}R \, \mathsf{h}_{00}^2 - \mathcal{L}_R \sim \frac{1}{64} \sqrt{g} \left[\dot{g}_{ij} \dot{g}^{ij} + (g^{ij} \dot{g}_{ij})^2 \right] \mathsf{h}_{00}^2 + \frac{\sqrt{g}}{16} \, {}^{(3)}R \, \mathsf{h}_{00}^2 - \mathcal{L}_R \,.$$
(7.48)

Finally, we substitute $h_{\mu\nu} = 2h_{\mu\nu}$:

$$\mathcal{H} \sim \frac{1}{16} \sqrt{g} \left[\dot{g}_{ij} \dot{g}^{ij} + (g^{ij} \dot{g}_{ij})^2 \right] h_{00}^2 + \frac{\sqrt{g}}{4} {}^{(3)}R h_{00}^2 - \mathcal{L}_R \,.$$
(7.49)

From this expression, we see that appropriate curvature terms must be supplied by \mathcal{L}_R in order for the h_{00}^2 terms to disappear. To see that this is actually possible, we write down the explicit form of \mathcal{L}_R for the background metric (Eq.(7.30)). Necessary formulae are

$$R = {}^{(3)}R + g^{ij}\ddot{g}_{ij} + \frac{3}{4}\dot{g}_{ij}\dot{g}^{ij} + \frac{1}{4}(g^{ij}\dot{g}_{ij})^2, \qquad (7.50)$$

$$R_{00} = -\frac{1}{2}g^{ij}\ddot{g}_{ij} - \frac{1}{4}\dot{g}_{ij}\dot{g}^{ij}, \qquad (7.51)$$

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from which the h_{00}^2 terms involved in Eq.(7.25) are obtained as

$$\frac{a_1}{2}R_{\mu\nu\rho\sigma}h^{\mu\rho}h^{\nu\sigma}\sim 0\,,\tag{7.52}$$

$$\frac{a_2}{2}R_{\mu\nu}h^{\mu\rho}h^{\nu}_{\rho} \sim -\frac{a_2}{2}R_{00}h^2_{00} = \frac{a_2}{2} \left[\frac{1}{2}g^{ij}\ddot{g}_{ij} + \frac{1}{4}\dot{g}_{ij}\dot{g}^{ij}\right]h^2_{00}, \qquad (7.53)$$

$$\frac{a_3}{2}Rh_{\mu\nu}h^{\mu\nu} \sim \frac{a_3}{2}Rh_{00}^2 = \frac{a_3}{2} \Big[{}^{(3)}R + g^{ij}\ddot{g}_{ij} + \frac{3}{4}\dot{g}_{ij}\dot{g}^{ij} + \frac{1}{4}(g^{ij}\dot{g}_{ij})^2 \Big] h_{00}^2 , \qquad (7.54)$$

$$\frac{b_1}{2}Rh^2 \sim \frac{b_1}{2}Rh_{00}^2 = \frac{b_1}{2} \Big[{}^{(3)}R + g^{ij}\ddot{g}_{ij} + \frac{3}{4}\dot{g}_{ij}\dot{g}^{ij} + \frac{1}{4}(g^{ij}\dot{g}_{ij})^2 \Big] h_{00}^2 , \qquad (7.55)$$

$$b_2 R_{\mu\nu} h^{\mu\nu} h \sim -b_2 R_{00} h_{00}^2 = b_2 \Big[\frac{1}{2} g^{ij} \ddot{g}_{ij} + \frac{1}{4} \dot{g}_{ij} \dot{g}^{ij} \Big] h_{00}^2 \,. \tag{7.56}$$

The reduced Hamiltonian is then expressed as

$$\mathcal{H} = \frac{\sqrt{g}}{16} \Big\{ 4^{(3)}R + \dot{g}_{ij}\dot{g}^{ij} + (g^{ij}\dot{g}_{ij})^2 - 2(a_2 + 2b_2)(2g^{ij}\ddot{g}_{ij} + \dot{g}_{ij}\dot{g}^{ij}) \\ - 2(a_3 + b_1) \Big[4^{(3)}R + 4g^{ij}\ddot{g}_{ij} + 3\dot{g}_{ij}\dot{g}^{ij} + (g^{ij}\dot{g}_{ij})^2 \Big] \Big\} h_{00}^2 , \qquad (7.57)$$

and we find that the necessary and sufficient conditions for the coefficients of four independent terms ${}^{(3)}R$, $g^{ij}\ddot{g}_{ij}$, $\dot{g}_{ij}\dot{g}^{ij}$ and $(g^{ij}\dot{g}_{ij})^2$ to disappear are given by the conditions (7.26) and (7.27). They are the conditions we promised to show in the beginning of this section so that the action (7.23)–(7.25) describes a massive spin 2 field with correct DoF in an arbitrary curved background.

7.3 Analysis based on the Lagrangian

In this section we reproduce the results in the previous section directly from the Lagrangian without resort to the ADM decomposition. We again set the background metric to the form (7.30) by using the diffeo-invariance. Then the FP Lagrangian can be written in the following form, by decomposing $h_{\mu\nu}$ and their covariant derivatives to the temporal and spatial components and by integrating by parts appropriately:

$$\mathcal{L} = \sqrt{g} \left[\frac{1}{2} C^{ijkl} \dot{h}_{ij} \dot{h}_{kl} + \frac{1}{2} M^{ijkl} h_{ij} h_{kl} + D^{ij} \dot{h}_{ij} h_{00} + E^{ij} h_{ij} h_{00} + F^{ijk} \dot{h}_{ij} h_{0k} + G^{ijk} h_{ij} h_{0k} + H^i h_{0i} h_{00} + \frac{1}{2} I^{ij} h_{0i} h_{0j} + \frac{1}{2} J (h_{00})^2 \right].$$
(7.58)

Here, dots denote derivatives with respect to t. C^{ijkl} does not include curvatures or spatialderivative operators. I^{ij} does not include spatial-derivative operators but may include curvatures (as well as m^2). Note that the FP kinetic term $\mathcal{L}_{\mathcal{E}}$ does not contain terms of the form $\dot{h}_{00}\dot{h}_{ij}$. Completing the square with respect to \dot{h}_{ij} leads to

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$$+\frac{1}{2}M^{ijkl}h_{ij}h_{kl} + E^{ij}h_{ij}h_{00} + \frac{1}{2}J(h_{00})^2 + G^{ijk}h_{ij}h_{0k} + H^ih_{0i}h_{00} + \frac{1}{2}I^{ij}h_{0i}h_{0j} \\ -\frac{1}{2}(C^{-1})_{ijkl}\Big(D^{ij}h_{00} + F^{ijm}h_{0m}\Big)\Big(D^{kl}h_{00} + F^{kln}h_{0n}\Big)\Big].$$
(7.59)

The condition for this Lagrangian to give the proper constraints is, as discussed in the previous section, that the terms of the form h_{00}^2 or $h_{00} h_{0i}$ do not survive after the Legendre transformation is made with respect to \dot{h}_{ij} . This is translated in the Lagrangian formalism as the condition that the second and third lines of Eq.(7.59) do not give terms of the form h_{00}^2 or $h_{00} h_{0i}$. This condition can be written as

$$J - DC^{-1}D = 0, (7.60)$$

$$H^{i} - (DC^{-1}F)^{i} = 0. (7.61)$$

In the following, we directly compute the LHS of Eqs.(7.60) and (7.61), and show that Eq.(7.61) is always satisfied but Eq.(7.60) requires the conditions Eq.(7.26) and Eq.(7.27).⁸

With the metric (7.30), the connections are given by

$$\Gamma_{00}^{0} = \Gamma_{0i}^{0} = \Gamma_{00}^{i} = 0, \quad \Gamma_{ij}^{0} = \frac{1}{2}\dot{g}_{ij}, \quad \Gamma_{0j}^{i} = \frac{1}{2}g^{ik}\dot{g}_{kj}, \quad (7.62)$$

and Γ_{jk}^{i} agrees with the connection associated with g_{ij} . Accordingly, the covariant derivatives take the forms

$$\nabla_{0}h_{00} = \dot{h}_{00},$$

$$\nabla_{i}h_{00} = \partial_{i}h_{00} - 2\Gamma_{i0}^{j}h_{0j},$$

$$\nabla_{0}h_{0i} = \dot{h}_{0i} - \Gamma_{0i}^{j}h_{0j},$$

$$\nabla_{j}h_{0i} = \partial_{j}h_{0i} - \Gamma_{j0}^{k}h_{ki} - \Gamma_{ji}^{0}h_{00} - \Gamma_{ji}^{k}h_{0k},$$

$$\nabla_{0}h_{ij} = \dot{h}_{ij} - \Gamma_{0i}^{k}h_{kj} - \Gamma_{0j}^{k}h_{ki},$$

$$\nabla_{k}h_{ij} = \partial_{k}h_{ij} - \Gamma_{ki}^{0}h_{0j} - \Gamma_{kj}^{0}h_{0i} - \Gamma_{ki}^{l}h_{lj} - \Gamma_{kj}^{l}h_{li}.$$
(7.63)

We now write the FP Lagrangian with non-minimal couplings in the following form:

$$\mathcal{L} = \sqrt{-g} \left[-\frac{1}{2} \nabla_{\lambda} h_{\mu\nu} \nabla^{\lambda} h^{\mu\nu} + \nabla^{\mu} h_{\mu\nu} \nabla_{\lambda} h^{\lambda\nu} - \nabla^{\mu} h_{\mu\nu} \nabla^{\nu} h + \frac{1}{2} \nabla_{\mu} h \nabla^{\mu} h - \frac{m^2}{2} (h_{\mu\nu} h^{\mu\nu} - h^2) \right] \\ + \frac{\tilde{a}_1}{2} R_{\mu\rho\nu\sigma} h^{\mu\nu} h^{\rho\sigma} + \frac{\tilde{a}_2}{2} R^{\mu}_{\ \lambda} h_{\mu\nu} h^{\lambda\nu} + \frac{\tilde{a}_3}{2} R h_{\mu\nu} h^{\mu\nu} + \frac{\tilde{b}_1}{2} R h^2 + \tilde{b}_2 R_{\mu\nu} h^{\mu\nu} h \right], \quad (7.64)$$

⁸After the first manuscript was accepted for publication, we found that a similar analysis was made in [109].

where the parameters are related with those in the previous section, Eq.(7.25), as

$$\tilde{a}_1 = a_1 + 2, \quad \tilde{a}_2 = a_2 + 2, \quad \tilde{a}_3 = a_3 - 1,
\tilde{b}_1 = b_1 + \frac{1}{2}, \quad \tilde{b}_2 = b_2 - 1.$$
(7.65)

By substituting Eqs.(7.63) to (7.64), the coefficients in Eq.(7.58) are expressed as

$$C^{ijkl} = \frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk}) - g^{ij} g^{kl}, \qquad (7.66)$$

$$(C^{-1})_{ijkl} = \frac{1}{2} (g_{ik}g_{jl} + g_{il}g_{jk}) - \frac{1}{2} g_{ij}g_{kl}, \qquad (7.67)$$

$$D^{ij} = \frac{1}{2} (g^{ik} \Gamma^j_{k0} + g^{jk} \Gamma^i_{k0}) - g^{ij} g^{kl} \Gamma^0_{kl}, \qquad (7.68)$$

$$F^{ijk}h_{0k} = 2(g^{ij}g^{kl} - g^{ik}g^{jl})\partial_k h_{0l} + O(\Gamma^2), \qquad (7.69)$$

$$H^{i}h_{0i} = \dot{g}^{ij}\partial_{i}h_{0j} + g^{ij}g^{kl}\dot{g}_{kl}\partial_{i}h_{0j} + O(\Gamma^{2})$$
(7.70)

$$J = 2g^{ij}g^{kl}(\Gamma^{0}_{ik}\Gamma^{0}_{jl} - \Gamma^{0}_{ij}\Gamma^{0}_{kl}) + \frac{1}{2}g^{ij}\dot{g}_{ij}g^{kl}\Gamma^{0}_{kl}$$

$$+ \dot{g}^{ij}\Gamma^{0}_{ij} + g^{ij}\dot{\Gamma}^{0}_{ij} + 2\tilde{\alpha}R_{00} + 2\tilde{\beta}R, \qquad (7.71)$$

where

$$\tilde{\alpha} = -\left(\frac{\tilde{a}_2}{2} + \tilde{b}_2\right) = -\left(\frac{a_2}{2} + b_2\right),\tag{7.72}$$

$$\tilde{\beta} = \frac{\tilde{a}_3}{2} + \frac{b_1}{2} = \frac{a_3}{2} + \frac{b_1}{2} - \frac{1}{4} .$$
(7.73)

One can easily check that the condition (7.61) is automatically satisfied (up to higher-order terms). On the other hand, the LHS of Eq.(7.60) can be rewritten to the form

$$J - DC^{-1}D = \frac{1}{2}(1 - 2\tilde{\alpha} + 4\tilde{\beta})g^{ij}\ddot{g}_{ij} + \frac{1}{4}(1 - 2\tilde{\alpha} + 6\tilde{\beta})\dot{g}^{ij}\dot{g}_{ij} + \frac{1}{2}\tilde{\beta}(g^{ij}\dot{g}_{ij})^2 + 2\tilde{\beta}^{(3)}R, \quad (7.74)$$

which vanishes only when $\tilde{\alpha} = 1/2$ and $\tilde{\beta} = 0$, i.e.,

$$\tilde{a}_2 + 2\tilde{b}_2 = -1 \rightarrow a_2 + 2b_2 = -1,$$
(7.75)

$$\tilde{a}_3 + \tilde{b}_1 = 0 \rightarrow a_3 + b_1 = \frac{1}{2}.$$
 (7.76)

We thus have reproduced the conditions (7.26) and (7.27) without using the ADM formalism. The procedure in this section is a simpler algorithm, and might have some application to the analysis of higher spin theories.

7.4 Connection to massive gravity

Actually, there is a well-known theory of massive spin 2 particles. That is the so-called massive gravity theory [110] [111], whose consistency has been proven based on the analysis of the DoF [112] [113] (for a review, see [114] [115]). We now discuss its relation to our results.⁹

The massive gravity is a non-linear theory, which has a spin 2 massive field $\hat{g}_{\mu\nu}$ and a fixed reference metric $f_{\mu\nu}$. Here we will consider a classical solution and the fluctuation around it. In general, the classical solution $g_{\mu\nu}$ is determined after $f_{\mu\nu}$ and an initial condition are specified. However, because we are interested in the fluctuation around the classical solution, it is better to regard $f_{\mu\nu}$ as a function of the classical solution $g_{\mu\nu}$. Then the consistency of the EoM for the fluctuation field is automatically guaranteed due to that of the full non-linear theory. We will see that the quadratic Lagrangian for the fluctuation indeed satisfies the conditions (7.26) and (7.27). However, it has only one free parameter, although the massive gravity theory in general has two free parameters.

