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Partial Sums of Certain Analytic Functions

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Abstract

It is well-known that Koebe function $f(z)=\dfrac{z}{(1-z)^2}$ is the extremal function for the class $\mathcal{S}^*$ of starlike functions in the open unit disk $\mathbb{U}$, and that the function $g(z)=\dfrac{z}{1-z}$ is the extremal function for the class $\mathcal{K}$ of convex functions in the open unit disk $\mathbb{U}$. But the partial sum $f_n(z)$ of $f(z)$ is not starlike in $\mathbb{U}$, and the partial sum $g_n(z)$ of $g(z)$ is not convex in $\mathbb{U}$. The object of the present paper is to discuss for starlikeness and convexty of the partial sums $f_n(z)$ and $g_n(z)$.

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\mathbb{U}$. Let $\mathcal{S}^*(\alpha)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \qquad (z\in\mathbb{U}) \tag{1.2}$$

for some $\alpha(0\leqq\alpha<1)$. A function $f(z)$ in $\mathcal{S}^*(\alpha)$ is said to be starlike of order α in $\mathbb{U}$. Furthermore, let $\mathcal{K}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

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$$(1.3) \qquad \operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in \mathbb{U})$$

for some $\alpha(0 \leqq \alpha < 1)$. A function $f(z)$ belonging to $\mathcal{K}(\alpha)$ is said to be convex of order α in $\mathbb{U}$.

By the definitions for the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, we note that $f(z) \in \mathcal{S}^*(\alpha)$ if and only if $zf'(z) \in \mathcal{K}(\alpha)$ and denote by $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ and $\mathcal{K}(0) \equiv \mathcal{K}$.
It is well-known that Koebe function $f(z)$ given by

$$(1.4) \qquad f(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} kz^k$$

is the extremal function for the class S^*, and that the function

$$(1.5) \qquad g(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k$$

is the extremal function for the class $\mathcal{K}$.
For a function $f(z)$ given by (1.1), we introduce the partial sum of $f(z)$ by

$$(1.6). \qquad f_n(z) = z + \sum_{k=2}^{n} a_k z^k .$$

For the partial sums $f_n(z)$ of $f(z) \in \mathcal{S}^*$, Szegö [4] showed the following result.

Theorem 1.1. *If $f(z) \in \mathcal{S}^*$, then $f_n(z) \in \mathcal{S}^*$ for $|z| < \frac{1}{4}$, and $f_n(z) \in \mathcal{K}$ for $|z| < \frac{1}{8}$.*

Further, Padmanabhan [3] proved the following theorem.

Theorem 1.2. *If $f(z)$ is 2-valently starlike in $\mathbb{U}$, then $f_n(z)$ is 2-valently starlike for $|z| < \frac{1}{6}$.*

Recently, Li and Owa [1] derived some interesting results for partial sums of the Libera integral operator which is defined by

$$L(f)(z) = \frac{2}{z}\int_0^z f(t)dt.$$

for $f(z) \in \mathcal{A}$, and Owa [2] considered partial sums for the extremal functions of the classes $\mathcal{S}^*$ and $\mathcal{K}$.

Remark 1.1. *If $f(z) \in \mathcal{S}$, then $f_n(z) \notin \mathcal{S}$ for $|a_n| \geqq \frac{1}{n}$.*

Proof. Note that

$$f_n'(z) = 1 + \sum_{k=2}^{\infty} k a_k z^{k-1} = n a_n \left\{ z^{n-1} + \frac{(n-1)a_{n-1}}{n a_n} z^{n-2} + \cdots + \frac{1}{n a_n} \right\} = 0$$

for $z = z_j$ $(j = 1, 2, 3, \cdots, n-1)$
Therefore, we have

$$\left| \prod_{j=1}^{n-1} z_j \right| = \left| (-1)^{n-1} \frac{1}{n a_n} \right| \leqq 1.$$

This shows that there exists a point $z_j \in \mathbb{U}$ such that $|z_j| < 1$. Thus we say that $f_n(z) \notin \mathcal{S}$ for $|a_n| \geqq \frac{1}{n}$. □

Noting that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S}$, we also have
(i) $f_n(z) \notin \mathcal{S}^*$ for $f(z) = \dfrac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} k z^k \in \mathcal{S}^*$.
(ii) $g_n(z) \notin \mathcal{K}$ for $g(z) = \dfrac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k \in \mathcal{K}$.

2 Partial sums $f_3(z)$ and $g_3(z)$

For Koebe function $f(z)$ given by

$$f(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} k z^k$$

which is the extremal function for the class $\mathcal{S}^*$, we consider the partial sum $f_3(z) = z + 2z^2 + 3z^3$.

