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Kyoto University
GEOMETRIC PROPERTIES OF SOLUTIONS OF A CLASS OF DIFFERENTIAL EQUATIONS

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Abstract

The main object of this paper is to investigate several geometric properties of the solutions of the second-order linear differential equation:

$$w''(z) + a(z)w'(z) + b(z)w(z) = 0,$$

where the functions $a(z)$ and $b(z)$ are analytic in the open unit disk $U$. Relevant connections of the results presented in this paper with those given earlier by (for example) M.S. Robertson, S.S. Miller, and H. Saitoh are also considered.

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1. Introduction

Let $A$ denote the class of functions $f$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\mathcal{U} := \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}.$$  

Also let $S$, $S^*$, and $S^*(\alpha)$ denote the subclasses of $A$ consisting of functions which are, respectively, univalent, starlike with respect to the origin, and starlike of order $\alpha$ in $\mathcal{U}$ ($0 \leq \alpha < 1$). Thus, by definition, we have (see, for details, [2] and [8]; see also [7] and [11])

$$S^*(\alpha) = \{ f : f \in A \text{ and } \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha \ (z \in \mathcal{U}; 0 \leq \alpha < 1) \}$$

and

$$S^* := S^*(\alpha) |_{\alpha=0} = S^*(0).$$

For functions $f \in A$ with $f'(z) \neq 0 (z \in \mathcal{U})$, we define the Schwarzian derivative of $f(z)$ by

$$S(f, z) := \left( \frac{f'(z)}{f(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2$$

($f \in A; f'(z) \neq 0 (z \in \mathcal{U})$).

We begin by recalling the following result of Miller [4].

**Theorem A** (Miller [4]). Let the function $p(z)$ be analytic in $\mathcal{U}$ with

$$|zp(z)| < 1 \quad (z \in \mathcal{U}).$$

Also let $v(z)$ denote the unique solution of the following initial-value problem:

$$v''(z) + p(z) v(z) = 0 \quad (v(0) = 0; v'(0) = 1)$$

in $\mathcal{U}$. Then

$$\left| \frac{zv'(z)}{v(z)} - 1 \right| < 1 \quad (z \in \mathcal{U})$$

and $v(z)$ is a starlike conformal map of the unit disk $\mathcal{U}$.

Theorem A is related rather closely to some earlier results of Robertson [9] and Nehari [6], which we recall here as Theorem B and Theorem C below.

**Theorem B** (Robertson [9]). Let $zp(z)$ be analytic in $\mathcal{U}$ and

$$\Re \{ z^2 p(z) \} \leq \frac{\pi^2}{4} |z|^2 \quad (z \in \mathcal{U}).$$
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Then the unique solution $W = W(z)$ of the following initial-value problem:

$$W''(z) + p(z)W(z) = 0 \quad (W(0) = 0; \ W'(0) = 1)$$

(1.8)
is univalent and starlike in U. The constant $\frac{\pi^2}{4}$ in the inequality (1.7) is the best possible.

**Theorem C (Nehari [6]).** If $f \in A$ satisfies the following inequality involving its Schwarzian derivative defined by (1.4):

$$|S(f, z)| \leq \frac{\pi^2}{2} \quad (z \in U),$$

(1.9)
then $f \in S$. The result is sharp for the function $f(z)$ given by

$$f(z) = \frac{e^{iz} - 1}{i\pi} \quad (i := \sqrt{-1}).$$

(1.10)

**Remark 1.** By setting

$$p(z) = \frac{1}{2}S(f, z) \quad (z \in U)$$

(1.11)
and using (1.9), we obtain the inequality (1.7). Obviously, therefore, the hypothesis in Theorem C is stronger than that in Theorem B.

In the present paper, we aim at investigating several geometric properties of the solutions of the following initial-value problem which involves a general family of second-order linear differential equations:

$$w''(z) + a(z)w'(z) + b(z)w(z) = 0 \quad (w(0) = 0; w'(0) = 1),$$

(1.12)
where the functions $a(z)$ and $b(z)$ are analytic in U (see [3]). We also show how our results are related to those of (for example) Robertson [9], Miller [4], and Saitoh [10].