The action of massive gravity is given by

$$S = \int d^4x \sqrt{-\hat{g}} \left[\frac{1}{2} \hat{R} - m^2 \sum_{n=0}^{4} \alpha_n e_n(\mathbb{K}) \right] \,, \tag{7.77}$$

$$(\mathbb{K})^{\mu}_{\ \nu} \equiv (\sqrt{\hat{g}^{-1}f})^{\mu}_{\ \nu} - \delta^{\mu}_{\ \nu} \ . \tag{7.78}$$

Here $f_{\mu\nu}$ is the reference metric and not a dynamical variable. $\sqrt{\hat{g}^{-1}f}$ denotes the square root as a matrix: $((\sqrt{\hat{g}^{-1}f})^2)^{\mu}{}_{\nu} = \hat{g}^{\mu\lambda}f_{\lambda\nu}$. $e_n(\mathbb{K})$ is the elementary symmetric polynomial of degree n in the eigenvalues of \mathbb{K} . They are represented as follows ($[\mathbb{X}] \equiv \text{tr}\mathbb{X}$):

$$e_{0}(\mathbb{K}) = 1 ,$$

$$e_{1}(\mathbb{K}) = [\mathbb{K}] ,$$

$$e_{2}(\mathbb{K}) = \frac{1}{2}([\mathbb{K}]^{2} - [\mathbb{K}^{2}]) ,$$

$$e_{3}(\mathbb{K}) = \frac{1}{6}([\mathbb{K}]^{3} - 3[\mathbb{K}][\mathbb{K}^{2}] + 2[\mathbb{K}^{3}]) ,$$

$$e_{4}(\mathbb{K}) = \frac{1}{24}([\mathbb{K}]^{4} - 6[\mathbb{K}]^{2}[\mathbb{K}^{2}] + 3[\mathbb{K}^{2}]^{2} + 8[\mathbb{K}][\mathbb{K}^{3}] - 6[\mathbb{K}^{4}]) .$$
(7.79)

Several conditions are imposed on the parameters α_n $(n = 0, \dots, 4)$ in order to satisfy the following requirements. We first set $\hat{g}_{\mu\nu} = g_{\mu\nu} + 2h_{\mu\nu}$, and expand the Lagrangian with respect to the fluctuation $h_{\mu\nu}$ around $g_{\mu\nu}$. We then require that the first-order terms in $h_{\mu\nu}$ vanish, and that the second-order terms involving m^2 take the same form as the FP mass term in the flat background. A straightforward calculation leads to the conditions $\alpha_1 = \alpha_0$, $\alpha_2 = \alpha_0 - 1$,

 $^{^{9}}$ They have developed the massive gravity theory further to construct a theory called bimetric gravity [116]. However, because our purpose is to discuss spin 2 particles in the gravitational background, it is more appropriate to consider its original form.

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and we find that the reference metric $f_{\mu\nu}$ is expressed by $g_{\mu\nu}$ as

$$f_{\mu\nu} = g_{\mu\nu} + \frac{2}{m^2} R_{\mu\nu} - \frac{1}{3m^2} g_{\mu\nu} R + O\left(\frac{R^2}{m^4}\right) \,. \tag{7.80}$$

Since K is of first or higher order both in $h_{\mu\nu}$ and in the curvature, α_4 does not contribute to the quadratic Lagrangian.

After some calculation, we obtain the Lagrangian for the fluctuation

$$\mathcal{L} = \sqrt{-g} \left[h_{\mu\nu} \mathcal{E}^{\mu\nu\rho\sigma} h_{\rho\sigma} - \frac{m^2}{2} (h_{\mu\nu} h^{\mu\nu} - h^2) + \frac{2(\alpha_0 - \alpha_3) - 5}{2} R^{\mu\nu} h_{\mu\lambda} h_{\nu}^{\lambda} + \frac{-4(\alpha_0 - \alpha_3) + 11}{12} R h_{\mu\nu} h^{\mu\nu} + \frac{\alpha_0 - \alpha_3 - 2}{3} R h^2 - (\alpha_0 - \alpha_3 + 2) R^{\mu\nu} h_{\mu\nu} h \right],$$
(7.81)

which has the form of the action (7.23)–(7.25) with

$$a_1 = 0$$
, $a_2 = 2(\alpha_0 - \alpha_3) - 5$, $a_3 = -\frac{2(\alpha_0 - \alpha_3)}{3} + \frac{11}{6}$,
 $b_1 = \frac{2(\alpha_0 - \alpha_3)}{3} - \frac{4}{3}$, $b_2 = -(\alpha_0 - \alpha_3) + 2$. (7.82)

The coefficients (7.82) indeed satisfy Eq.(7.76), but depend only on a single parameter $\alpha_0 - \alpha_3$. We thus may conclude that the Lagrangian in sections 7.2 and 7.3 gives a more general description than the massive gravity theory, at least for the free FP field in weak gravitational backgrounds.

7.5 Spin 3 case

Next, we discuss a massive spin 3 theory in the general background.

The variables to describe a massive spin 3 field consist of a traceful, rank-3 symmetric tensor $G_{\mu\nu\lambda}$ and an auxiliary scalar D. Denoting the trace of $G_{\mu\nu\lambda}$ by $G_{\mu} \equiv g^{\nu\lambda}G_{\mu\nu\lambda}$, the Lagrangian can be written in the form

$$\mathcal{L} = \mathcal{L}_{\min} + \mathcal{L}_R \tag{7.83}$$

with

$$\mathcal{L}_{\min} = \sqrt{-g} \left[-\frac{1}{2} \nabla_{\mu} G_{\nu\lambda\rho} \nabla^{\mu} G^{\nu\lambda\rho} + \frac{3}{2} \nabla^{\alpha} G_{\alpha\mu\nu} \nabla_{\beta} G^{\beta\mu\nu} - 3 \nabla^{\mu} G_{\mu\nu\lambda} \nabla^{\nu} G^{\lambda} \right. \\ \left. + \frac{3}{2} \nabla_{\mu} G_{\nu} \nabla^{\mu} G^{\nu} + \frac{3}{4} \nabla^{\mu} G_{\mu} \nabla^{\nu} G_{\nu} + \frac{1}{4} \partial_{\mu} D \nabla^{\mu} D \right. \\ \left. - \frac{m^2}{2} \left(G_{\mu\nu\lambda} G^{\mu\nu\lambda} - 3 G_{\mu} G^{\mu} \right) + m^2 D^2 - \frac{m}{2} \nabla^{\mu} G_{\mu} D \right],$$
(7.84)

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$$\mathcal{L}_{R} = \sqrt{-g} \left[\frac{a}{2} R_{\mu\nu\lambda\rho} G^{\mu\lambda\alpha} G^{\nu\rho}{}_{\alpha} + \frac{b_{1}}{2} R_{\mu\nu} G^{\mu\alpha\beta} G^{\nu}{}_{\alpha\beta} + b_{2} R_{\mu\nu} G^{\mu\nu\alpha} G_{\alpha} + \frac{b_{3}}{2} R_{\mu\nu} G^{\mu} G^{\nu} + \frac{c_{1}}{2} R G_{\mu\nu\lambda} G^{\mu\nu\lambda} + \frac{c_{2}}{2} R G_{\mu} G^{\mu} + \frac{c_{3}}{2} R D^{2} \right].$$
(7.85)

We will set the background metric to take the form

$$ds^{2} = -dt^{2} + g_{ij}(t) dx^{i} dx^{j}$$
(7.86)

and assume that all the fields depend only on time t. This setup greatly reduces the amount of necessary calculation, and, as we have observed in the preceding sections, should be sufficient for investigating how the DoF are removed due to constraints.

The coefficients in Eq.(7.84) are determined such that only the spatial, traceless part of the tensor $G_{\mu\nu\lambda}$ is dynamical in the flat Minkowski space. To confirm this, it is convenient to introduce the following parametrization for the temporal components of $G_{\mu\nu\lambda}$ in the background metric (7.86):

$$G_{000} = X + 3F$$
, $G_{00i} = V_i$, $G_{0ij} = \tilde{G}_{0ij} + \frac{1}{3}g_{ij}F$, (7.87)

where F is the trace of G_{0ij} , $F = g^{jk}G_{0jk}$, and \tilde{G}_{0ij} is the traceless part of G_{0ij} . One can easily show that \tilde{G}_{0ij} have a nonvanishing quadratic mass term and no kinetic terms, which means that \tilde{G}_{0ij} can be removed from the Lagrangian algebraically (and thus are not dynamical variables). It is also easy to see for the case of flat Minkowski space, that the Legendre transformation from \dot{G}_{ijk} , \dot{X} , \dot{D} to their conjugate momenta P^{ijk} , P_X , P_D yields only the linear terms for V_i and F, which means that V_i and F play the role of multiplier fields.

In the flat Minkowski case, the multipliers V_i and F actually yield the constraints that remove all the DoF except for the spatial, traceless part of the tensor $G_{\mu\nu\lambda}$. To see this, we note that the dynamics of (G_{ijk}, P^{ijk}, V_i) is totally decoupled from that of (X, P_X, D, P_D, F) in our setup. We first discuss the subsystem (G_{ijk}, P^{ijk}, V_i) . The primary and secondary constraints with respect to V_i are found to be

$$\kappa_1^i \equiv 3m^2 \delta_{jk} G^{ijk} = 0, \qquad (7.88)$$

$$\kappa_2^i \equiv -\frac{3}{4}m^2 \delta_{jk} P^{ijk} = 0, \qquad (7.89)$$

which have a nonvanishing Poisson bracket, $\{\kappa_1^i, \kappa_2^i\} = (15/4)m^4 \neq 0$. Thus, the multipliers V_i remove the DoF of the trace part of G_{ijk} and P^{ijk} , and V_i itself is determined by the equation $\dot{\kappa}_2^i = 0$. As for the subsystem (X, P_X, D, P_D, F) , the multiplier F yields four constraints (primary, secondary, tertiary and quaternary), which are expressed as

$$\chi_1 \equiv 2mP_D + 2m^2 X = 0, \qquad (7.90)$$

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$$\chi_2 \equiv 4m^2 P_X + 4m^3 D = 0, \qquad (7.91)$$

$$\chi_3 \equiv -12m^3 P_D - 2m^4 X = 0, \qquad (7.92)$$

$$\chi_4 \equiv -4m^4 P_X - 24m^5 D = 0. \tag{7.93}$$

Their Poisson brackets take the form $\{\chi_1, \chi_2\} = \{\chi_1, \chi_3\} = 0, \{\chi_1, \chi_4\} = -40m^6 \neq 0$, and $\det\{\chi_a, \chi_b\} \neq 0$ (a, b = 1, ..., 4). Thus, the multiplier F removes the DoF of (X, P_X, D, P_D) , and F itself is determined by the equation $\dot{\chi}_4 = 0$.

We now require that the same mechanism also work for the background (7.86). One can easily show that the quadratic terms in V_i and F are given by

$$\mathcal{H}|_{V_{i},F}^{(\text{quad})} = \sqrt{g} \left[\left(-\frac{3}{4} g^{ij} \ddot{g}_{ij} - \frac{3}{8} \dot{g}^{ij} \dot{g}_{ij} \right) V_{k} V^{k} + \left(-\frac{3}{2} g^{ik} \ddot{g}_{kj} - \frac{3}{4} \dot{g}^{ik} \dot{g}_{kj} \right) V_{i} V^{j} + \left(\frac{31}{6} g^{ij} \ddot{g}_{ij} + \frac{31}{12} \dot{g}^{ij} \dot{g}_{ij} \right) F^{2} \right] - \mathcal{L}_{R}|_{V_{i},F}^{(\text{quad})}$$
(7.94)

with

$$\begin{aligned} \mathcal{L}_{R} \Big|_{V_{i},F}^{(\text{quad})} / \sqrt{g} \\ &= \Big[\frac{1}{2} (b_{1} + b_{2} + 3c_{1} + c_{2}) g^{ij} \ddot{g}_{ij} + \frac{1}{8} (2b_{1} + 2b_{2} + 9c_{1} + 3c_{2}) \dot{g}^{ij} \dot{g}_{ij} + \frac{1}{8} (3c_{1} + c_{2}) (g^{ij} \dot{g}_{ij})^{2} \Big] V_{k} V^{k} \\ &+ \Big[\frac{1}{4} (-2a + b_{1} + b_{3}) g^{ik} \ddot{g}_{kj} + \frac{1}{4} (-a + b_{1} + b_{3}) \dot{g}^{ik} \dot{g}_{kj} + \frac{1}{8} (b_{1} + b_{3}) g^{kl} \dot{g}_{kl} g^{im} \dot{g}_{mj} \Big] V_{i} V^{j} \\ &+ \Big[\Big(\frac{5}{9} a - \frac{43}{18} b_{1} - \frac{8}{3} b_{2} - b_{3} - 5c_{1} - 2c_{2} \Big) g^{ij} \ddot{g}_{ij} \\ &+ \Big(\frac{19}{72} a - \frac{11}{9} b_{1} - \frac{7}{6} b_{2} - \frac{1}{2} b_{3} - \frac{15}{4} c_{1} - \frac{3}{2} c_{2} \Big) \dot{g}^{ij} \dot{g}_{ij} \\ &+ \Big(-\frac{a}{72} - \frac{b_{1}}{36} + \frac{1}{6} b_{2} - \frac{5}{4} c_{1} - \frac{1}{2} c_{2} \Big) (g^{ij} \dot{g}_{ij})^{2} \Big] F^{2} \,. \end{aligned}$$

$$(7.95)$$

These quadratic terms must vanish in order for the V_i and F to give four primary constraints, and we find that the parameters in the non-minimal couplings must take the following values:

$$a = 3, \quad b_1 = -\frac{30}{37}, \quad b_2 = -\frac{51}{74}, \quad b_3 = \frac{30}{37}, \quad c_1 = \frac{119}{222}, \quad c_2 = -\frac{119}{74}.$$
 (7.96)

However, the primary and secondary constraints χ_1 , χ_2 take the forms

$$\chi_1 = 2mP_D + 4g^{ij}\dot{g}_{ij}P_X + \sqrt{g}\left(2m^2 - \zeta\right)X,$$

$$\chi_2 = -4mg^{ij}\dot{g}_{ij}P_D + (4m^2 - 2\zeta)P_X + \sqrt{g}\left(4m^3 + 2c_3mR\right)D - \sqrt{g}\left(6m^2g^{ij}\dot{g}_{ij} - 8\eta g^{ij}\dot{g}_{ij}\right)X,$$

where ζ and η are functions of the curvature. In order for the constraints to give the tertiary and quaternary constraints, the Poisson bracket $\{\chi_1, \chi_2\}$ must vanish. However, this does not hold at the next order $m^3 \times (R/m^2)$ for generic backgrounds. This means that a massive spin field cannot described in the arbitrary curved spacetime. We have to consider the way to overcome this problem. There are two possible ways: the restriction of the spacetime manifolds to specific ones, and introduction of additional particles of different spin. The former is, however, less desirable since the theory will be very background-dependent. On the other hand, the latter is nontrivial once one considers the mixing (or bi-linear) terms for them. Yet, it is hard to guess that the finite number of fields can be a remedy. This remains to be a future work.