Theorem 2.1. *The partial sum $f_3(z) = z + 2z^2 + 3z^3$ of Koebe function $f(z) = \dfrac{z}{(1-z)^2}$ satisfies*

$$\text{(2.1)} \qquad \operatorname{Re}\left(1 + \frac{z f_3''(z)}{f_3'(z)}\right) > \alpha(r) = 3 - \frac{2(1-2r)}{1 - 4r + 9r^2} > 2\left(1 - \frac{\sqrt{10}}{5}\right) = 0.7350\ldots.$$

Where

$$0 \leqq r < \frac{7 - 2\sqrt{10}}{9} = 0.0750\ldots.$$

Proof. We consider α such that

$$\text{(2.2)} \qquad \operatorname{Re}\left\{1+\frac{zf_3''(z)}{f_3'(z)}\right\} = \operatorname{Re}\left\{3-\frac{2(1+2z)}{1+4z+9z^2}\right\} > \alpha$$

for $0 \leqq r < \dfrac{7-2\sqrt{10}}{9} = 0.0750\cdots$.
It follows that

$$\text{(2.3)} \quad \operatorname{Re}\left\{\frac{1+2z}{1+4z+9z^2}\right\} = \frac{1}{2}+\frac{(1-9r^2)(1+9r^2+4r\cos\theta)}{2(1-2r^2+81r^4+8r(1+9r^2)\cos\theta+36r^2\cos^2\theta)} < \frac{3-\alpha}{2},$$

that is, that

$$\text{(2.4)} \qquad \operatorname{Re}\left\{\frac{(1-9r^2)(1+9r^2+4r\cos\theta)}{1-2r^2+81r^4+8r(1+9r^2)\cos\theta+36r^2\cos^2\theta}\right\} < 2-\alpha.$$

Let the function $g(t)$ be given by

$$\text{(2.5)} \qquad g(t) = \frac{(1-9r^2)(1+9r^2+4rt)}{1-2r^2+81r^4+8r(1+9r^2)t+36r^2t^2} \quad (t=\cos\theta).$$

Then, we have

$$\text{(2.6)} \qquad g'(t)$$
$$= \frac{-4r(1+3r)(1-3r)(1+38r^2-81r^3+162r^4+18r(1+4r+9r^2)t+36r^2t^2)}{(1-2r^2+81r^4+8r(1+9r^2)t+36r^2t^2)^2}.$$

Letting

$$\text{(2.7)} \qquad h(t) = 1+38r^2-81r^3+162r^4+18r(1+4r+9r^2)t+36r^2t^2,$$

we see that

(i) $h(t) < 0 \implies g'(t) > 0$,
(ii) $h(t) > 0 \implies g'(t) < 0$,

and

(iii) $h(t) = 0$ for $t=t_1, t=t_2 \quad (t_1 > t_2)$.

It is easy to see that $t_2 < -1$. Since

$$t_1 = \frac{-3(1+4r+9r^2) + \sqrt{5 - 72r + 154r^2 + 972r^3 + 81r^4}}{12r}, \tag{2.8}$$

our condition $0 \leqq r < \dfrac{7-2\sqrt{10}}{9}$ implies that $t_1 \leqq -1$, so that, $h(t) \geqq 0$. Consequently, we conclude that

$$g(t) \leqq g(-1) = \frac{1-9r^2}{1-4r+9r^2} < \frac{2\sqrt{10}}{5} \leqq 2-\alpha, \tag{2.9}$$

that is,

$$\alpha = 2 - \frac{1-9r^2}{1-4r+9r^2} = 3 - \frac{2(1-2r)}{1-4r+9r^2}.$$

Thus, we have

$$\mathrm{Re}\left\{1 + \frac{zf_3''(z)}{f_3'(z)}\right\} > \alpha(r)$$

and

$$\alpha(r) = 3 - \frac{2(1-2r)}{1-4r+9r^2} \tag{2.10}$$

for $0 \leqq r \leqq \dfrac{7-2\sqrt{10}}{9}$. □

Next, for the function

$$g(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k$$

which is the extremal function for the class $\mathcal{K}$, we consider the partial sum $g_3(z) = z + z^2 + z^3$.

Theorem 2.2. *The partial sum $g_3(z) = z + z^2 + z^3$ of the function $g(z) = \dfrac{z}{1-z}$ satisfies*

$$\mathrm{Re}\left(\frac{zg_3'(z)}{g_3(z)}\right) > \alpha(r) = 3 - \frac{2-r}{1-r+r^2} > \frac{4-\sqrt{5}}{2} = 0.9919\ldots. \tag{2.11}$$