2. A Class of Bounded Functions

Let $B_J$ denote the class of bounded functions

$$w(z) = \sum_{n=1}^{\infty} c_n z^n,$$

(2.1)
analytic in U, for which

$$|w(z)| < J \quad (z \in U; J > 0).$$

(2.2)
If $g(z) \in B_J$, then we can show (by using the Schwarz lemma [1]) that the function $w(z)$ defined by

$$w(z) := z^{-\frac{1}{2}} \int_{0}^{z} g(t) t^{-\frac{1}{2}} dt$$

(2.3)
is also in the class $B_J$. Thus, in terms of derivatives, we have

$$\left| \frac{1}{2} w(z) + zw'(z) \right| < J \quad (z \in \mathbb{U}) \implies |w(z)| < J \quad (z \in \mathbb{U}).$$

(2.4)

Furthermore, by letting

$$h(u,v) := \frac{1}{2} u + v,$$

we can rewrite (2.4) in the form:

$$|h(w(z), zw'(z))| < J \quad (z \in \mathbb{U}) \implies |w(z)| < J \quad (z \in \mathbb{U}).$$

(2.6)

In this section, we show that the implication (2.6) holds true for functions $h(u,v)$ in the class $\mathcal{H}_J$ given by Definition 1 below (see also [5]).

**Definition 1.** Let $\mathcal{H}_J$ be the class of complex functions $h(u,v)$ satisfying each of the following conditions:

(i) $h(u,v)$ is continuous in a domain $D \subset \mathbb{C} \times \mathbb{C}$;

(ii) $(0,0) \in D$ and $|h(0,0)| < J \quad (J > 0)$;

(iii) $|h(Ke^{i\theta}, Ke^{i\theta})| \geq J$ whenever $(Ke^{i\theta}, Ke^{i\theta}) \in D \quad (\theta \in \mathbb{R}; \ K \geq J > 0)$.

**Example 1.** It is easily seen that the function

$$h(u,v) = \alpha u + v \quad (\Re(\alpha) \geq 0; \ \mathbb{D} = \mathbb{C} \times \mathbb{C})$$

(2.7)

is in the class $\mathcal{H}_J$.

**Definition 2.** Let $h \in \mathcal{H}_J$ with the corresponding domain $D$. We denote by $B_J(h)$ the class of functions $w(z)$ given by (2.1), which are analytic in $\mathbb{U}$ and satisfy each of the following conditions:

(i) $(w(z), zw'(z)) \in D$;

(ii) $|h(w(z), zw'(z))| < J \quad (z \in \mathbb{U}; \ J > 0)$.

The function class $B_J(h)$ is not empty. Indeed, for any given function $h \in \mathcal{H}_J$, we have

$$w(z) = c_1 z \in B_J(h)$$

(2.8)

for sufficiently small $|c_1|$ depending on $h$.

**Theorem D** (Saitoh [10]). For any $h \in \mathcal{H}_J$,

$$B_J(h) \subset B_J \quad (h \in \mathcal{H}_J; \ J > 0).$$
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**Remark 2.** Theorem D shows that, if \( h \in \mathcal{H}_J \) (with the corresponding domain \( D \)) and if \( w(z) \), given by (2.1), is analytic in \( U \) and

\[
(w(z), zw'(z)) \in D,
\]
then the implication (2.4) holds true.

Theorem D leads us immediately to the following result, which was also given by Saitoh [10].

**Theorem E** (Saitoh [10]). Let \( h \in \mathcal{H}_J \) and let the function \( b(z) \) be analytic in \( U \) with

\[
|b(z)| < J \quad (z \in U; J > 0).
\]
If the initial-value problem:

\[
h(w(z), zw'(z)) = b(z) \quad (w(0) = 0)
\]
has a solution \( w(z) \) analytic in \( U \), then

\[
|w(z)| < J \quad (z \in U; J > 0).
\]

### 3. Main Results and Their Consequences

One of our main results is contained in the following theorem.

**Theorem 1.** Let the functions \( a(z) \) and \( b(z) \) be analytic in \( U \) with

\[
\left| z^2 \left\{ b(z) - \frac{1}{2}a'(z) - \frac{1}{4}|a(z)|^2 \right\} \right| < J \quad (z \in U; J > 0)
\]
and

\[
\Re \{za(z)\} > -2J \quad (z \in U; J > 0).
\]
Also let \( w(z) \) denote the solution of the initial-value problem (1.12) in \( U \). Then

\[
1 - J - \frac{1}{2}\Re \{za(z)\} < \Re \left( \frac{zw'(z)}{w(z)} \right) < 1 + J - \frac{1}{2}\Re \{za(z)\}
\]
\[
(z \in U; J > 0).
\]