7.6 Summarizing remarks

In this chapter we have obtained the Lagrangian that describes a free massive spin 2 particle propagating in the general gravitational background to the first order in the curvature. The Lagrangian contain non-minimal couplings. the coefficients have three free parameters, and, in particular, the coupling constant associated with the Riemann tensor is arbitrary. The Lagrangian with such parameters includes that of the linearized massive gravity theory.

On the other hand, there is no consistent Lagrangian for a massive spin 3 field, even though we have derived the restriction of non-minimal couplings as necessary conditions. The same problem is likely to occur with arbitrary fields of spin larger than 2. If we can describe them coupled with gravity, it is expected to be a theory containing infinitely many MHSF. The IIB matrix model or string theory are indeed such formulation. In particular, the operator interpretation of the matrix model inevitably result in the introduction of infinitely many fields with infinitely high spin. These situation is perfectly consistent, and to make further investigation of the formulation of such MHSF will give an important clue to understanding the structure of the effective field theory of the IIB matrix model.

Chapter 8

Summary and Conclusion of the Thesis

In this thesis, we have shown the study both on the IIB matrix model as a Planck-scale physics, and on low-energy physics that are expected to be induced from the former. The underlying philosophy was that we can approach in two ways a quantum theory to describe gravity, spacetimes and matters. One is to study a candidate for the nonperturbative formulation of string theory, namely the IIB matrix model. The other is to analyze field theory at high-energy region and get some inspection about its UV completion.

In Chapter 2, we have reviewed the IIB matrix model and explained why it reproduces various theory including gauge field theory and string theory. In Chapter 3, we investigate the gravitational interaction with the noncommutative interpretation, which is well adopted in many works. In turn, in Chapter 4, we have introduced the operator interpretation and exhibited its convincing features to describe gravity and spacetimes.

In Chapter 5, we have shown several studies around the stability of the IIB matrix model with the operator interpretation. There we have presented three results. First, the original operator interpretation is the minimal. The principal bundle is essential and we seems to have any truncation of the class of operators anymore. Second, the U(N) symmetry for the matrix contains, in terms of coefficient fields of derivative operators, higher spin gauge transformations for some class of fields. Therefore it contains higher spin field, and should be identified to DoF in string theory in tensionless limit. However, EoM for them contain higher derivatives. Third, when we take into account radiative correction, the IIB matrix model itself remains massless theory while it will get massive DoF with the supersymmetries broken. More bosonic DoF than fernionic ones imply the generation of stable (non-tachyonic) mass terms, which may be an origin of masses at low-energy effective field theory.

On the other hand, Chapters 6 and 7 have been devoted to analysis on field theory. We have studied and investigated possible generalizations of the hill-climbing inflation, that can

connect MPP and Higgs inflation. Note that the former is suggested in the work on the IIB matrix model. As another issue, we have attempt to construct the lagrangian for MHSF in curved spacetimes, which leads to the investigation to the structure of the effective field theory of the IIB matrix model. We have shown that while spin 2 field can be described with the non-minimal couplings, the spin 3 field cannot consistently live on arbitrary spacetime. This and the observation of arbitrarily high spin field suggest the necessity of introducing infinitely many higher spin fields, just as in the IIB matrix model.

All of our results strongly support the existence of underlying connection between the matrix model as a theory of quantum gravity and field theory describing our universe. Of course, there are many issues to be studies in the future. The most serious one is the definition of the action for the IIB matrix model, with the operator interpretation. To settle this problem is a straightforward way to further analyze the stability and low-energy effective theory. While we should keep working on such aspects of the matrix model, the study on cosmological and theoretical aspect of field theory will provides many hints to pursue its UV completion, which can be the IIB matrix model. Such a complementary program is critical to proceed toward the construction of quantum gravity theory.

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Appendix

A A Note on Proposal for Gauge-invariant Regularization of Chiral Gauge Theory

If the IIB matrix model can reproduce SM, it should address well-defined high-energy behavior of matter DoF as well as gravitational and spacetime ones. In other words, it should provide regularizing framework for the SM. With this observation, among the several sectors in the SM, the most problematic one is the chiral sector. It has a symmetry of $SU(2)_L \times U(1)_Y$. Recently, it was reported in [40, 41] that the IIB matrix model can yield the chiral gauge theory which is defined on background branes (spacetime) as a classical solution. However, the high-energy behavior of chiral fermion, in particular its regularization, still remains to be analyzed.

On the other hand, the regularization of chiral gauge theory is very problematic issue even within the framework of field theory. It is mainly because the theory has an anomaly and no regulator exists that maintain the gauge-invariance.

Therefore in this appendix, we investigate this issue from the point of field theory. In contrast to the approach with which one should derive consistent field theory from the matrix model, we will attempt to figure out the connection of field theory to the matrix model, pursuing its sophisticated regularization. Although our present result is far from such a goal, we are able to obtain some implication for chiral gauge theory at high-energy region.

A.1 A problems around the regularization of chiral gauge theory

If one naïvely regularizes chiral gauge theory by applying the ordinary scheme such as the dimensional regularization or Pauli-Villars (PV) regularization, one obtains the anomaly from one-loop Feynman diagrams. In that case, however, one has to add local finite counter terms to the action in order to recover the gauge symmetry, even when the theory is free from anomaly. We call the counter terms the fake anomaly. The computation of the necessary counter terms is quite complicated in general, and it is natural to ask whether there is some regularization scheme that yields no fake anomaly. At the level of the one-loop Feynman diagram, it has been known that the covariant regularizations serve as such schemes. They are equivalent to regularizations of the change of the fermion measure covariantly [120], and the diagram gives

only the covariant anomaly.¹ On the other hand, there is no such formulation known at the Lagrangian level; the regularized Lagrangian to realize some covariant regularization remains to be constructed. So far it has been partially achieved in various works. For example, the generalized PV regularization [122] is one of the sophisticated formulation, which regularizes the chiral gauge theory covariantly as long as the gauge anomaly is absent. It also reproduces the Abelian anomaly with the correct coefficient [123]. It has been shown that when one applies the generalized PV regularization, the regularized contribution can be regarded as a sort of the covariant regularization [124]. However, some improvement is still required since this scheme fails to regularize the parity-odd contribution.

On the other hand, there is a theory called the domain-wall (DW) fermion [42], where a chiral fermion is induced from a higher dimensional Dirac fermion with a topological defect in its mass. Its relationship to the generalized PV regularization was discussed in [125]. The theories have been thoroughly investigated in the context of the lattice gauge theory [126–140]. In particular, it has been proposed in [135, 136] that the lattice regularization of chiral gauge theory can be realized by combining the DW fermion and the gradient flow of the gauge field [141, 142].

Although the DW fermion was originally discussed in the context of the lattice gauge theory, it should be useful to pursue a perturbative formulation as well. In this paper, we propose a regularization with which no fake anomaly emerges by defining chiral gauge theory through the DW fermion. The significant point is that the theory is vector-like and its regularized version is expected to induce chiral gauge theory without fake anomaly. One might guess that the problem is readily solved by combining the DW fermion with the dimensional regularization. However, it is not the case for the following reason. For example, consider a $(4 + \varepsilon + 1)$ dimensional DW fermion. Since its kinetic term respects the $(4 + \epsilon + 1)$ -dimensional Lorentz invariance, the 5-th and ε -th components of loop momenta p and the corresponding gamma matrices enter the loop integrand as a combination $\gamma_5 p_5 + p_{\varsigma}$. It can be rewritten as $\gamma_5 (p_5 + \gamma_5 p_{\varsigma})$. Here, γ_5 is the chiral operator with respect to the 4-dimensional induced chiral gauge theory. Because $\gamma_5 p_{\varepsilon}$ commutes with all the 4-dimensional gamma matrices, we can replace it with its eigenvalues $\pm (p_{\varepsilon}^2)^{1/2}$ in each eigenspace to find that the integrand depends on $p_5 \pm (p_{\varepsilon}^2)^{1/2}$. As a result, p_{ε} , the integration over that would lead to the finiteness of the loop integration in the usual dimensional regularization, is absorbed in the integration over p_5 . It leaves the overall loop integration unregularized. In order to circumvent this difficulty around the dimensional regularization and the DW fermion, we propose the partially dimensional regularization (PDR), where we apply the dimensional regularization to not all of field contents. In the case of the DW fermion, the dimensional regularization is applied only to the gauge field, while fermions are regularized with the PV regularization. As will be demonstrated in this paper, this PDR

¹Although the consistent and covariant anomalies differ by the Bardeen-Zumino current [121], it vanishes when the theory is free from any gauge anomaly. Therefore one needs no counter term in anomaly-free cases.

might be useful to calculate the effective action.

This paper is organized as follows. In Sec.A.2, we briefly review the DW fermion and the induced chiral fermion loops on the domain-wall. There the vacuum polarization was discussed as an explicit example to see how we regularized fermion loops with the PV regularization. Then in Sec.A.3, we show that this scheme gives the consistent anomaly in the 2-dimensional case. In Sec.A.4, we present how the PDR works as a regularization with an example of the 4-dimensional Yukawa theory. It is shown that the non-conservation of ε -th components of momenta results in the regularization. In Sec.A.5, we analyze the case where the PDR is applied to the 5-dimensional DW fermion. In the theory the induced chiral gauge theory is 4-dimensional. We investigate the self-energy of the fermion and discuss the renormalization. Finally in Sec.A.6 we give the summary and several open questions.

A.2 A brief review of the domain-wall fermions

Let us have a brief review of some results in our previous work [140], where we have stated that one can regularize a chiral fermion loop with a sort of PV fields. Although the main topic of the work was related to the gradient flow, the discussion around the regularization also holds in the case where the flow is switched off.

Consider the (2d + 1)-dimensional action for a DW fermion:

$$S = \int \frac{d^{2d}p}{(2\pi)^{2d}} ds \Big[\bar{\psi}(-p,s) \left(i \not\!\!p + \not\!\!A + \gamma_5 \partial_s - \epsilon(s) M \right) \psi(p,s) + \bar{\phi}(-p,s) \left(i \not\!\!p + \not\!\!A + \gamma_5 \partial_s + M \right) \phi(p,s) \Big] + \int \frac{d^{2d}p}{(2\pi)^{2d}} \frac{1}{4} F_{\mu\nu}(-p) F_{\mu\nu}(p),$$
(1)

where we have represented the fields in the momentum space with respect to 2d directions, and in the coordinate space with respect to the (2d + 1)th direction. We have denoted the coordinate for this direction as s. ψ is the ordinary DW fermion, while ϕ is an auxiliary bosonic spinor field which is necessary for subtracting undesirable bulk contributions. γ_5 is the gamma matrix corresponding to the s-direction, or the chiral operator from the viewpoint of the 2d-dimensional domain-wall.² $F_{\mu\nu}$ is a 2d-dimensional gauge field strength, and we define its couplings to ψ and ϕ through copying it along the s-direction. It implies $A_{2d+1} = 0$. The

²Although we consider a (2d + 1) -dimensional theory we have assigned the subscript "five" to the matrix, since we would like to eventually describe a four-dimensional chiral gauge theory and the matrix will correspond to the usual chiral operator.



Figure 1: The diagram of the vacuum polarization involving a fermion loop. Although we compute it as a (2d + 1)-dimensional diagram, one can integrate out the s and s'-dependence of the localized mode to obtain an effectively 2d-dimensional quantity.

propagators for ψ and ϕ are obtained from the action (1), and are given by

$$G_{\psi}(p,s,s') = \begin{cases} S^{(+)} + D^{(+)} & (0 < s, s') \\ D^{(-+)} & (s < 0 < s') \\ D^{(+-)} & (s' < 0 < s) \\ S^{(-)} + D^{(-)} & (s, s' < 0) \end{cases}, \quad G_{\phi}(p,s,s') = S^{(-)}, \tag{2}$$

where

$$S^{\pm}(p,s,s') = -\left[\theta(s-s')\frac{1}{2}\left(\frac{i\not\!\!p\pm M}{\sqrt{p^2+M^2}} - \gamma_5\right)e^{-(s-s')\sqrt{p^2+M^2}} + \theta(s'-s)\frac{1}{2}\left(\frac{i\not\!p\pm M}{\sqrt{p^2+M^2}} + \gamma_5\right)e^{-(s'-s)\sqrt{p^2+M^2}}\right],$$
(3)

$$D^{\mp\pm}(p,s,s') = \mp \sqrt{p^2 + M^2} \frac{i\not\!p}{p^2} \frac{\gamma_5}{2} \left(\frac{i\not\!p \pm M}{\sqrt{p^2 + M^2}} \pm \gamma_5\right) e^{\pm(s-s')\sqrt{p^2 + M^2}}.$$
 (5)

Here $S^{(\pm)}$ are the usual propagators for (2d + 1)-dimensional fermions with their masses $\pm M$. On the other hand, $D^{(\pm)}$ and $D^{(\mp\pm)}$ are ones for the massless modes localized on the domainwall, as is seen from the factors of exponential. They correspond to the 2*d*-dimensional chiral fermion. Using the propagators, the vacuum polarization with an external momentum k_{μ} (Fig.(1)) is calculated as

$$\int \frac{d^{2d}p}{(2\pi)^{2d}} \int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} ds' \Pi_{\mu\nu} = \int \frac{d^{2d}p}{(2\pi)^{2d}} \int_{0}^{+\infty} ds \int_{0}^{+\infty} ds' \Pi_{\mu\nu}^{blk} + \int \frac{d^{2d}p}{(2\pi)^{2d}} \Pi_{\mu\nu}^{DW}, \quad (6)$$

where

$$\Pi_{\mu\nu}^{blk} = \text{Tr}[S^{(+)}\gamma_{\mu}S^{(+)\prime}\gamma_{\nu}] - \text{Tr}[S^{(-)}\gamma_{\mu}S^{(-)\prime}\gamma_{\nu}],\tag{7}$$

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$$\Pi_{\mu\nu}^{DW} = \int_{-\infty}^{0} ds \int_{-\infty}^{0} ds' \Big\{ \operatorname{Tr}[D^{(-)}\gamma_{\mu}D^{(-)'}\gamma_{\nu}] + \operatorname{Tr}[D^{(-)}\gamma_{\mu}S^{(-)'}\gamma_{\nu}] + \operatorname{Tr}[S^{(-)}\gamma_{\mu}D^{(-)'}\gamma_{\nu}] \Big\} \\ + \int_{-\infty}^{0} ds \int_{0}^{+\infty} ds' \Big\{ \operatorname{Tr}[D^{(-+)}\gamma_{\mu}D^{(+-)'}\gamma_{\nu}] - \operatorname{Tr}[S^{(-)}\gamma_{\mu}S^{(-)'}\gamma_{\nu}] \Big\} \\ + \int_{0}^{\infty} ds \int_{-\infty}^{0} ds' \Big\{ \operatorname{Tr}[D^{(+-)}\gamma_{\mu}D^{(-+)'}\gamma_{\nu}] - \operatorname{Tr}[S^{(-)}\gamma_{\mu}S^{(-)'}\gamma_{\nu}] \Big\} \\ + \int_{0}^{+\infty} ds \int_{0}^{+\infty} ds' \Big\{ \operatorname{Tr}[D^{(+)}\gamma_{\mu}D^{(+)'}\gamma_{\nu}] + \operatorname{Tr}[D^{(+)}\gamma_{\mu}S^{(+)'}\gamma_{\nu}] + \operatorname{Tr}[S^{(+)}\gamma_{\mu}D^{(+)'}\gamma_{\nu}] \Big\}.$$
(8)

Here the arguments of the factors with primes are (p', s', s) with p' = p - k, while those without primes are (p, s, s'). $\Pi_{\mu\nu}^{DW}$ and $\Pi_{\mu\nu}^{blk}$ represent the contributions which is and is not localized on the domain-wall, respectively. $\Pi_{\mu\nu}^{blk}$ vanishes in the region s < 0, where the massive modes of ψ and ϕ cancel each other. Furthermore, one can easily find out that $\Pi_{\mu\nu}^{blk}$ actually vanishes even in the region s > 0 for the following observation. The parity-even contributions of ψ and ϕ to $\Pi_{\mu\nu}^{blk}$ are canceled, since they depend on even powers of M and take the same forms except for the overall sign. On the other hands, the parity-odd contributions would yield the Charn-Simons term, which vanishes in our setup where A_{μ} is independent of s and $A_{2d+1} = 0$. Therefore, one concludes that

$$\Pi^{blk}_{\mu\nu} \equiv 0. \tag{9}$$

At the same time, however, $\Pi^{DW}_{\mu\nu}$ has a contribution of the massive modes S^{\pm} as appears in Eq.(8). It is the result of the interactions to the modes localized on the domain-wall as in Eq.(8).