Where

$$0 \leqq r < \frac{7-3\sqrt{5}}{2} = 0.1458\ldots.$$

Proof. We consider α such that

$$\operatorname{Re}\left\{\frac{zg_3'(z)}{g_3(z)}\right\} = \operatorname{Re}\left\{3 - \frac{2+z}{1+z+z^2}\right\} > \alpha \tag{2.12}$$

for $0 \leqq r < \dfrac{7-3\sqrt{5}}{2} = 0.1458\cdots$.
This implies that

$$\operatorname{Re}\left\{\frac{2+z}{1+z+z^2}\right\} = 1 + \frac{(1-r^2)(1+r^2+r\cos\theta)}{1-r^2+r^4+4r^2\cos^2\theta+2r(1+r^2)\cos\theta} < 3-\alpha, \tag{2.13}$$

that is ,

$$\operatorname{Re}\left\{\frac{(1-r^2)(1+r^2+r\cos\theta)}{1-r^2+r^4+4r^2\cos^2\theta+2r(1+r^2)\cos\theta}\right\} < 2-\alpha. \tag{2.14}$$

Let the function $g(t)$ be given by

$$g(t) = \frac{(1-r^2)(1+r^2+rt)}{1-r^2+r^4+4r^2t^2+2r(1+r^2)t}. \quad (t=\cos\theta). \tag{2.15}$$

Then, we have

$$g'(t) = \frac{r(r+1)(r-1)(1+5r^2+r^4+4r^2t^2+8r(1+r^2)t)}{(1-r^2+r^4+4r^2t^2+2r(1+r^2)t)^2}. \tag{2.16}$$

Defining the function $h(t)$ by

$$h(t) = 1+5r^2+r^4+4r^2t^2+8r(1+r^2)t, \tag{2.17}$$

we see that

(i) $h(t) < 0 \implies g'(t) > 0$,
(ii) $h(t) > 0 \implies g'(t) < 0$,

and

(iii) $h(t) = 0$ for $t = t_1, t = t_2 \quad (t_1 > t_2)$.

Note that $t_2 < -1$. Since

$$t_1 = \frac{-2(1+r^2)+\sqrt{3(1+r^2+r^4)}}{2r} < 0, \tag{2.18}$$

our condition $0 \leqq r < \dfrac{7-3\sqrt{5}}{2}$ of the theorem implies that $t_1 \leqq -1$, so that, $h(t) \geqq 0$. Consequently, we conclude that

$$(2.19) \qquad g(t) \leqq g(-1) = \frac{1-r^2}{1-r+r^2} < \frac{4-\sqrt{5}}{2} \leqq 2-\alpha,$$

that is,

$$\alpha = 2 - \frac{1-r^2}{1-r+r^2} = 3 - \frac{2-r}{1-r+r^2}.$$

Thus, we have

$$\mathrm{Re}\left\{\frac{zg_3'(z)}{g_3(z)}\right\} > \alpha(r),$$

and

$$(2.20) \qquad \alpha(r) = 3 - \frac{2-r}{1-r+r^2}$$

for $0 \leqq r < \dfrac{7-3\sqrt{5}}{2} = 0.1458\cdots$.

□

Using the same method in the above, we also derive the following result.

Theorem 2.3. *The partial sum $f_3(z) = z+2z^2+3z^3$ of Koebe function $f(z) = \dfrac{z}{(1-z)^2}$ satisfies*

$$(2.21) \qquad \mathrm{Re}\left\{\frac{zf_3'(z)}{f_3(z)}\right\} > \alpha(r) = 3 - \frac{2(1-r)}{1-2r+3r^2} > \frac{3(89-16\sqrt{22})}{137} = 0.3055\ldots.$$

Where

$$0 \leqq r < \frac{5-\sqrt{22}}{3} = 0.1031\ldots.$$

Theorem 2.4. *The partial sum $g_3(z) = z + z^2 + z^3$ of the function $g(z) = \dfrac{z}{1-z}$ satisfies*

$$(2.22) \quad \mathrm{Re}\left\{1 + \frac{zg_3''(z)}{g_3'(z)}\right\} > \alpha(r) = 3 - \frac{2(1-r)}{1-2r+3r^2} > \frac{3(89-16\sqrt{22})}{137} = 0.3055\ldots.$$

Where

$$0 \leqq r < \frac{5-\sqrt{22}}{3} = 0.1031\ldots.$$

3 Partial sums $f_4(z)$ and $g_4(z)$

For the Koebe function

$$f(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} kz^k$$

which is the extremal function for the class S^*, we consider the partial sum $f_4(z) = z + 2z^2 + 3z^3 + 4z^4$.