**Proof.** First of all, by means of the transformation:

\[
w(z) = \exp \left( -\frac{1}{2} \int_0^z a(t) \, dt \right) \cdot v(z),
\]
we can rewrite the initial-value problem (1.12) in the normal form:

\[
v''(z) + \left\{ b(z) - \frac{1}{2}a'(z) - \frac{1}{4}|a(z)|^2 \right\} v(z) = 0
\]
\[
(v(0) = 0; \ v'(0) = 1).
\]
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If we now put

$$u(z) = \frac{zv'(z)}{v(z)} - 1 \quad (z \in U),$$  \hspace{1cm} (3.6)

then $u(z)$ is analytic in $U$ with $u(0) = 0$, and (3.5) becomes

$$[u(z)]^2 + u(z) + zu'(z) = -z^2 \left\{ b(z) - \frac{1}{2}a'(z) - \frac{1}{4} |a(z)|^2 \right\} \quad (u(0) = 0)$$  \hspace{1cm} (3.7)

or, equivalently,

$$h(u(z), zu'(z)) = -z^2 \left\{ b(z) - \frac{1}{2}a'(z) - \frac{1}{4} |a(z)|^2 \right\} \quad (u(0) = 0),$$  \hspace{1cm} (3.8)

where, for convenience,

$$h(\xi, \eta) := \xi^2 + \xi + \eta.$$  \hspace{1cm} (3.9)

It is easily observed from (3.1), (3.2), and (3.8) that $h(\xi, \eta) \in \mathcal{H}_J$, that is, that

(i) $h(\xi, \eta)$ is continuous in $D = \mathbb{C} \times \mathbb{C}$;
(ii) $(0, 0) \in D$ and $|h(0, 0)| = 0 < J \quad (J > 0)$;
(iii) For $(Je^{i\theta}, Ke^{i\theta}) \in D \quad (\theta \in \mathbb{R}; \ K \geq J > 0),$

$$|h(Je^{i\theta}, Ke^{i\theta})| = |J e^{i2\theta} + Je^{i\theta} + Ke^{i\theta}|$$

$$= |J e^{i\theta} + J + K| \geq J.$$

Thus, by applying Theorem $E$, we find from the hypothesis (3.1) of Theorem 1 that

$$|u(z)| < J \quad (z \in U; \ J > 0),$$

which, in view of the relationship (3.6), yields

$$1 - J < \Re \left( \frac{zv'(z)}{v(z)} \right) < 1 + J \quad (z \in U; \ J > 0).$$  \hspace{1cm} (3.10)

Next, by logarithmically differentiating (3.4) in its equivalent form:

$$v(z) = \exp \left( \frac{1}{2} \int_0^z a(t) \ dt \right) \cdot w(z),$$

we have

$$\frac{zv'(z)}{v(z)} = \frac{zu'(z)}{w(z)} + \frac{1}{2}za(z),$$  \hspace{1cm} (3.11)

so that (3.10) becomes

$$1 - J < \Re \left( \frac{zu'(z)}{w(z)} \right) + \frac{1}{2} \Re \{ za(z) \} < 1 + J$$  \hspace{1cm} (3.12)

$$\quad (z \in U; \ J > 0),$$

which obviously yields the assertion (3.3) of Theorem 1.
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Remark 3. If, in Theorem 1, we have
\[ \Re \{za(z)\} \leq 2(1 - J) \quad (z \in U; \ J > 0), \]
so that
\[ 0 \leq 1 - J - \frac{1}{2} \Re \{za(z)\} < 1 \quad (z \in U; \ J > 0), \]
then the assertion (3.3) immediately yields
\[ w(z) \in S^* \left( 1 - J - \frac{1}{2} \Re \{za(z)\} \right) \]
in conjunction with the definition (1.2).

Example 2. If we let
\[ a(z) = -2Jz \quad \text{and} \quad b(z) = J^2 z^2 \quad (J > 0) \]
in Theorem 1, then the solution of the initial-value problem:
\[ w''(z) - 2Jzw'(z) + J^2 z^2 w(z) = 0 \]
\[ (w(0) = 0; \ w'(0) = 1) \]
is given by
\[ w(z) = \frac{1}{\sqrt{J}} \exp \left( \frac{1}{2} Jz^2 \right) \cdot \sin \left( z\sqrt{J} \right). \]
In this case, if we further assume that
\[ 0 < J \leq \frac{1}{2}, \]
then
\[ w(z) \in S^* (1 - 2J) \quad \left( 0 < J \leq \frac{1}{2} \right), \]
so that, in particular, we have
\[ J = \frac{1}{2}: \quad w(z) = \sqrt{2} \exp \left( \frac{1}{4} z^2 \right) \cdot \sin \left( \frac{z}{\sqrt{2}} \right) \in S^*, \]
\[ J = \frac{1}{3}: \quad w(z) = \sqrt{3} \exp \left( \frac{1}{6} z^2 \right) \cdot \sin \left( \frac{z}{\sqrt{3}} \right) \in S^* \left( \frac{1}{3} \right), \]
\[ J = \frac{1}{4}: \quad w(z) = 2 \exp \left( \frac{1}{8} z^2 \right) \cdot \sin \left( \frac{z}{2} \right) \in S^* \left( \frac{1}{2} \right), \]
and so on.