After some calculation we obtain the following result:

$$\Pi^{DW}_{\mu\nu} = \frac{1}{2} \left\{ \operatorname{Tr} \left[\frac{i\not\!p}{p^2} \gamma_{\mu} \frac{i\not\!p'}{p'^2} \gamma_{\nu} \right] - \operatorname{Tr} \left[\frac{i\not\!p + M}{p^2 + M^2} \gamma_{\mu} \frac{i\not\!p' + M}{p'^2 + M^2} \gamma_{\nu} \right] \right\} + \frac{1}{4} \operatorname{Tr} \left[\frac{i\not\!p}{p^2} \hat{\gamma}_5 \gamma_{\mu} \frac{i\not\!p'}{p'^2} \gamma_{\nu} \right] + \frac{1}{4} \operatorname{Tr} \left[\frac{i\not\!p}{p^2} \gamma_{\mu} \frac{i\not\!p'}{p'^2} \hat{\gamma}_5' \gamma_{\nu} \right]$$
(10)

with

$$\hat{\gamma}_5 = 2 \left(1 - \frac{\sqrt{p^2 + M^2} \left(\sqrt{p^2 + M^2} \sqrt{p'^2 + M^2} + M^2 \right)}{\sqrt{p'^2 + M^2} \left(\sqrt{p^2 + M^2} + \sqrt{p'^2 + M^2} \right)^2} \right) \frac{M}{\sqrt{p^2 + M^2}} \gamma_5, \tag{11}$$

and $\hat{\gamma}'_5$ being the same expression with p and p' exchanged. We emphasize that Eq.(10) is a nontrivial result of the combination and cancellation between terms from each lines in Eq.(8). In particular, the first line of Eq.(10) contains no root factor such as $\sqrt{p^2 + M^2}$, and is a local expression. It is regarded as the parity-even contribution with a single usual PV field. The leading divergent part is thus eliminated. On the contrary, the second line is highly nonlocal expression, and it can be interpreted as fermion loops with a deformed chiral operator Eq.(11) inserted either of the two vertices. When the loop momentum is sufficiently large, *i.e.* $p, p' \gg M$, the deformed operator is roughly evaluated as

$$\hat{\gamma}_5, \, \hat{\gamma}_5' \sim \frac{M}{\sqrt{p^2}} \, \gamma_5. \tag{12}$$

This means that the second line of Eq.(10) contains a suppression factor of p^{-1} on the loop divergence. Moreover, the factor makes it possible to expand the contribution with respect to M^2/p^2 and M^2/p'^2 , to obtain a series of M with the odd-order. By expanding the entire $\Pi^{DW}_{\mu\nu}$, we obtain the following form:

$$\Pi_{\mu\nu}^{DW} = \sum_{n\geq 1} a_n \, M^{2n} \operatorname{Tr}[A_n] + \sum_{n\geq 1} b_n \, M^{2n-1} \operatorname{Tr}[B_n].$$
(13)

Here a_n and b_n are numerical coefficients, A_n and B_n are matrices of $O(p^{-2n-2})$ and $O(p^{-2n-1})$, respectively. Each of B_n includes one γ , and they represent the parity-odd part. In particular, in the case of d = 1 and the resulting gauge theory is anomalous, they represent the anomaly.

In order to regularize $\Pi^{DW}_{\mu\nu}$, it is sufficient to introduce "PV pairs" and cancel the first few terms in Eq.(13). Therefore, the action to consider is

$$S \to \int \frac{d^{2d}p}{(2\pi)^{2d}} ds \Big[\bar{\psi}(-p,s) \left(i \not\!\!p + \not\!\!A + \gamma_5 \partial_s - \epsilon(s) M \right) \psi(p,s) \\ + \bar{\phi}(-p,s) \left(i \not\!\!p + \not\!\!A + \gamma_5 \partial_s + M \right) \phi(p,s) \\ + \sum_{i=1}^{m} \sum_{r=1}^{|C_i|} \bar{\psi}_{i,r}(-p,s) \left(i \not\!\!p + \not\!\!A + \gamma_5 \partial_s - \epsilon(s) M_i \right) \psi_{i,r}(p,s) \\ + \sum_{i=1}^{m} \sum_{r=1}^{|C_i|} \bar{\phi}_{i,r}(-p,s) \left(i \not\!\!p + \not\!\!A + \gamma_5 \partial_s + M_i \right) \phi_{i,r}(p,s) \Big] \\ + \int \frac{d^{2d}p}{(2\pi)^{2d}} \frac{1}{4} F_{\mu\nu}(-p) F_{\mu\nu}(p), \tag{14}$$

where the masses of the additional pairs, $\psi_{i,r}$ and $\phi_{i,r}$, are denoted as M_i . For each *i*, all of $\psi_{i,r}$'s are simultaneously either fermionic or bosonic, and the corresponding $\phi_{i,r}$'s follow the opposite statistics. We have introduced $\sum_{i=1}^{m} |C_i|$ PV pairs in total. We take the sign of an integer C_i positive when $\psi_{i,r}$'s are fermionic. The PV pairs yield the same form of contributions to the vacuum polarization as the original pair, changing $\Pi_{\mu\nu}^{(DW)}$ as

$$\Pi_{\mu\nu}^{DW} = \sum_{i=0}^{m} C_i \left(\sum_{n \ge 1} a_n \, M_i^{2n} \operatorname{Tr}[A_n] + \sum_{n \ge 1} b_n \, M_i^{2n-1} \operatorname{Tr}[B_n] \right)$$
(15)

with $M_0 = M$ and $C_0 = 1$. Therefore, the conditions to regularize $\Pi_{\mu\nu}^{DW}$ are summarized as

follows:

Note that the zeroth-order condition $\sum_{i=0}^{m} C_i = 0$ is not necessary because it corresponds to elimination of the leading divergence, which have been removed automatically in Eq.(10). Rather, in order to prevent the PV pairs from generating extra massless modes, we require

$$\sum_{i=1}^{m} C_i = 0.$$
 (17)

Eqs.(16) and Eq.(17) can be solved with respect to C_i and M_i when we introduce a sufficiently number of the PV pairs. The most important point of the above analysis is that the parity-odd part has a non-locally deformed chiral operator, and that it enables us to regularize that part in the same manner as the ordinary PV regularization. Therefore, we have obtained the way to regularize the vacuum polarization even in the two-dimensional theory is anomalous. Note that it works even for the two-dimensional theory with anomaly. The general loop diagrams are also regularized by Eq.(14).

Finally, we briefly comment on why the present regularization yields the gauge-invariant result in the anomaly-free case, as mentioned in [140]. If we take the limit $M \to \infty$, Eq.(10) takes the form of

$$\Pi_{\mu\nu}^{DW} \rightarrow \frac{1}{2} \operatorname{Tr} \left[\frac{i\not{p}}{p^{2}} \gamma_{\mu} \frac{i\not{p'}}{p'^{2}} \gamma_{\nu} \right] + \frac{1}{4} \operatorname{Tr} \left[\frac{i\not{p}}{p^{2}} \gamma_{5} \gamma_{\mu} \frac{i\not{p'}}{p'^{2}} \gamma_{\nu} \right] + \frac{1}{4} \operatorname{Tr} \left[\frac{i\not{p}}{p^{2}} \gamma_{\mu} \frac{i\not{p'}}{p'^{2}} \gamma_{5} \gamma_{\nu} \right]$$
$$= \frac{1}{2} \left(\operatorname{Tr} \left[\frac{i\not{p}}{p^{2}} \frac{1 + \gamma_{5}}{2} \gamma_{\mu} \frac{i\not{p'}}{p'^{2}} \gamma_{\nu} \right] + \operatorname{Tr} \left[\frac{i\not{p}}{p^{2}} \gamma_{\mu} \frac{i\not{p'}}{p'^{2}} \frac{1 + \gamma_{5}}{2} \gamma_{\nu} \right] \right).$$
(18)

Note that if one regularizes Eq.(18), it is regarded as a covariant regularization, a prescription where one regularizes one-loop diagrams with only one projection operator inserted at some vertex. Then it is natural that the regularized theory Eq.(14) gives the gauge-invariant parity-even part. Note that Eq.(18) includes the average over the points where the projection operator is inserted. The bose symmetry is thus automatically maintained. These facts imply that the theory has no fake anomaly.³ However, as is mentioned above, the form of Eq.(10) has been

³The present regularization is specifically interesting because it regularizes the parity-even part by PV regularization, while it deforms non-locally the parity-odd part with the same PV mass.

obtained as the result of all the relevant calculation (integration over *s*-direction, combination and cancellation of the terms). Although the DW fermion indeed realizes a good regularization, the relationship to the covariant regularizations still remains to be figured out.

A.3 Chiral anomaly

As is well known, chiral gauge theory has the abelian anomaly (fermion number anomaly). In this section, we derive the anomaly in our regularization. We find that the gauge invariant fermion number current is defined as

$$J_{\mu}(x) \equiv \int ds \Big[\bar{\psi}(x,s) \gamma_{\mu} \psi(x,s) + \bar{\phi}(x,s) \gamma_{\mu} \phi(x,s) + \sum_{i=1}^{m} \sum_{r=1}^{|C_i|} \Big[\bar{\psi}_{i,r}(x,s) \gamma_{\mu} \psi_{i,r}(x,s) + \bar{\phi}_{i,r}(x,s) \gamma_{\mu} \phi_{i,r}(x,s) \Big] \Big],$$
(19)

which is the Noether current for the U(1) rotation of the DW fermion, subtracting field, and all the regulator fields. We will show that this current reproduces the correct anomaly.

Evaluation of anomaly

In the following, we evaluate the expectation value of the current Eq.(19) in the presence of the background gauge field A_{μ} . We focus on the case of the two-dimensional Abelian chiral gauge theory. In this case, we need not introduce the PV pairs because the only divergent part in the fermion loop is already subtracted with ϕ .

The Feynman diagram corresponding to the current is shown in Fig.2. This expectation value can be expressed in terms of the vacuum polarization Eq.(6) as follows:

$$\left\langle J_{\mu}(k)\right\rangle_{A} = \int \frac{d^{2}p}{(2\pi)^{2}} \int ds \int ds' \Pi_{\mu\nu} A_{\nu}(k) \tag{20}$$

$$= \int \frac{d^2 p}{(2\pi)^2} \,\Pi^{DW}_{\mu\nu} A_{\nu}(k). \tag{21}$$

Recall that $\Pi_{\mu\nu}^{blk}$ vanishes as stated in the previous section. Using the fact that the parity-even part of $\Pi_{\mu\nu}^{DW}$ is transverse, the divergence of the current is expressed as

$$k_{\mu} \langle J_{\mu} \rangle_{A} = \int \frac{d^{2}p}{(2\pi)^{2}} k_{\mu} \Pi^{DW}_{\mu\nu} A_{\nu}$$
(22)

$$= \frac{1}{4}A_{\nu}\int \frac{d^2p}{(2\pi)^2} \operatorname{Tr}\left[\frac{1}{ip}\hat{\gamma}_5 \not k \frac{1}{ip'}\gamma_{\nu} + \frac{1}{ip} \not k \frac{1}{ip'}\hat{\gamma}_5'\gamma_{\nu}\right]$$
(23)

$$= \frac{-1}{4} A_{\nu} \int \frac{d^2 p}{(2\pi)^2} \left(f(p, p') + f(p', p) \right) \operatorname{Tr} \left[\left(\frac{1}{\not p} - \frac{1}{\not p'} \right) \gamma_5 \gamma_{\nu} \right],$$
(24)



Figure 2: A diagram representing the anomalous current

where

$$f(p_1, p_2) \equiv 2 \left(1 - \frac{\sqrt{(p_1)^2 + M^2} \left(\sqrt{(p_1)^2 + M^2} \sqrt{(p_2)^2 + M^2} + M^2 \right)}{\sqrt{(p_2)^2 + M^2} \left(\sqrt{(p_1)^2 + M^2} + \sqrt{(p_2)^2 + M^2} \right)^2} \right) \frac{M}{\sqrt{(p_1)^2 + M^2}}.$$
 (25)

By expanding f as

$$f(p,p') \sim f(p',p) \sim 2\left(\frac{3}{4} - \frac{M^2}{4(p^2 + M^2)}\right)\frac{M}{\sqrt{p^2 + M^2}} + \mathcal{O}(k^2/M^2)$$
 (26)

and taking the limit $k^2/M^2 \to 0$, we obtain

$$\int \frac{d^2 p}{(2\pi)^2} k_{\mu} \Pi^{DW}_{\mu\nu} \to -\int \frac{d^2 p}{(2\pi)^2} \left(\frac{3}{4} - \frac{M^2}{4(p^2 + M^2)}\right) \frac{M}{\sqrt{p^2 + M^2}} \operatorname{Tr}\left[\left(\frac{1}{\not{p}} - \frac{1}{\not{p}'}\right) \gamma_5 \gamma_{\nu}\right]$$
(27)

$$= \int \frac{d^2 p}{(2\pi)^2} \left(\frac{3}{4} - \frac{M^2}{4(p^2 + M^2)}\right) \frac{M}{\sqrt{p^2 + M^2}} \operatorname{Tr}\left[\frac{p'}{p'^2} \gamma_5 \gamma_\nu\right].$$
 (28)

Here the term proportional to 1/p has vanished in the last line because it is odd in p.