Theorem 3.1. *The partial sum $f_4(z) = z + 2z^2 + 3z^3 + 4z^4$ of Koebe function $f(z) = \dfrac{z}{(1-z)^2}$ satisfies*

$$\text{(3.1)} \qquad \operatorname{Re}\left\{\frac{zf_4'(z)}{f_4(z)}\right\} > \alpha(r) = 4 - \frac{3-4r+3r^2}{1-2r+3r^2-4r^3}.$$

Where $0 \leqq r \leqq r_0 < 1$ and

$$r_0 = \frac{3}{16} + \frac{\sqrt[3]{531+16\sqrt{1695}}}{16\sqrt[3]{9}} - \frac{37}{16\sqrt[3]{3(531+16\sqrt{1695})}} = 0.3545\ldots.$$

Proof. For $f_4(z) = z + 2z^2 + 3z^3 + 4z^4$, we have

$$\text{(3.2)} \qquad \operatorname{Re}\left\{\frac{zf_4'(z)}{f_4(z)}\right\} = \operatorname{Re}\left\{\frac{1+4z+9z^2+16z^3}{1+2z+3z^2+4z^3}\right\}$$

$$= 4 - \operatorname{Re}\left\{\frac{3+4z+3z^2}{1+2z+3z^2+4z^3}\right\}$$

$$= 4 - \operatorname{Re}\left\{\frac{3+4re^{i\theta}+3r^2e^{i2\theta}}{1+2re^{i\theta}+3r^2e^{i2\theta}+4r^3e^{i3\theta}}\right\} \quad (z = re^{i\theta}).$$

By using Mathematica, we know that the value of (3.2) takes its minimum value for $\theta = \pi$. This gives that

$$\text{(3.3)} \qquad \operatorname{Re}\left\{\frac{zf_4'(z)}{f_4(z)}\right\} \geqq 4 - \frac{3-4r+3r^2}{1-2r+3r^2-4r^3} \quad (0 \leqq r \leqq r_0).$$

Let the function $h(r)$ be given by

$$h(r) = 4 - \frac{3-4r+3r^2}{1-2r+3r^2-4r^3} \quad (0 \leqq r \leqq r_0).$$

Since $0 = h(r_0) \leqq h(r) \leqq 1$
for

$$r_0 = \frac{3}{16} + \frac{\sqrt[3]{531+16\sqrt{1695}}}{16\sqrt[3]{9}} - \frac{37}{16\sqrt[3]{3(531+16\sqrt{1695})}} = 0.3545\cdots,$$

we have

$$\text{(3.4)} \qquad \operatorname{Re}\left\{\frac{zf_4'(z)}{f_4(z)}\right\} > \alpha(r) = 4 - \frac{3-4r+3r^2}{1-2r+3r^2-4r^3}$$

which completes the proof of the theorem.

□

Next, we show

Theorem 3.2. *The partial sum $g_4(z) = z + z^2 + z^3 + z^4$ of the function $g(z) = \dfrac{z}{1-z}$ satisfies*

$$\text{(3.5)} \qquad \operatorname{Re}\left\{1 + \frac{zg_4''(z)}{g_4'(z)}\right\} > \alpha(r) = 4 - \frac{3-4r+3r^2}{1-2r+3r^2-4r^3}.$$

Where $0 \leqq r \leqq r_0 < 1$ and

$$r_0 = \frac{3}{16} + \frac{\sqrt[3]{531+16\sqrt{1695}}}{16\sqrt[3]{9}} - \frac{37}{16\sqrt[3]{3(531+16\sqrt{1695})}} = 0.3545\ldots.$$

Proof. For $g_4(z) = z + z^2 + z^3 + z^4$, we have

$$\text{(3.6)} \qquad \operatorname{Re}\left\{1 + \frac{zg_4''(z)}{g_4'(z)}\right\} = \operatorname{Re}\left\{1 + \frac{2z+6z^2+12z^3}{1+2z+3z^2+4z^3}\right\}$$

$$= 4 - \operatorname{Re}\left\{\frac{3+4z+3z^2}{1+2z+3z^2+4z^3}\right\}$$

$$= 4 - \operatorname{Re}\left\{\frac{3+4re^{i\theta}+3r^2e^{i2\theta}}{1+2re^{i\theta}+3r^2e^{i2\theta}+4r^3e^{i3\theta}}\right\} \quad (0 \leqq r \leqq r_0)$$

Further, an application of Mathematica shows that

$$\text{(3.7)} \qquad \operatorname{Re}\left\{1 + \frac{zg_4''(z)}{g_4'(z)}\right\} \geqq 4 - \frac{3-4r+3r^2}{1-2r+3r^2-4r^3} \quad (0 \leqq r \leqq r_0).$$

Defining the function $h(r)$ by

$$h(r) = 4 - \frac{3-4r+3r^2}{1-2r+3r^2-4r^3} \quad (0 \leqq r \leqq r_0),$$

we see that

$$0 \leqq h(r_0) \leqq h(r) \leqq 1$$

for

$$r_0 = \frac{3}{16} + \frac{\sqrt[3]{531+16\sqrt{1695}}}{16\sqrt[3]{9}} - \frac{37}{16\sqrt[3]{3(531+16\sqrt{1695})}} = 0.3545\cdots,$$

that is, that

$$\text{Re}\left\{1+\frac{zg_4''(z)}{g_4'(z)}\right\} > \alpha(r) = 4 - \frac{3-4r+3r^2}{1-2r+3r^2-4r^3} \tag{3.8}$$

for $0 \leqq r \leqq r_0$. □

Using the same method in the above, we also derive

Theorem 3.3. *The partial sum $f_4(z) = z+2z^2+3z^3+4z^4$ of Koebe function $f(z) = \dfrac{z}{(1-z)^2}$ satisfies*

$$\text{Re}\left\{1+\frac{zf_4''(z)}{f_4'(z)}\right\} > \alpha(r) = 4 - \frac{3-8r+9r^2}{1-4r+9r^2-16r^3}.$$