Example 3. For
\[ a(z) = -2Jz \quad \text{and} \quad b(z) = \lambda \quad (J > 0; \ \lambda \in \mathbb{C}), \]
the initial-value problem (1.12) becomes
\[ w''(z) - 2Jzw'(z) + \lambda w(z) = 0 \]
\[ (w(0) = 0; w'(0) = 1), \]
which, under the transformation:
\[ w(z) = \exp\left(\frac{1}{2}Jz^2\right) \cdot v(z), \]
assumes the \textit{normal} form:
\[ v''(z) + (\lambda + J - J^2z^2)v(z) = 0 \]
\[ (v(0) = 0; v'(0) = 1). \]

**Remark 4.** In their special case when \( J = \frac{1}{2} \), the differential equations in (3.19) and (3.21) can be identified with such classical differential equations as Hermite's equation and Weber's equation, respectively (\textit{cf.}, \textit{e.g.}, [1] and [12]).

Next we prove the following result for the solution of the initial-value problem (3.21).

**Theorem 2.** If
\[ |\lambda + J - J^2z^2| < T \quad (z \in \mathbb{U}; J, T > 0), \]
then the solution \( v(z) \) of the initial-value problem (3.21) satisfies the inequality:
\[ \left| \frac{zv'(z)}{v(z)} - 1 \right| < T \quad (z \in \mathbb{U}; T > 0). \]

**Proof.** Just as in our demonstration of Theorem 1, the function \( u(z) \), given by (3.6), is analytic in \( \mathbb{U} \), \( u(0) = 0 \), and \( [\text{cf. Equation (3.7)}] \)
\[ h(u(z), zu'(z)) = -z^2(\lambda + J - J^2z^2) \quad (u(0) = 0), \]
where \( h(\xi, \eta) \) is defined, as before, by (3.9).

Now it is easily seen from (3.22) and (3.24) that \( h(\xi, \eta) \in \mathcal{H}_T \), that is, that
(i) \( h(\xi, \eta) \) is continuous in \( \mathbb{D} = \mathbb{C} \times \mathbb{C} \);
(ii) \( (0, 0) \in \mathbb{D} \) and \( |h(0, 0)| = 0 < T \quad (T > 0) \);
(iii) For \( (Te^{i\theta}, Ke^{i\theta}) \in \mathbb{D} \quad (\theta \in \mathbb{R}; K \geqq T > 0) \),
\[ |h(Te^{i\theta}, Ke^{i\theta})| \geqq T. \]
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By applying Theorem E, we thus find from the hypothesis (3.22) of Theorem 2 that

$$|u(z)| < T \quad (z \in \mathbb{U}; \; T > 0),$$

which, in view of the relationship (3.6) again, leads us at once to the assertion (3.23) of Theorem 2.

**Remark 5.** If $0 < T \leq 1$, then Theorem 2 yields the following geometric property:

$$v(z) \in S^*(1-T) \quad (0 < T \leq 1)$$

for the solution $v(z)$ of the initial-value problem (3.21).

By putting $J = \frac{1}{2}$ and $T = 1$ in Theorem 2, we obtain the following known result.

**Corollary (Saitoh [10]).** If

$$|\lambda + \frac{1}{2} - \frac{1}{4}z^2| < 1 \quad (z \in \mathbb{U}),$$

then the solution $v(z)$ of Weber's differential equation:

$$v''(z) + \left(\lambda + \frac{1}{2} - \frac{1}{4}z^2\right)v(z) = 0$$

$$v(0) = 0; \; v'(0) = 1$$

is starlike in $\mathbb{U}$.

**Remark 6.** The solutions of Weber's differential equation in (3.26) are expressed as the parabolic cylinder (or Weber's) function $D_\lambda(z)$ defined by (cf., e.g., [1, p. 39 et seq.])

$$D_\lambda(z) := 2^{\lambda/2} \sqrt{\pi} \exp\left(-\frac{1}{4}z^2\right) \left[ \frac{1}{\Gamma(\frac{1-\lambda}{2})} \ {}_1F_1\left(-\frac{\lambda}{2}; \frac{3}{2}; -\frac{1}{2}z^2\right) - \frac{z\sqrt{2}}{\Gamma(\frac{1-\lambda}{2})} \ {}_1F_1\left(1-\lambda; \frac{3}{2}; \frac{1}{2}z^2\right) \right],$$

where $\ {}_1F_1(\alpha; \gamma; z)$ denotes the confluent hypergeometric function (see, for details, [1] and [12]).

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