By using the Feynman parametrization

$$\frac{1}{[p^2 + M^2]^n} \frac{1}{p'^2} = n \int_0^1 dx \, \frac{(1-x)^{n-1}}{[(1-x)(p^2 + M^2) + x(p-k)^2]^{1+n}} \tag{29}$$

for $n = \frac{1}{2}$ and $\frac{3}{2}$, Eq.(28) can be rewritten as the following expression:

with

$$q \equiv p - xk,$$
 $\Delta \equiv x(1-x)k^2 + (1-x)M^2.$ (31)

Its integration over q leads to the following form:

$$\frac{3M}{16\pi} \operatorname{Tr}\left[\not\!\!\!\!/ x_5 \gamma_{\nu}\right] \int_0^1 dx \, \left[\left(\frac{1-x}{\Delta}\right)^{\frac{1}{2}} - \frac{M^2}{3} \left(\frac{1-x}{\Delta}\right)^{\frac{3}{2}} \right] \tag{32}$$

$$\rightarrow -\frac{1}{4\pi} \frac{M}{|M|} k_{\mu} \epsilon_{\mu\nu} \qquad (\text{as } k/M \to 0).$$
(33)

The resulting expression for the divergence of the current is

$$\langle k_{\mu}J_{\mu}\rangle_{A} = \frac{-1}{4\pi} \frac{M}{|M|} \epsilon_{\mu\nu}k_{\mu}A_{\nu}(k).$$
(34)

Thus, we have obtained the gauge invariant form of the fermion number anomaly with the correct normalization.⁴ From Eq.(34), it can be seen that the signature of M corresponds to the chirality of the chiral fermion.

Comments on Eq.(34)

We give three comments on the anomaly Eq.(34). Firstly, because we have considered the Abelian case, the fermion number current takes the same form as the gauge current does. In non-Abelian cases, the former is defined also by Eq.(19). On the other hand, the latter would be defined as

$$J^{a}_{\mu}(x) \equiv \int ds \Big[\bar{\psi}(x,s) \gamma_{\mu} t^{a} \psi(x,s) + \bar{\phi}(x,s) \gamma_{\mu} t^{a} \phi(x,s) + \sum_{i=1}^{m} \sum_{r=1}^{|C_{i}|} \Big[\bar{\psi}_{i,r}(x,s) \gamma_{\mu} t^{a} \psi_{i,r}(x,s) + \bar{\phi}_{i,r}(x,s) \gamma_{\mu} t^{a} \phi_{i,r}(x,s) \Big] \Big].$$
(35)

Here t^a is the hermitian generator in the representation of the fermion multiplet. This would reproduce the consistent gauge anomaly in general dimensions.

Secondly, let us consider why the correct fermion number anomaly has been derived in our regularization. Because the *regularized* Lagrangian Eq.(14) is invariant under the U(1) rotation, the corresponding current J_{μ} seems to be conserved even at the *quantum level*. However, it is not the case. In order to see this, let the gauge field evolve slowly in the *s*-direction. It can be realized by, for example, the gradient flow:

$$\partial_s A_\mu = -\frac{\xi \ \epsilon(s)}{M} \frac{\delta S[A]}{\delta A_\mu},\tag{36}$$

where ξ is a flow parameter and $\xi = 0$ corresponds to switching off the flow. In this situation $(\xi \neq 0)$, the parity-odd part of $\Pi_{\mu\nu}^{blk}$ in Eq.(21) does not vanish but contributes as the parity anomaly. Thus Eq.(34) must be replaced by

$$\langle k_{\mu}J_{\mu}\rangle_{A} = \frac{-1}{4\pi} \frac{M}{|M|} \epsilon_{\mu\nu} k_{\mu} A_{\nu}(k,s=0) + \frac{2M}{|M|} \int_{0}^{\infty} ds \ k_{\mu} \frac{\delta S_{CS}}{\delta A_{\mu}(k,s)},\tag{37}$$

where S_{CS} is the three-dimensional Chern-Simons action. The second term in Eq.(37) can be

⁴The coefficient $\frac{1}{4\pi}$ shows that it is the consistent anomaly. It would be replaced with $\frac{1}{2\pi}$ if it were the covariant one. It is interesting that the result is consistent one although our regularization is related to the covariant regularization as in Eq.(18).

rewritten as a surface term:

$$\frac{2M}{|M|} \int_0^\infty ds \ k_\mu \frac{\delta S_{CS}}{\delta A_\mu(k,s)} = \left[\frac{-1}{4\pi} \frac{M}{|M|} \epsilon_{\mu\nu} k_\mu A_\nu(k,s)\right]_{s=0}^{s=\infty}.$$
(38)

The contribution at s = 0 is canceled by the first term in Eq.(37) (See Ref. [143].) and thus we obtain

$$\langle k_{\mu}J_{\mu}\rangle_{A} = \frac{-1}{4\pi} \frac{M}{|M|} \epsilon_{\mu\nu} k_{\mu} A_{\nu}(k, s = \infty).$$
(39)

Because the gauge field becomes pure gauge at $s = \infty$, this quantity vanishes and then the current is conserved. In our case $(\xi \to 0)$, however, A_{μ} does not evolve: $A_{\mu}(k, s = \infty) = A_{\mu}(k)$. Thus Eq.(39) does not vanish and is equal to the previous result Eq.(34), which has been obtained by setting $\Pi_{\mu\nu}^{blk} = 0$ from the beginning. Consequently, we can understand the origin of the anomaly as follows. In the three-dimensional region except for the boundary, the current is conserved as seen from the regularized Lagrangian. However, this system is not closed under the condition $A_{\mu}(x,s) = A_{\mu}(x)$, and thus the current flows out to the infinity in s-direction. In the view point of the chiral mode on the domain-wall, this current non-conservation is seen as the anomaly.⁵

Finally, let us consider another definition of the fermion number current. Naively, in the ordinary 2n-dimensional system consisting of one chiral fermion, the U(1) current

$$\bar{\psi}_L \gamma_\mu P_L \psi_L \tag{40}$$

seems to be equal to the chiral U(1) current

$$\bar{\psi}_L \gamma_5 \gamma_\mu P_L \psi_L. \tag{41}$$

Therefore, it appears that in our regularization we can define the fermion number current alternatively as

$$J^{5}_{\mu}(x) \equiv \int ds \Big[\bar{\psi}(x,s) \gamma_{5} \gamma_{\mu} \psi(x,s) + \bar{\phi}(x,s) \gamma_{5} \gamma_{\mu} \phi(x,s) + \sum_{i=1}^{m} \sum_{r=1}^{|C_{i}|} \Big[\bar{\psi}_{i,r}(x,s) \gamma_{5} \gamma_{\mu} \psi_{i,r}(x,s) + \bar{\phi}_{i,r}(x,s) \gamma_{5} \gamma_{\mu} \phi_{i,r}(x,s) \Big] \Big].$$
(42)

However, this is not the case, since the regularized Lagrangian is not invariant under the transformation $\psi \to e^{i\alpha\gamma_5}\psi$. Indeed one can check that the divergence of this current differs from Eq.(34) by both parity-odd and even terms.

⁵When one considers a finite size system (-L < s < L) and imposes the periodic boundary condition, the current is conserved because another chiral fermion will be induced on the anti-domain wall s = L that *absorbs* the flowing-out current.

A.4 Partially dimensional regularization – Example–

In this section, we propose the PDR as a novel useful regularization and demonstrate how it works. Let us consider the 4D Yukawa theory as an example. Here we assume that the fermion is regularized by the ordinary PV fields. The PDR changes the spacetime dimension of the scalar field from 4 to $(4 + \epsilon)$ while keeping the fermion in the 4-dimensional space. We will show that the two parameters, the PV mass and ϵ , play the roles of the ultraviolet (UV) cutoffs and regularize all the UV divergences in the theory. In Sec.A.5, we will apply the PDR to the gauge sector in chiral gauge theory.

Yukawa theory

The regularized action is given by^6

$$S = \int d^{4+\epsilon} x \left[\frac{1}{2} \left(\partial_{\mu} \phi(x, x_{\epsilon}) \right)^{2} + \frac{\mu^{2}}{2} (\phi(x, x_{\epsilon}))^{2} + \frac{\lambda}{4!} (\phi(x, x_{\epsilon}))^{4} \right] \\ + \int d^{4} x \ \bar{\psi}(x) \left[\partial \!\!\!/ - m + g \phi(x, 0) \right] \psi(x),$$
(43)

where x_{ϵ} denotes the ϵ -dimensional coordinate. Note that we have omitted the PV fields for the fermion. In this system, the fermion $\psi(x)$ lives on the 4-dimensional brane embedded in the $(4 + \epsilon)$ -dimensional space, $x_{\epsilon} = 0$, while the scalar lives in the $(4 + \epsilon)$ -dimensional space.

From the Lagrangian Eq.(43), we obtain the Feynman rules as follows:

$$\overline{\psi(x)}\overline{\psi}(y) = \int \frac{d^4p}{(2\pi)^4} \ e^{ip \cdot (x-y)} \frac{1}{ip - m},\tag{44}$$

$$\overline{\phi(x,x_{\epsilon})\phi(y,y_{\epsilon})} = \int \frac{d^{4+\epsilon}k}{(2\pi)^{4+\epsilon}} e^{ik\cdot(x-y)+ik_{\epsilon}(x_{\epsilon}-y_{\epsilon})} \frac{1}{k^2+(k_{\epsilon})^2+\mu^2},$$
(45)



⁶We denote the Dirac fermion and scalar field by $\psi(x)$ and $\phi(x, x_{\epsilon})$, respectively, only in this section. The reader should not confuse them with the DW fermion and the subtracting field.

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Figure 3: A diagram including a scalar internal line with the Yukwa vertices at the ends

Here \hat{k} denotes the $(4 + \epsilon)$ -dimensional momentum, i.e. $\hat{k} = (k, k_{\epsilon})$, and we have written explicitly the δ -functions enforcing the conservation of momentum. Note that at the Yukawa vertex, Eq. (47), the δ -function is only for the 4-dimensions, and hence the ϵ -dimensional momentum k_{ϵ} of the scalar $\phi(\hat{k})$ is not conserved.

Under the above rules, we have to pay a special attention when dealing with a diagram including an internal scalar line with the Yukawa vertices at the ends. As an example, let us consider a diagram shown in Fig.3. The ϵ -dimensional momentum k_{ϵ} flows along the scalar line but does not along the fermion ones. In the Fourier space, the internal line gives

$$\int \frac{d^{\epsilon}k_{\epsilon}}{(2\pi)^{\epsilon}} \frac{1}{k^2 + (k_{\epsilon})^2 + \mu^2}$$
(48)

The integral over k_{ϵ} appears because it is not constrained by any δ -functions. Carrying out the integral, we obtain

$$\frac{\Gamma(1-\epsilon/2)}{(4\pi)^{\epsilon/2}} \frac{1}{[k^2+\mu^2]^{1-\epsilon/2}}.$$
(49)

The exponent $(1 - \epsilon/2)$ in the denominator plays a role of regularization for loop diagrams, as will be seen in the next subsubsection.

One-loop calculation

We show that how our method can regularize the divergences in the one-loop diagrams. Clearly, diagrams consisting of only the scalar lines, which are shown in Fig.4, are regularized in the same manner as in the ordinary dimensional regularization. On the other hand, a fermion loop, shown in Fig.5, is regularized by the PV fields ⁷. Therefore, what we are interested in is ones involving both the scalar and fermion internal lines. (See Fig.6.)

Let us consider the fermion self-energy diagram Fig.6(a), which we denote by $\Sigma(p)$. Using

⁷Note that the ϵ -dimensional momentum in the external ϕ lines is not conserved, that is $k_{\epsilon} \neq k'_{\epsilon}$, because the Yukawa vertex conserves momentum only in the 4-dimensions. However, this does not affect the unitarity as the cutoff is removed ($\epsilon \rightarrow 0$).



Figure 4: One-loop diagrams consisting of only the scalar lines



Figure 5: A fermion loop diagram



Figure 6: One-loop diagrams including both of the scalar and fermion internal lines
Eq.(49), we have

$$\Sigma(p) = g^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{i(\not p - \not k) - m} \frac{\Gamma(1 - \epsilon/2)}{(4\pi)^{\epsilon/2}} \frac{1}{[k^2 + \mu^2]^{1 - \epsilon/2}}$$
(50)

$$= -g^{2} \frac{\Gamma(1-\epsilon/2)}{(4\pi)^{\epsilon/2}} \int \frac{d^{4}k}{(2\pi)^{4}} \frac{i(\not p - \not k) + m}{(p-k)^{2} + m^{2}} \frac{1}{[k^{2} + \mu^{2}]^{1-\epsilon/2}}.$$
(51)

Its integration over k leads to the following expression:

$$\Sigma(p) = -g^2 \frac{\Gamma(-\epsilon/2)}{(4\pi)^{2+\epsilon/2}} \int_0^1 dx \ x^{-\epsilon/2} \left[i(1-x) \not p + m \right] (\Delta')^{\epsilon/2}$$
(52)

with

$$\Delta' \equiv x(1-x)p^2 + xm^2 + (1-x)\mu^2.$$
(53)

Here x is the Feynman parameter. This expression Eq.(52) is finite as long as $0 < \epsilon < 2$, and thus the UV divergence is regularized in our model. Furthermore, one can check that its difference from the result in the ordinary dimensional scheme is finite constants.

Next, we investigate the other diagram (Fig.6(b)), which is the one-loop vertex correction $\Gamma(p, k)$. Using Eqs.(44)–(47) and (49), we have

$$\Gamma(p,k) = g^3 \int \frac{d^4l}{(2\pi)^4} \frac{1}{i(\not p - \not l + \not k) - m} \frac{1}{i(\not p - \not l) - m} \frac{\Gamma(1 - \epsilon/2)}{(4\pi)^{\epsilon/2}} \frac{1}{[l^2 + \mu^2]^{1 - \epsilon/2}}.$$
 (54)

After some calculation, we obtain the following result:

$$\Gamma(p,k) = \frac{g^3}{16\pi^2} \int_{xyz} \left(\frac{-2}{(4z\pi)^{\epsilon/2}} \Gamma(-\epsilon/2) (\Delta'')^{\epsilon/2} + \frac{\left[i(z\not\!\!p + (1-x)\not\!\!k) + m\right] \left[i(z\not\!\!p - x\not\!\!k) + m\right]}{\Delta''} \right)$$
(55)

with

$$\int_{xyz} \equiv \int_0^1 dx \int_0^1 dy \int_0^1 dz \ \delta(x+y+z-1),$$
(56)

$$\Delta'' \equiv x(1-x)(p+k)^2 + y(1-y)p^2 - 2xy(p+k) \cdot p + (x+y)m^2 + z\mu^2.$$
(57)

Also in this case, the UV divergence is controlled thanks to ϵ .

Thus, the combination of the PDR and the PV regularization can regularize all the one-loop divergences. It is obvious that this is also true about higher-order loop diagrams. Note that the PDR is essentially as a sort of the analytic regularization [144–146], since the integration over k_{ε} gives k of the fractional power.