Where $0 \leqq r \leqq r_1 < 1$ and

$$r_1 = \frac{9}{64} + \frac{\sqrt[3]{4257+64\sqrt{18681}}}{64\sqrt[3]{9}} - \frac{269}{64\sqrt[3]{3(4257+64\sqrt{18681})}} = 0.1933\ldots.$$

Theorem 3.4. *The partial sum $g_4(z) = z+z^2+z^3+z^4$ of the function $g(z) = \dfrac{z}{1-z}$ satisfies*

$$\text{Re}\left\{\frac{zg_4'(z)}{g_4(z)}\right\} > \alpha(r) = 4 - \frac{3-2r+r^2}{(1-r)(1+r^2)}.$$

Where $0 \leqq r \leqq r_2 < 1$ and

$$r_2 = \frac{1}{4} + \frac{\sqrt[3]{5(9+4\sqrt{6})}}{4\sqrt[3]{9}} - \frac{\sqrt[3]{25}}{4\sqrt[3]{3(9+4\sqrt{6})}} = 0.6058\ldots.$$

4 Appendix

In this section, we try to describe the image domain of the disk by the partial sums for the theorems in Section 2 and Section 3.

Example 4.1. By Theorem 2.1, we take the partial sum $f_3(z) = z + 2z^2 + 3z^3$ for $|z| = r$ with

$$0 \leqq r < \frac{7 - 2\sqrt{10}}{9} = 0.0750\ldots.$$

The image domain of $f_3(z)$ is shown in Fig.4.1.

```
<< Graphics`ComplexMap`
f3[z_] = z + 2 z^2 + 3 z^3
Pmf3 := PolarMap[f3, {0, (7 - 2 Sqrt[10])/9}, {0, 2 π}];
Show[Pmf3]
```

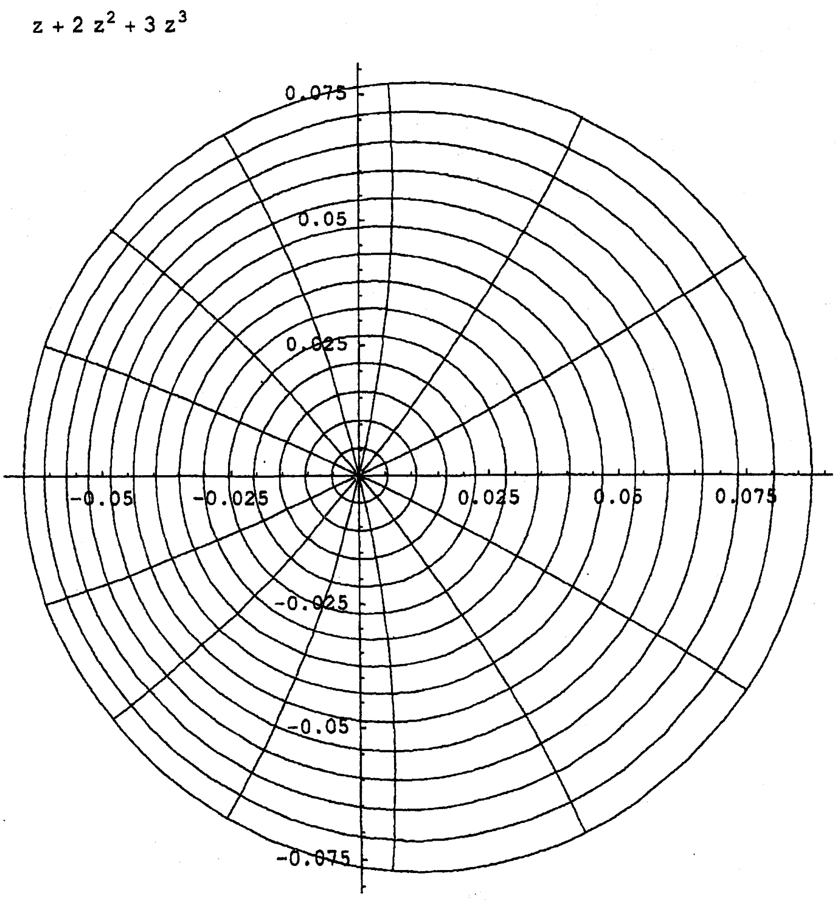

Fig.4.1

Example 4.2. By Theorem 2.2, we take the partial sum $g_3(z) = z + z^2 + z^3$ for $|z| = r$ with

$$0 \leqq r < \frac{7-3\sqrt{5}}{2} = 0.1458\ldots.$$

The image domain of $g_3(z)$ is given by Fig.4.2.

```
<< Graphics`ComplexMap`
g3[z_] = z + z^2 + z^3
Pmg3; = PolarMap[g3, {0, (7 - 3 √5)/2}, {0, 2 π}];
Show[Pmg3]
```

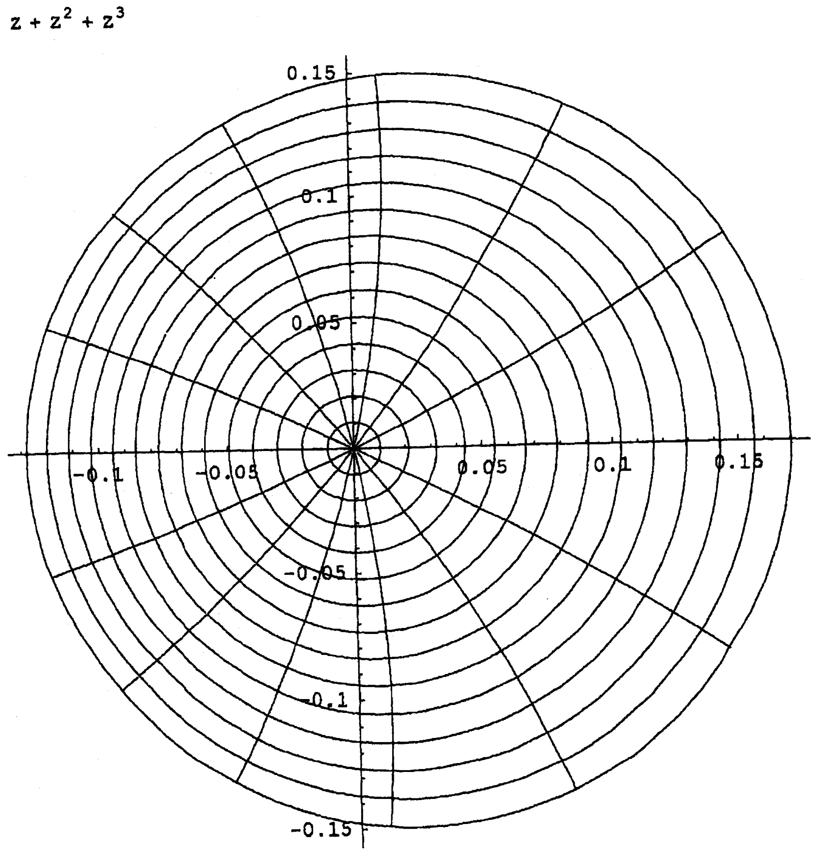

Fig.4.2

Example 4.3. By Theorem 2.3, we consider the partial sum $f_3(z) = z + 2z^2 + 3z^3$ for $|z| = r$ with

$$0 \leqq r < \frac{5 - \sqrt{22}}{3} = 0.1031\ldots.$$

We give the image domain of $f_3(z)$ in Fig.4.3.

```
<< Graphics`ComplexMap`
f3[z_] = z + 2 z^2 + 3 z^3
Pmf3 := PolarMap[f3, {0, (5 - √22)/3}, {0, 2 π}];
Show[Pmf3]
```