A.5 PDR for chiral gauge theory

In Sec.A.2 and A.3, we have regularized the fermion sector only. For practical calculations, however, we must also regularize the gauge sector. In this section, we apply the PDR to the

gauge sector while keeping the DW fermion on the 5-dimensional space. After that, we will show that all the one-loop UV divergences are regularized by the combination of the PDR and the PV regularization, and that they can be renormalized by local counter terms. For simplicity, we consider the case where the resulting physical theory is 4-dimensional.

The regularized model

Firstly, we replace the 4-dimensional gauge field $A_{\mu}(x)$ in Eq.(14) with the $(4 + \epsilon)$ -dimensional one $A_M(x, x_{\epsilon})$. Here x and x_{ϵ} are the coordinates of the 4-dimensional and ϵ -dimensional space, respectively, and the lower index M runs from 1 to $4 + \epsilon$. Next, we extend the gauge field in the s-direction, and thus the gauge field lives in the $(4 + \epsilon + 1)$ -dimensional space. Note that it is independent of s, that is $A_M(x, x_{\epsilon}, s) = A_M(x, x_{\epsilon})$. Furthermore, we set the $(4 + \epsilon + 1)$ th component of the field to 0: $A_{4+\epsilon+1} = 0$. On the other hand, the DW fermion (and all the regulator fields) live on the 5-dimensional brane $x_{\epsilon} = 0$ embedded in the $(4 + \epsilon + 1)$ -dimensional space. Therefore, the regularized action of this system is given by the following equation:

$$S_{reg} = \int d^4x ds \ \bar{\psi} \left[\not{D} + \gamma_5 \partial_s - M \epsilon(s) \right] \psi + \int d^4x ds \ \bar{\phi} \left[\not{D} + \gamma_5 \partial_s + M \right] \phi$$

$$+ \sum_{i=1}^m \sum_{r=1}^{|C_i|} \int d^4x ds \ \bar{\psi}_{i,r} \left[\not{D} + \gamma_5 \partial_s - M \epsilon(s) \right] \psi_{i,r}$$

$$+ \sum_{i=1}^m \sum_{r=1}^{|C_i|} \int d^4x ds \ \bar{\phi}_{i,r} \left[\not{D} + \gamma_5 \partial_s + M \right] \phi_{i,r}$$

$$+ \frac{1}{4g^2} \int d^{4+\epsilon}x \ \operatorname{tr}(F_{MN}(x, x_{\epsilon}))^2, \tag{58}$$

with

$$D_{\mu} = \partial_{\mu} - iA_{\mu}(x, 0) \qquad (\mu = 1, \cdots, 4).$$
 (59)

Fig.7 shows this model in the coordinate space.

In terms of G_{ψ} and G_{ϕ} that have been defined by Eqs.(2)-(5), the Feynman rules are expressed as follows:

$$\overline{\psi(x,s)\psi(y,s')} = \int \frac{d^4p}{(2\pi)^4} \ e^{ip \cdot (x-y)} \ G_{\psi}(p,s,s') \tag{60}$$

$$\overline{\phi(x,s)}\overline{\phi}(y,s') = \int \frac{d^4p}{(2\pi)^4} \ e^{ip \cdot (x-y)} \ G_{\phi}(p,s,s') \tag{61}$$

$$\overline{A^{a}_{\mu}(x,x_{\epsilon})A^{b}_{\nu}(y,y_{\epsilon})} = \int \frac{d^{4+\epsilon}}{(2\pi)^{4+\epsilon}} e^{ik\cdot(x-y)+ik_{\epsilon}\cdot(x_{\epsilon}-y_{\epsilon})} \frac{\delta^{ab}\delta_{\mu\nu}}{k^{2}+k_{\epsilon}^{2}} \qquad (\text{Feynman gauge})$$
(62)



Figure 7: The set up of our model Eq.(58) in the coordinate space. ψ 's and ϕ 's live on the (4 + 1)dimensional brane $x_{\varepsilon} = 0$ (solid brane). Their massless modes are localized on the 4-dimensional domain-wall (the thick line). On the other hand, A_M is defined as the gauge field on the $(4 + \varepsilon)$ dimensional brane s = 0 (the dotted brane), and it is duplicated along the s-direction.



Here we have written explicitly the delta-functions enforcing momentum conservation. We also have the gauge boson's three- and four-point vertices which are the same form as in the ordinary dimensional regularization. Note that the ϵ -dimensional momentum is not conserved at the gauge-fermion coupling. This situation is similar to the the Yukawa theory in the previous section.

Before calculating the one-loop diagrams, let us consider a diagram containing an internal line of A_{μ} with the gauge-fermion vertices at the ends. Similarly to the previous section, the internal line gives

$$\int \frac{d^{\epsilon}k_{\epsilon}}{(2\pi)^{\epsilon}} \frac{\delta_{\mu\nu}\delta^{ab}}{k^2 + (k_{\epsilon})^2} = \frac{\Gamma(1 - \epsilon/2)}{(4\pi)^{\epsilon/2}} \frac{\delta_{\mu\nu}\delta^{ab}}{(k^2)^{1 - \epsilon/2}}.$$
(64)

The exponent $(1 - \epsilon/2)$ plays a role of regularization as in Sec.A.4.

One-loop calculation

Firstly, we consider the gauge boson loop diagrams and the fermion loop diagram (See Fig.8 and Fig.9). It is obvious that the UV divergences of the former are regularized and renormalized to the $(4 + \epsilon)$ -dimensional gauge kinetic term as in the ordinary dimensional scheme. The latter are regularized by the subtracting field and the PV pairs as in Sec.A.2. Here note that the ϵ -



Figure 8: The gauge boson one-loop diagrams



Figure 9: The vacuum polarization diagram

dimensional momenta in the external gauge boson lines are not conserved: $k_{\epsilon} \neq k'_{\epsilon}$ because only 4-dimensional momentum is conserved at the fremion-fermion-gauge boson vertex. Therefore the UV divergence is renormalized by the 4-dimensional local counter term

$$\delta_g \int d^4x \, \mathrm{tr}(F_{\mu\nu})^2. \tag{65}$$

This term has the $(4 + \varepsilon)$ -dimensional gauge-invariance even when ε is finite, and does not cause any problem in the limit $\varepsilon \to 0$.

Next, we consider the DW fermion's self-energy diagram (Fig.10). This diagram induces radiative corrections to the massless and the bulk propagators because the external lines contain the both modes. Obviously, it is not easy to investigate what counter terms are necessary. Therefore, we adopt the following strategy. We start with analyzing the self-energy in the region far from the domain wall, i.e. $s \gg M^{-1}$, where the massless mode is suppressed and only the bulk modes remain in the diagram. Furthermore, the argument of renormalization is parallel to the ordinary 4-dimensional Dirac fermion because the gauge field does not vary in the s-direction. After that, we consider the case that M is large compared with the external momentum. This limit is equivalent to focusing on the correction to the massless propagator. As will see below, the counter terms that are necessary for the bulk modes in the region $s \gg M^{-1}$ are sufficient to renormalize the massless mode.

For $s \gg M^{-1}$, we can evaluate the diagram in the 5-dimensional Fourier space because the propagator of the bulk modes are translation invariant. Noting that the momentum of the A. A Note on Proposal for Gauge-invariant Regularization of Chiral Gauge Theory 111



Figure 10: The fermion self-energy diagram

gauge field does not have the 5-th component, we have

$$\int \frac{d^4k}{(2\pi)^4} (i\gamma_\mu t^a) \frac{1}{i(\not\!p - \not\!k) + i\gamma_5 p_5 - M} (i\gamma_\nu t^b) \frac{\Gamma(1 - \epsilon/2)}{(4\pi)^{\epsilon/2}} \frac{\delta_{\mu\nu} \delta^{ab}}{(k^2)^{1 - \epsilon/2}}$$
(66)

$$= (ig)^{2} t^{a} t^{a} \frac{\Gamma(1-\epsilon/2)}{(4\pi)^{\epsilon/2}} \int \frac{d^{4}k}{(2\pi)^{4}} \frac{2i(\not p - \not k) - 4M'}{(p-k)^{2} + |M'|^{2}} \frac{1}{(k^{2})^{1-\epsilon/2}}$$
(67)

with

$$M' \equiv -i\gamma_5 p_5 + M. \tag{68}$$

Here we have denoted the 5-th component of momentum of the fermion by p_5 . Note that the expression (67) seems to be the self-energy in the 4-dimensional theory with the effective mass M'. Therefore it is obvious that the UV divergences can be renormalized by the ordinary procedure in the 4-dimensions. In order to obtain the counter terms that eliminate the UV divergences, we expand Eq.(67) by p:

$$(ig)^{2}t^{a}t^{a}\frac{\Gamma(1-\epsilon/2)}{(4\pi)^{\epsilon/2}}\int \frac{d^{4}k}{(2\pi)^{4}} \frac{2}{(k^{2})^{1-\epsilon/2}} \times \left[\frac{-ik-2M'}{k^{2}+|M'|^{2}} + \frac{ip}{k^{2}+|M'|^{2}} + \frac{-2ik-4M'}{(k^{2}+|M'|^{2})^{2}} p \cdot k + \mathcal{O}(p^{2})\right].$$
(69)

We concentrate on the UV divergent terms and obtain

$$(ig)^{2}t^{a}t^{a}\frac{\Gamma(1-\epsilon/2)}{(4\pi)^{\epsilon/2}}\int \frac{d^{4}k}{(2\pi)^{4}} \frac{1}{(k^{2})^{1-\epsilon/2}} \left[\frac{-4M'}{k^{2}+|M'|^{2}}+i\not\!\!p\frac{k^{2}}{(k^{2}+|M'|^{2})^{2}}\right].$$
(70)

After some calculation, we obtain the following expression:

$$\rightarrow (ig)^2 t^a t^a \ \Gamma(-\epsilon/2) \left[\frac{-M'}{4\pi^2} + \frac{ip}{16\pi^2} \right] \quad (\text{as } \epsilon \rightarrow 0).$$
(72)

It can be seen that the UV divergences are regularized by ϵ as in Sec.A.4. Consequently, the



Figure 11: The fermion self-energy diagram with external legs

necessary counter terms are found to be 8

$$\int d^4x \int_{s>0} ds \ \bar{\psi} \left[\delta_Z \not{\partial} - \delta_M M' \right] \psi, \tag{73}$$

where

$$\delta_Z = \frac{(ig)^2 t^a t^a}{8\pi^2} \frac{1}{\epsilon},\tag{74}$$

$$\delta_M = \frac{(ig)^2 t^a t^a}{2\pi^2} \frac{1}{\epsilon}.$$
(75)

Note that Eqs.(74) and (75) are mass-independent renormalizations. In particular, the mass is renormalized multiplicatively. Therefore, we expect that Eqs.(74) and (75) hold even if the mass M depends on s:

$$\int d^4x \int ds \ \bar{\psi} \left[\delta_Z \partial \!\!\!/ + \delta_M \left(\gamma_5 \partial_s - M \epsilon(s) \right) \right] \psi. \tag{76}$$

In other words, these counter terms would also be sufficient to renormalize the UV divergences of the massless mode.

Let us check this point. In order to focus on the self-energy of the massless mode, we attach the external legs to it (See Fig.11) and take the limit $M \gg p$. In this situation, the external propagators $G_{\psi}(p, s, s')$ and $G_{\psi}(p, t', t)$ reduce to

$$G_{\psi}(p,t',t) \to -\frac{i \not\!\!\!/ M}{p^2} P_R \ e^{-(|t|+|t'|)|M|}$$
(77)

where we have assumed M > 0. (For M < 0, the chirality projection should be replaced with P_L .) Eqs.(77) and (78) are nothing but the propagators of the chiral fermion localized on the domain wall. Because the self-energy is put between P_R and P_L , we can drop the terms except

⁸The difference between the coefficients of $\partial and \gamma_5 \partial_s$ reflects the lack of the 5-dimensional Lorentz invariance. On the other hand, $\gamma_5 \partial_s$ and M share the same coefficient because they are regarded as the effective mass in the 4-dimensions.

for those proportional to $\not p - \not k$. This means that no additional mass counter terms for the massless mode are needed, and hence we can focus on the wave function renomalization. We evaluate the self-energy part keeping the 5-th coordinate in the real space as follows:

$$\frac{\Gamma(1-\epsilon/2)}{(4\pi)^{\epsilon/2}} \int \frac{d^4k}{(2\pi)^4} \int ds dt \ (ig)^2 t^a t^a \ \frac{1}{(k^2)^{1-\epsilon/2}} \gamma_\mu G_\psi(p-k,t,s) \gamma_\mu \ e^{-(|s|+|t|)M}.$$
(79)

Substituting Eqs.(2)-(5) into the internal propagator $G_{\psi}(p-k,t,s)^9$ and using the symmetry under the reflection $s \leftrightarrow -s, t \leftrightarrow -t$, we obtain the following expression:

$$(ig)^{2}t^{a}t^{a}\frac{2\Gamma(1-\epsilon/2)}{(4\pi)^{\epsilon/2}} \int \frac{d^{4}k}{(2\pi)^{4}} \frac{1}{(k^{2})^{1-\epsilon/2}} \\ \left[\int_{0}^{\infty} ds \int_{0}^{\infty} dt \; \frac{ip'M(\sqrt{p'^{2}+M^{2}}+M)}{p'^{2}(\sqrt{p'^{2}+M^{2}})} \; e^{-(s+t)(\sqrt{p'^{2}+M^{2}}+M)} \right. \\ \left. + \int_{0}^{\infty} ds \int_{s}^{\infty} dt \; \frac{ip'}{\sqrt{p'^{2}+M^{2}}} \; e^{-t(\sqrt{p'^{2}+M^{2}}+M)-s(M-\sqrt{p'^{2}+M^{2}})} \right. \\ \left. + \int_{0}^{\infty} dt \int_{t}^{\infty} ds \; \frac{ip'}{\sqrt{p'^{2}+M^{2}}} \; e^{-s(\sqrt{p'^{2}+M^{2}}+M)-t(M-\sqrt{p'^{2}+M^{2}})} \right. \\ \left. + \int_{0}^{\infty} ds \int_{-\infty}^{0} dt \frac{ip'(\sqrt{p'^{2}+M^{2}}-M)}{p'^{2}} \; e^{-(s-t)(\sqrt{p'^{2}+M^{2}}+M)} \right].$$
(80)

Here, the first and fourth terms come from the massless mode in $G_{\psi}(p-k,t,s)$ while the second and third ones from the bulk modes. Note that the latter also contribute to the wave function renormalization of the massless mode. Carrying out the integrals over s and t, we obtain a simple result:

$$\frac{-2g^2}{M} t^a t^a \frac{\Gamma(1-\epsilon/2)}{(4\pi)^{\epsilon/2}} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^{1-\epsilon/2}} \frac{i\not\!p'}{p'^2}
= \frac{-g^2}{8\pi^2 M} t^a t^a \, i\not\!p \int_0^1 dx \, (1-x) \left(\frac{(1-x)p^2}{4\pi}\right)^{\epsilon/2} \, \Gamma(-\epsilon/2).$$
(81)

The divergent term in Eq.(81),

can be renormalized by the counter term Eq.(76). Indeed, putting the counter term between the external legs and integrating over s, we obtain

$$\int ds \,\delta_Z \,i\not\!p \,e^{-2|s|M} = \frac{(ig)^2 t^a t^a}{8\pi^2 M} i\not\!p \frac{1}{\epsilon},\tag{83}$$

which cancels the UV divergent term Eq.(82).