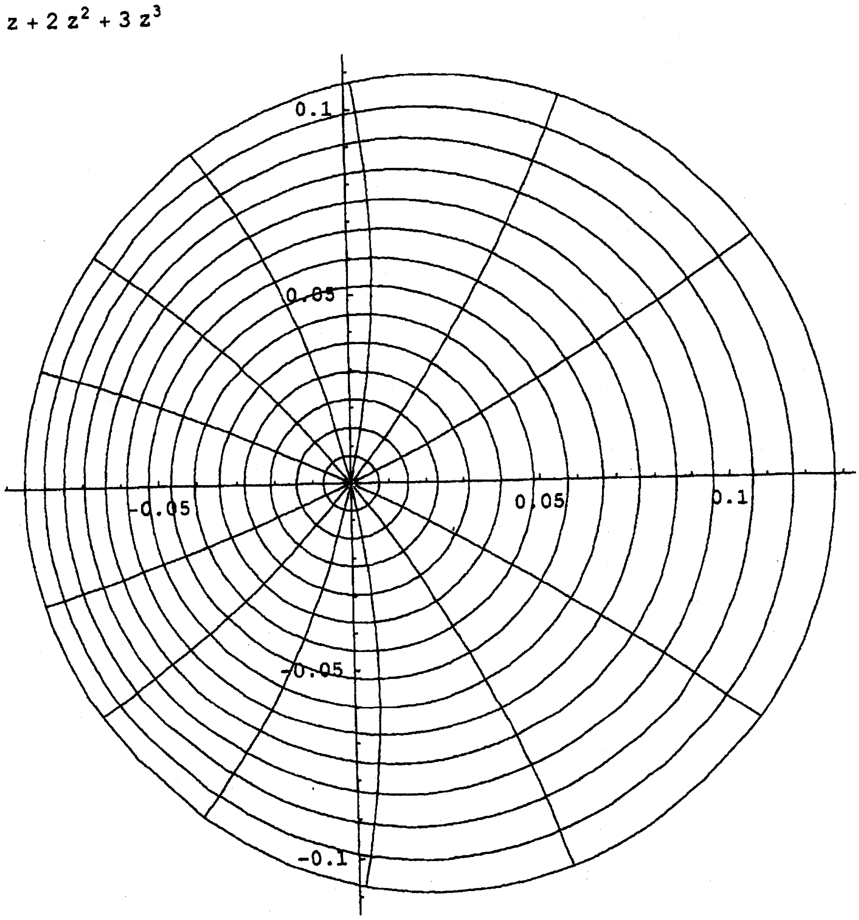

Fig.4.3

Example 4.4. By Theorem 2.4, we take the partial sum $g_3(z) = z + z^2 + z^3$ for $|z| = r$ with

$$0 \leqq r < \frac{7 - 3\sqrt{5}}{2} = 0.1458\ldots.$$

The image domain of $g_3(z)$ is shown in Fig.4.4.

```
<< Graphics`ComplexMap`
g3[z_] = z + z^2 + z^3
Pmg3 := PolarMap[g3, {0, (5 - Sqrt[22])/3}, {0, 2 π}];
Show[Pmg3]
```

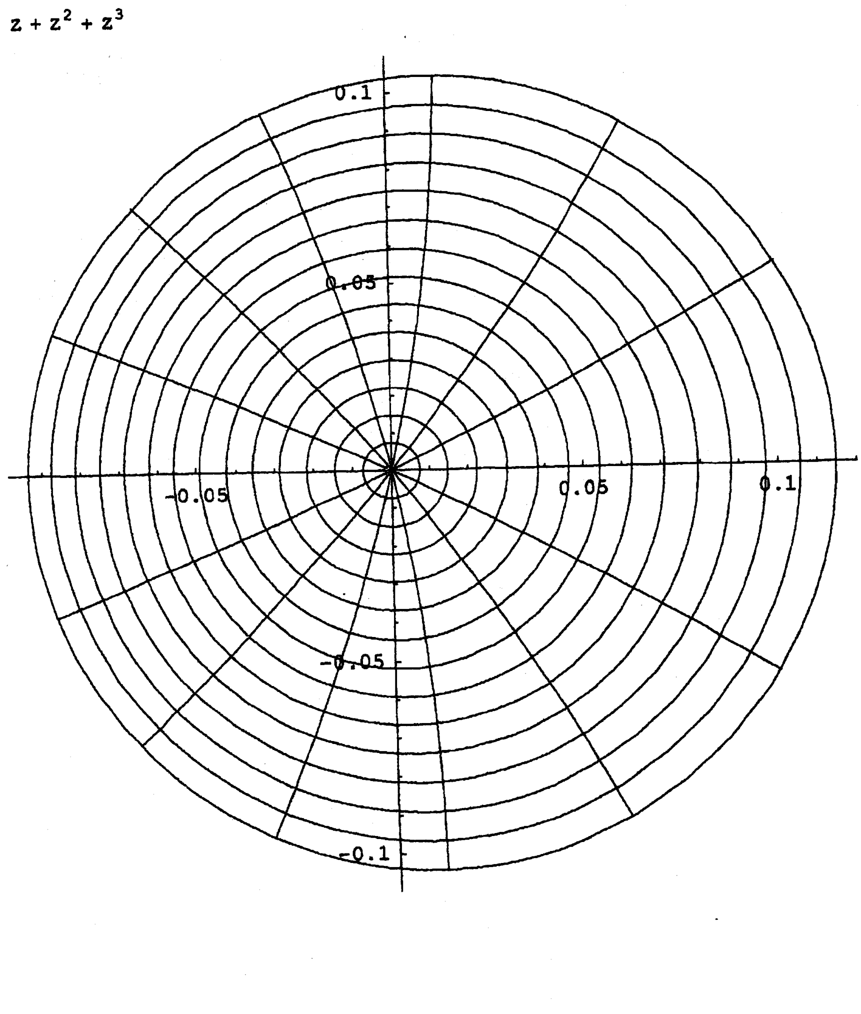

Fig.4.4

Example 4.5. By Theorem 3.1, we consider the partial sum $f_4(z) = z + 2z^2 + 3z^3 + 4z^4$ for $|z| = r$ with $0 \leqq r \leqq r_0 < 1$ and

$$r_0 = \frac{3}{16} + \frac{\sqrt[3]{531 + 16\sqrt{1695}}}{16\sqrt[3]{9}} - \frac{37}{16\sqrt[3]{531 + 16\sqrt{1695}}} = 0.3545\ldots.$$