Thus, we conclude that the UV divergences in the DW fermion self-energy can be renor-

 $^{^{9}}$ Note that one cannot take the same limit as Eqs.(77) and (78) for the internal propagator because the momentum in the internal line should be integrated.

malized by the counter term Eq.(76). It is easy to check that the vertex correction diagram would be renormalized in the same manner:

$$\int d^4x \int ds \,\,\delta'_Z \bar{\psi}(x,s) \mathcal{A}(x,x_\epsilon=0)\psi(x,s). \tag{84}$$

A.6 Summarizing remarks

In this appendix we have proposed a regularization that incorporates with the DW fermion. It regulates fermion loops by introducing the PV pairs, and regulates the other loops by applying the PDR. By considering three-dimensional theory, we have explicitly shown that the regularization works both in parity-even and odd parts, and obtained the correct Abelian anomaly in the two dimensional chiral gauge theory. It is a good regularization of chiral gauge theory on the DW since it produces no fake anomaly. A significant point of this formulation is that it is given at the Lagrangian level, that is, the theory is well-defined from the starting point, and we need not introduce counter terms to recover the gauge symmetry. Furthermore, our regularization might provide a rather easy method for the explicit calculation of loop corrections. We also have investigated the renormalization of the DW fermion. There we have checked that once we add the counter terms to the full theory, then the renormalization of the massless mode is automatically achieved.

Although we expect that the method keeps gauge invariance to all-order, it should be rigorously proved. We will study the Ward identity in the regularized theory.

Unfortunately, the above study seems to tell us few clues to revealing the connection between field theory and the matrix model. However, it is remarkable that we have found out an extra dimension serves a sophisticated structure to regulate chiral gauge theory. The existence of extra dimension is also predicted in the preceding work on the IIB matrix model [41]. Therefore, by investigating the regularization of chiral gauge theory in this direction, it will be possible to clarify that connection.

We are going to report the progress in these studies elsewhere.

B Change of effective action and amplitude by diffeomorphism

One might take seriously the discrepancy between the two actions, Eqs. (3.14) and (3.15). If some procedure makes the one-loop effective action identical to Eq.(3.15), then one can regard the propagation of the NC U(1) gauge field as that of the metric fluctuation at the action level. For example, let us consider the diffeomorphism transformation (3.17). Then the metric fluctuation $h^{\mu\nu}$ transforms as

$$\delta h^{\mu\nu} = \partial^{\mu}\xi^{\nu} + \partial^{\nu}\xi^{\mu}, \tag{85}$$

$$\xi^{\mu} = \theta^{\mu\alpha} A_{\alpha}, \tag{86}$$

although the diffeomorphism transformation is not actually realized in the NC U(1) gauge theory. Assuming that the above change of $h^{\mu\nu}$ is verified, we obtain

$$h^{\mu\nu} = \Lambda_{\rm NC}^{-2} \times \left(\theta^{\mu\alpha} F_{\alpha}^{\ \nu} + \theta^{\nu\alpha} F_{\alpha}^{\ \mu} + \frac{1}{2} \delta^{\mu\nu} \theta^{\alpha\beta} F_{\alpha\beta}\right) + \Lambda_{\rm NC}^{-4} \times \mathcal{O}(A^2). \tag{87}$$

In the viewpoint of the NC U(1) gauge theory, such a transformation produces additional interaction terms to the action. If we take this new action as the tree-level action, one-loop effective action of A_{μ} is given by

$$\Gamma_{\Phi}^{(y)} \sim -\frac{1}{32\pi^2 g^2} \int d^4 y \bigg[\frac{\Lambda^4}{4\Lambda_{\rm NC}^4} \theta^{\mu\nu} F_{\mu\nu} \theta^{\lambda\rho} F_{\lambda\rho} + \frac{\Lambda^2}{24\Lambda_{\rm NC}^4} F^{\mu\nu} \partial \circ \partial F_{\mu\nu} \bigg], \tag{88}$$

where we extract the quadratic parts in A_{μ} and drop higher-order terms in $\Lambda^2/\Lambda_{\rm NC}^2$. This coincides with the expanded EH action Eq.(3.15). This can be understood as follows. In the coordinates yielding Eq.(87), the leading term in A_{μ} vanishes in the trace of $h^{\mu\nu}$. Therefore the diffeomorphism-invariant measure in the functional space approximately agree with the flat measure:

$$||\delta\Phi||^2 = \int d^4x \ \sqrt{G}\delta\Phi(x)^2 \simeq \int d^4x \delta\Phi(x)^2,\tag{89}$$

As a result, the path integral over Φ in the NC U(1) gauge theory gives the diffeomorphisminvariant effective action Eq.(3.15), as far as we keep track of the lowest-order in A_{μ} .

Once we adopts the second term in Eq.(88) as the kinetic term of A_{μ} , we can do similar calculation to that in section 3. By considering the interaction corresponding to Eq.(87), the scattering amplitude is given by

$$\mathcal{M}_A^{(y)} = \frac{6}{\Lambda^2} \frac{1}{k^2 \tilde{k}^2} (p \cdot \tilde{k}) (q \cdot \tilde{k}) (4p \cdot q + k^2) + (s \text{ channel}) + (u \text{ channel}).$$
(90)

This result shows that the amplitude does not match Eq.(3.29) even if the effective action agrees with the action obtained from the EH action after the substitution (87). It is mainly because of the lack of the degrees of freedom. However, if we assume the average leading to the average (3.32), we can obtain the result identical to the gravitational one:

$$\mathcal{M}_{A}^{(y)} = \frac{2}{\Lambda^2} \left[2 \left\{ \frac{1}{k^2} 2(p \cdot q)^2 + p \cdot q \right\} + \frac{1}{4} k^2 \right] + (s \leftrightarrow t) + (t \leftrightarrow u) = -\frac{2}{\Lambda^2} \left(\frac{su}{t} + \frac{tu}{s} + \frac{ts}{u} \right).$$
(91)

C Multi-local action from the matrix model with loop corrections

In Chapter 5, we have calculated the mass terms in the one-loop effective action. In the calculation, we treat the classical field as infinite-order derivative operators $A_a(x, p, g, t)$. On the other hand, matrices are understood for the beginning as integral kernels or bi-local fields $A_a(x, y)$. They are related by Eq.(4.2). Therefore, in principle, We can calculate the effective action in terms of the bi-local fields directly. In this appendix, we show such an analysis and present the extraordinary form of the effective action.

In the calculation of loop corrections, we write the matrix as the following form:

$$A_{(a)}(x,g;y,h) = A^0_{(a)}(x,g;y;h) + \phi_{(a)}(x,g;y,h).$$
(92)

Since the matrices is regarded as operators acting on $C^{\infty}(E_{\text{prin}})$, they are represented as bi-local fields on E_{prin} . We expand the classical field $A^0_{(a)}$ around the flat spacetime:

$$A^{0}_{(a)}(x,g;y;h) = \left[i\partial_{(a)} + B_{(a)}(x,g) + \frac{i}{2}\{h^{\ b}_{(a)}(x,g),\partial_{b}\} + \frac{i}{2}\{\bar{\omega}^{bc}_{(a)},\mathcal{O}_{bc}\} + \cdots\right]\delta(x-y)\delta_{gh},$$
(93)

where δ_{gh} is defined as the delta function on the group manifold:

$$\int dh f(h) \delta_{gh} = f(g). \tag{94}$$

In the expansion, we have written explicitly lower spin modes $B_{(a)}$, $h_{(a)}^{b}$, $\bar{\omega}_{(a)}^{bc}$. By using them we will take the background field method. On the other hand, the field to be integrated $\phi_{(a)}(x, g; y, h)$ is treated as the bi-local field. This treatment enables us to deal with numerous local DoF by packing them into a single bi-local field. It is another method than the calculation in Chapter 5.

We substitute Eq.(92) to Eq.(4.28) and expand it. In particular, we focus on the typical quadratic term of the fluctuation: 10

$$S_{\phi^2} = -\frac{1}{2} \operatorname{Tr} \left(\left[A^0_{(a)}, \phi_{(b)} \right]^2 \right).$$
(95)

For convenience, we introduce the coordinate systems with indices in parentheses:

$$\xi^{(a)} = R^{(a)}{}_{b}(g^{-1})x^{b}, \quad \eta^{(a)} = R^{(a)}{}_{b}(g^{-1})y^{b}.$$
(96)

Here, $R^{(a)}_{\ b}$ is the matrix elements of the vector representation of Spin(d). By expanding

¹⁰We suppose that the other quadratic term $Tr([A^{0(a)}, \phi_{(a)}]^2)$ is removed by some gauge transformation.

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Eq.(95) around the classical field, we obtain

$$S_{\phi^2} = \frac{1}{2} \int d^d x d^d y dg dh \left[\left(\frac{\partial}{\partial \xi_{(a)}} + \frac{\partial}{\partial \eta_{(a)}} - i\mathcal{A}^{(a)}(y,h;x,g) \right) \phi^{(a)}(y,h;x,g) + \left(\frac{\partial}{\partial \xi^{(a)}} + \frac{\partial}{\partial \eta^{(a)}} - i\mathcal{A}_{(a)}(;x,g;y,h) \right) \phi_{(a)}(x,g;y,h) \right],$$
(97)

where

$$\mathcal{A}_{(a)}(x,g;y,h) = B_{(a)}(x,g) - B_{(a)}(y,h) + \frac{i}{2} \{ h^{b}_{(a)}(x,g), \frac{\partial}{\partial x^{b}} \} + \frac{i}{2} \{ h^{b}_{(a)}(y,h), \frac{\partial}{\partial y^{b}} \} + \frac{i}{2} \{ \bar{\omega}^{bc}_{(a)}(x,g), \mathcal{O}^{(g)}_{bc} \} + \frac{i}{2} \{ \bar{\omega}^{bc}_{(a)}(y,h), \mathcal{O}^{(y)}_{bc} \} + \cdots .$$
(98)

Noting that $\phi(y,h;x,g) = \phi^*(x,g;y,h)$ due to its hermicity, and that $\phi(\xi;\eta) = \phi(\xi;\xi - (\xi - \eta))$ in terms of relative coordinates $\xi - \eta$, the propagator for the fluctuation is given by

$$\begin{aligned}
\mathcal{G}(x_1, g_1; y_1, h_1 | x_2, g_2; y_2, h_2) &\equiv \langle \phi(x_1, g_1; y_1, h_1) \phi^*(x_2, g_2; y_2, h_2) \rangle \\
&= G(\xi_1 - \xi_2) \delta((\xi_1 - \eta_1) - (\xi_2 - \eta_2)) \delta_{g_1 g_2} \delta_{h_1 h_2} \\
&= G(x_1 - x_2) \delta\left(R^{(a)}_{\ \ b} (g_1^{-1}) (x_1 - x_2)^b - R^{(a)}_{\ \ b} (h_1^{-1}) (y_1 - y_2)^b) \right) \delta_{g_1 g_2} \delta_{h_1 h_2} \\
\end{aligned}$$
(99)

Here, $G(x_1 - x_2)$ is the propagator for an ordinary massless salar field, which is of course Lorentz invariant. The propagator eq.(99) is expressed as a double line in Feynman diagrams. It should be paid attention that the propagator is invariant under the translations in x and yseparately.

Let us analyze loop corrections. An one-loop *n*-point function is understood as the loop of the propagator with *n* external fields inserted. We should take into account which of the single line loops (written by (x, g) and (y, h), respectively) we insert the external field into. A general *n*-point loop diagram is composed of *n* propagators with *n* vertices:

$$(n-\text{pt. loop}) = \int d^d x_1 \cdots d^d x_n d^d y_1 \cdots d^d y_n dg_1 \cdots dg_n dh_1 \cdots dh_n \prod_{i=1}^n \mathcal{P}_i, \qquad (100)$$

$$\mathcal{P}_{i} = \mathcal{F}_{i} \mathcal{F}_{i}' \mathcal{G}(x_{i}, g_{i}; y_{i}, h_{i} | x_{i+1}, g_{i+1}; y_{i+1}, h_{i+1}).$$
(101)

Here, we have made cyclicity in the index as $x_{n+1} \equiv x_1$ and so on. \mathcal{F}_i and \mathcal{F}'_i are background operators and written as

$$\mathcal{F}_{i} = c_{IJK}^{i} T_{i}^{I}(\{x_{j}\}) S_{i}^{J}(\{g_{j}\}) D_{i}^{K}(\{\partial/\partial x_{j}\}, \{\mathcal{O}_{ab}^{(g_{j})}\}),$$

$$\mathcal{F}_{i}^{\prime} = c_{IJK}^{\prime i} T_{i}^{\prime I}(\{y_{j}\}) S_{i}^{\prime J}(\{h_{j}\}) D_{i}^{\prime K}(\{\partial/\partial y_{j}\}, \{\mathcal{O}_{ab}^{(h_{j})}\}).$$
(102)

We have used reduced notation I, J and K for indices without parentheses. On the other hand, we have truncated ones in parentheses because they do not mix under Lorentz transformation. T_i^I and T_i^{II} are composite factors of fields which depend only on the spacetime coordinates:

$$A_I^{\{1\}}(x_1), \cdots, A_I^{\{n\}}(x_n)$$
(103)

Formally, they are polynomials of local composite fields. S_i^J and $S_i'^J$ are some functions on the Lorentz group. D_i^K and $D_i'^K$ are polynomials of derivative operators. In particular, when one takes into account the decomposition Eq.(4.46), The dependence on the spacetime and group coordinates can be factorized. c_{IJK}^i and $c_{IJK}'^i$ are Lorentz invariant tensors. This invariance are guaranteed by the fact that the indices in Eq.(93) are contracted so that the entire operator is Lorentz invariant.

In fact, the dependence on one of the *n* coordinates can be factorized due to translation invariance of the propagator. We denote such a coordinate as $(x, y) = (x_n, y_n)$. Then the relative coordinates are naturally introduced:

$$\tilde{x}_i = x_i - x, \quad \tilde{y}_i = y_i - y \quad (i = 1, \cdots, n-1).$$
 (104)

The propagator is written as

$$\mathcal{G}(x_i, g_i; y_i, h_i | x_{i+1}, g_{i+1}; y_{i+1}, h_{i+1}) = G(\tilde{x}_i - \tilde{x}_{i+1}) \delta\left(R^{(a)}{}_b(g_i^{-1})(\tilde{x}_i - \tilde{x}_{i+1})^b - R^{(a)}{}_b(h_i^{-1})(\tilde{y}_i - \tilde{y}_{i+1})^b)\right) \delta_{g_i g_{i+1}} \delta_{h_i h_{i+1}}.$$
(105)

It is independent of (x, y). The factor dependent on the coordinates are expanded around (x, y) to turn into the following form:

$$A_{I}^{\{i\}}(x_{i}) = \sum_{s=0}^{\infty} \frac{1}{s!} A_{I,a_{1}\cdots a_{s}}^{\{i\}}(x) \tilde{x}_{i}^{a_{1}}\cdots \tilde{x}_{i}^{a_{s}},$$

$$A_{I}^{\prime\{i\}}(y_{i}) = \sum_{s=0}^{\infty} \frac{1}{s!} A_{I,a_{1}\cdots a_{s}}^{\prime\{i\}}(y) \tilde{y}_{i}^{a_{1}}\cdots \tilde{y}_{i}^{a_{s}},$$
 (106)

where $A_{I,a_1\cdots a_s}^{\{i\}} = \partial_{a_1}\cdots \partial_{a_s} A_I^{\{i\}}$. We can also rewrite the derivatives as

$$\frac{\partial}{\partial x_i^a} = \frac{\partial}{\partial \tilde{x}_i^a} \quad (i = 1, \cdots, n-1), \quad \frac{\partial}{\partial x_n^a} = \frac{\partial}{\partial x^a} - \sum_{i=1}^{n-1} \frac{\partial}{\partial \tilde{x}_i^a},$$
$$\frac{\partial}{\partial y_i^a} = \frac{\partial}{\partial \tilde{y}_i^a} \quad (i = 1, \cdots, n-1), \quad \frac{\partial}{\partial y_n^a} = \frac{\partial}{\partial y^a} - \sum_{i=1}^{n-1} \frac{\partial}{\partial \tilde{y}_i^a}.$$
(107)

Since they act on the propagator, $\partial/\partial x^a$ and $\partial/\partial y^a$ yield no contribution. Therefore *n*-point

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diagram Eq.(101) takes the following form:

$$C_{I}^{I_{1}\cdots I_{n}}C_{J}^{\prime J_{1}\cdots J_{n}}\int d^{d}x d^{d}y \left(\hat{A}(x)\cdots \hat{A}(x)\right)^{I} \left(\hat{A}(y)\cdots \hat{A}(y)\right)^{J}$$

$$\times \int d^{d}\tilde{x}_{1}\cdots d^{d}\tilde{x}_{n-1}d^{d}\tilde{y}_{1}\cdots d^{d}\tilde{y}_{n-1}dg_{1}\cdots dg_{n}dh_{1}\cdots dh_{n}$$

$$\times \prod_{i=1}^{n} \tilde{\mathcal{F}}_{iI_{i}}\tilde{\mathcal{F}}_{iJ_{i}}^{\prime}\mathcal{G}(\tilde{x}_{i},g_{i};\tilde{y}_{i},h_{i}|\tilde{x}_{i+1},g_{i+1};\tilde{y}_{i+1},h_{i+1})$$
(108)

Here $C_{I}^{I_{1}\cdots I_{n}}$ and $C_{J}^{\prime J_{1}\cdots J_{n}}$ are Lorentz invariant tensors, and

$$\tilde{\mathcal{F}}_{iI_{i}} = c_{JKL}^{i} \tilde{f}_{iI_{i}}^{J}(\{\tilde{x}_{j}\}) \tilde{S}_{i}^{K}(\{g_{j}\}) \tilde{D}_{i}^{L}(\{\partial/\partial \tilde{x}_{j}\}, \{\mathcal{O}_{ab}^{\{g_{j}\}}\}), \\
\tilde{\mathcal{F}}_{iJ_{i}}' = c_{JKL}' \tilde{f}_{iJ_{i}}' \{\{\tilde{y}_{j}\}\} \tilde{S}_{i}'^{K}(\{h_{j}\}) \tilde{D}_{i}'^{L}(\{\partial/\partial \tilde{y}_{j}\}, \{\mathcal{O}_{ab}^{\{h_{j}\}}\}).$$
(109)

 c_{JKL}^i and $c_{JKL}'^i$ are also invariant tensors. $\tilde{f}_{iI_i}^J$ and $\tilde{f}_{iJ_i}'^J$ are polynomials of \tilde{x}_j and \tilde{y}_j , respectively. \tilde{S}_i^K and $\tilde{S}_i'^K$ are functions on Lorentz group. \tilde{D}_i^L and $\tilde{D}_i'^L$ are polynomials of derivatives. We can consider the independent Lorentz transformation on $\{\tilde{x}_i\}$ and $\{\tilde{y}_i\}$:

$$\begin{cases} \tilde{x}_i^a \longrightarrow R^a_{\ b}(u)\tilde{x}_i^b, \quad g_i \longrightarrow ug_i, \\ \tilde{y}_i^a \longrightarrow R^a_{\ b}(v)\tilde{y}_i^b, \quad h_i \longrightarrow vh_i. \end{cases}$$
(110)

They result in the transformations of indices I_1, \dots, I_n and J_1, \dots, J_n , respectively. However, the integral measure of spacetime coordinates and the propagator in Eq.(108) are Lorentz invariant. By integrating it over $\{\tilde{x}_i\}, \{g_i\}, \{\tilde{y}_i\}$ and $\{h_i\}$, it yields Lorentz invariant constants. In other words, I_1, \dots, I_n and J_1, \dots, J_n on $C_I^{I_1 \dots I_n}$ and $C_J^{\prime J_1 \dots J_n}$ are, after the integration, contracted to form Lorentz invariant quantity individually. Therefore, the rest set of indices Iand J are contracted within them, respectively. As a result, the *n*-point loop diagram takes the form of

$$(n-\text{pt. loop}) = \int d^d x d^d y \left(\hat{A}(x) \cdots \hat{A}(x) \right) \left(\hat{A}(y) \cdots \hat{A}(y) \right)$$
$$= \int d^d x \left(\hat{A}(x) \cdots \hat{A}(x) \right) \int d^d y \left(\hat{A}(y) \cdots \hat{A}(y) \right).$$
(111)

It implies that one-loop correction terms in the effective action is written by products of two integrals of local quantities.

Generalization of the above discussion to multi-loops corrections is qualitatively observed. we can conclude that loops of k single lines result in products of k integral of local quantities. The effective action should keep the original unitary symmetry, and hence each factor is invariant under local Lorentz transformation and diffeomorphism. To summarize, the effective action is given by the following form:

$$\Gamma = \hat{c}_i s_i + \hat{c}_{ij} s_i s_j + \hat{c}_{ijk} s_i s_j s_k + \cdots, \qquad (112)$$

$$s_i = \int d^d x \sqrt{g} O_i(x), \tag{113}$$

where $O_i(x)$ is some scalar and *i* is a mere label. \hat{c}_i , \hat{c}_{ij} and so on are constants, which is suppressed according to the number of indices. It is because a constant with more indices comes from more loops. Eq.(113) seems to violate locality. However, it is not the case. In the path integral, the integrand $\exp(i\Gamma)$ can be formally Fourier transformed with respect to $\{s_i\}$, and make the form of the path integral as

$$Z = \int \mathcal{D}\phi \, e^{i\Gamma} = \int \left(\prod_i d\lambda_i\right) \int \mathcal{D}\phi \, F(\lambda) e^{i\lambda_i s_i},\tag{114}$$

where the repeated indices are summed over, and ϕ represents all of dynamical fields symbolically. $F(\lambda)$ is some function whose Fourier transformation is $\exp(i\Gamma)$. The effective action Γ describes the theory with multiverse or wormholes [117, 118]. From the viewpoint of local field theory, Eq.(114) is interpreted as a theory with its coupling constants averaged by some weight function. Therefore, it is expected that observed coupling constants are actually the averaged values. It can provide the resolution to fine-tuning problems in particle physics and cosmology [119]. The type of the above path integral is, however, very difficult to study by analytic calculation. On the other hand, we can discuss the resolution of several fine-tuning problem by adopting the viable approximation [33]. Among those discussion, it is suggested that such a theory naturally gives the theoretical origin of MPP. Let us have a brief review of this mechanism in the following.

Once we set initial and final conditions at t = 0 and ∞ , the path integral is rewritten as

$$Z = \int_{\phi|_{t=0}=\phi_i}^{\phi|_{t=\infty}=\phi_f} \mathcal{D}\phi \, e^{i\Gamma}$$

= $\int \left(\prod_i d\lambda_i\right) \int_{\phi|_{t=0}=\phi_i}^{\phi|_{t=\infty}=\phi_f} \mathcal{D}\phi \, F(\lambda) e^{i\lambda_i s_i}$
= $\int \left(\prod_i d\lambda_i\right) F(\lambda) \langle f| \, e^{-i\int_0^\infty dt \hat{H}(\lambda;t)} \, |i\rangle .$ (115)

 $|i\rangle$ and $\langle f|$ is the initial and final states corresponding to the configuration ϕ_i and ϕ_f , respectively. Note that our universe has been at a state with very low energy density in the most part of its history. We can approximate the energy density to that of the ground state $\varepsilon(\lambda)$. Suppose that t_0 denotes the time the universe was relaxed to the state where the approximation

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is valid. By defining the relaxed state

$$|\psi(t_0;\lambda)\rangle \equiv e^{-i\int_0^{t_0} dt \hat{H}(\lambda;t)} |i\rangle, \qquad (116)$$

Eq.(115) is estimated as follows:

$$Z \sim \int \left(\prod_{i} d\lambda_{i}\right) F(\lambda) e^{-i\varepsilon(\lambda) \int_{t_{0}}^{\infty} dt Vol_{3}(t)} \langle f|\psi(t_{0};\lambda)\rangle$$
$$\sim \int \left(\prod_{i} d\lambda_{i}\right) F(\lambda) e^{-i\varepsilon(\lambda) Vol_{4}} \langle f|\psi(t_{0};\lambda)\rangle, \qquad (117)$$

where $Vol_3(t)$ is the space volume at t, and Vol_4 is the spacetime volume of our universe.

We can further approximate Eq.(117) with the fact that Vol_4 is very large, as is explained below. For simplicity, we will treat the case with a single parameter for λ . Consider a system including the scalar potential with two local minima. SM is probably such a system, according to the experimental restriction on the form of Higgs effective potential. Let us represent the minimal points as $\phi_1(\lambda)$ and $\phi_2(\lambda)$. Without loss of generality, we can assume that the energy density corresponding to one of the minima $\varepsilon(\phi_1(\lambda))$ monotonically increases as λ increases, and that of the other $\varepsilon(\phi_2(\lambda))$ monotonically decreases. If both of the locally minimal energy coincide to each other at $\lambda = \lambda_0$, then the global minimum point jumps from ϕ_1 to ϕ_2 . Accordingly, the vacuum energy density changes from $\varepsilon(\phi_1)$ to $\varepsilon(\phi_2)$. Unless $\phi_1(\lambda_0) \neq \phi_2(\lambda_0)$, the vacuum energy has a cusp at $\lambda = \lambda_0$; $\varepsilon'(\lambda_0 - 0) \neq \varepsilon'(\lambda_0 + 0)$. In this case, the integral over λ in Eq.(117) picks up the configuration with $\lambda = \lambda_0$ as the dominant contribution. It is proved by evaluating the integral explicitly. Assuming the quantity

$$\bar{F}(\lambda) \equiv F(\lambda) \left\langle f | \psi(t_0; \lambda) \right\rangle.$$
(118)

has finite support, the integral over λ in the partition function is approximated as the following:

$$\begin{split} Z &\sim \int_{-\infty}^{\infty} d\lambda \bar{F}(\lambda) e^{-i\varepsilon(\lambda)Vol_{4}} \\ &= \int_{-\infty}^{\lambda_{0}} d\lambda \bar{F}(\lambda) e^{-i\varepsilon(\lambda)Vol_{4}} + \int_{\lambda_{0}}^{\infty} d\lambda \bar{F}(\lambda) e^{-i\varepsilon(\lambda)Vol_{4}} \\ &= \int_{-\infty}^{\varepsilon(\lambda_{0})} d\varepsilon \left(\frac{d\varepsilon}{d\lambda}\right)^{-1} \bar{F}(\lambda(\varepsilon)) e^{-i\varepsilon(\lambda)Vol_{4}} + \int_{\varepsilon(\lambda_{0})}^{\infty} d\varepsilon \left(\frac{d\varepsilon}{d\lambda}\right)^{-1} \bar{F}(\lambda(\varepsilon)) e^{-i\varepsilon(\lambda)Vol_{4}} \\ &= \left[\frac{i}{Vol_{4}} \left(\frac{d\varepsilon}{d\lambda}\right)^{-1} \bar{F}(\lambda(\varepsilon)) e^{-i\varepsilon(\lambda)Vol_{4}}\right]_{-\infty}^{\varepsilon(\lambda_{0})} + \left[\frac{i}{Vol_{4}} \left(\frac{d\varepsilon}{d\lambda}\right)^{-1} \bar{F}(\lambda(\varepsilon)) e^{-i\varepsilon(\lambda)Vol_{4}}\right]_{\varepsilon(\lambda_{0})}^{\infty} \\ &+ O\left(\frac{1}{Vol_{4}^{2}}\right) \\ &\sim \frac{i}{Vol_{4}} \left[\left(\frac{d\varepsilon}{d\lambda}\right)_{\lambda_{0}-0}^{-1} - \left(\frac{d\varepsilon}{d\lambda}\right)_{\lambda_{0}+0}^{-1}\right] \bar{F}(\lambda_{0}) e^{-i\varepsilon(\lambda_{0})Vol_{4}} \end{split}$$

$$=\frac{i}{Vol_4}\left[\left(\frac{d\varepsilon}{d\lambda}\right)_{\lambda_0-0}^{-1} - \left(\frac{d\varepsilon}{d\lambda}\right)_{\lambda_0+0}^{-1}\right]F(\lambda_0)e^{-i\varepsilon(\lambda_0)Vol_4}\left\langle f|\psi(t_0;\lambda_0)\right\rangle.$$
(119)

Note that, at $\lambda = \lambda_0$, the vacuum of the system is degenerated by the two configuration $\phi_1(\lambda_0)$ and $\phi_2(\lambda_0)$. Therefore, Eq.(119) shows the theoretical origin of MPP in the context of the multi-local action Eq.(113), in the sense that the coupling "constant" λ in Eq.(114) has been fixed to λ_0 .

The essential points of the above analysis were the two assumption. First, the energy density of the universe is sufficiently close to that of the ground state during most part of its history. Second, the ground state energy has a cusp at some value of the parameter. As for the latter in particular, similar analysis is applicable to systems with $\varepsilon(\lambda)$ which shows more general behavior. In other words, if $\varepsilon(\lambda)$ has special points (non-analytic point like cusps, or stationary points), then the dominant contributions to the integration over λ in Eq.(117) come from those values. This provides a mechanism of fixing coupling constants to specific values, and may lead to a resolution of fine-tuning problem. It is remarkable that the mechanism involve the whole of the history of our universe through Vol_4 .

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