Then we show the image domain of $f_4(z)$ in Fig.4.5.

```
<< Graphics`ComplexMap`
f4[z_] = z + 2 z^2 + 3 z^3 + 4 z^4
Pmf4 := PolarMap[f4, {0, r0}, {0, 2 π}];
```

$$\texttt{r0} = \frac{3}{16} + \frac{\sqrt[3]{531 + 16\sqrt{1695}}}{16\sqrt[3]{9}} - \frac{37}{16\sqrt[3]{3\,(531 + 16\sqrt{1695})}}$$

```
Show[Pmf4]
```

$z + 2\,z^2 + 3\,z^3 + 4\,z^4$

$$\frac{3}{16} + \frac{(531 + 16\sqrt{1695})^{1/3}}{16\;3^{2/3}} - \frac{37}{16\,(3\,(531 + 16\sqrt{1695}))^{1/3}}$$

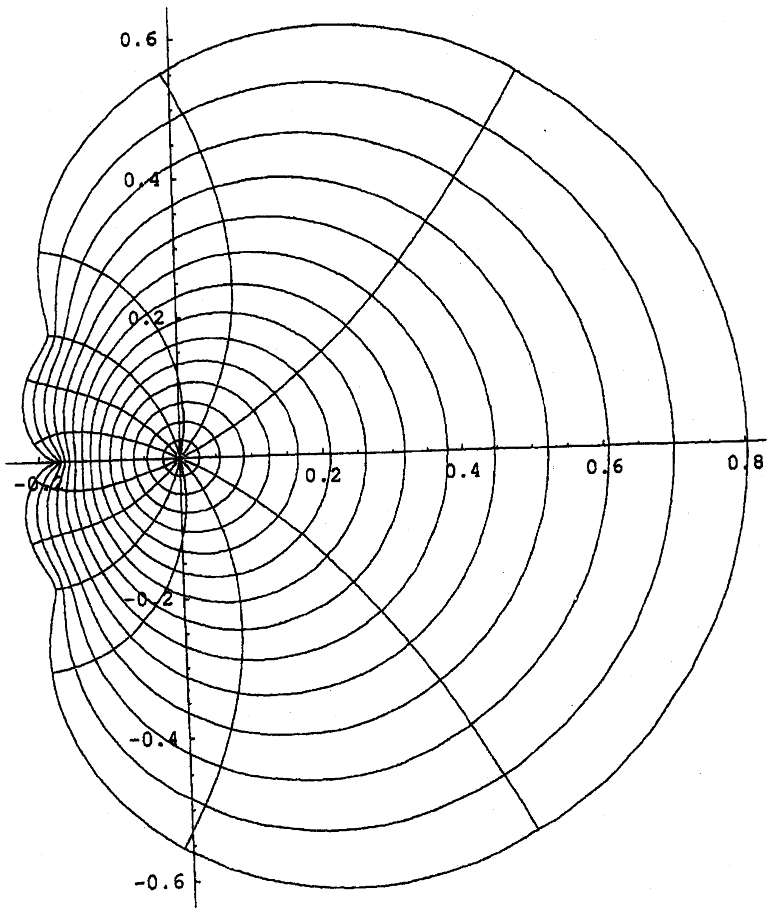

Fig.4.5

Example 4.6. By Theorem 3.2, we consider the partial sum $g_4(z) = z + z^2 + z^3 + z^4$ for $|z| = r$ with $0 \leqq r \leqq r_0 < 1$ and

$$r_0 = \frac{3}{16} + \frac{\sqrt[3]{531 + 16\sqrt{1695}}}{16\sqrt[3]{9}} - \frac{37}{16\sqrt[3]{3(531 + 16\sqrt{1695})}} = 0.3545\ldots.$$

Then we have the image domain of $g_4(z)$ by Fig.4.6.

```
<< Graphics`ComplexMap`
g4[z_] = z + z^2 + z^3 + z^4
Pmg4 := PolarMap[g4, {0, r0}, {0, 2 π}];
```

$$\mathtt{r0} = \frac{3}{16} + \frac{\sqrt[3]{531 + 16\sqrt{1695}}}{16\sqrt[3]{9}} - \frac{37}{16\sqrt[3]{3\,(531 + 16\sqrt{1695})}}$$

```
Show[Pmg4]
```

$$z + z^2 + z^3 + z^4$$

$$\frac{3}{16} + \frac{(531 + 16\sqrt{1695})^{1/3}}{16\,3^{2/3}} - \frac{37}{16\,(3\,(531 + 16\sqrt{1695}))^{1/3}}$$

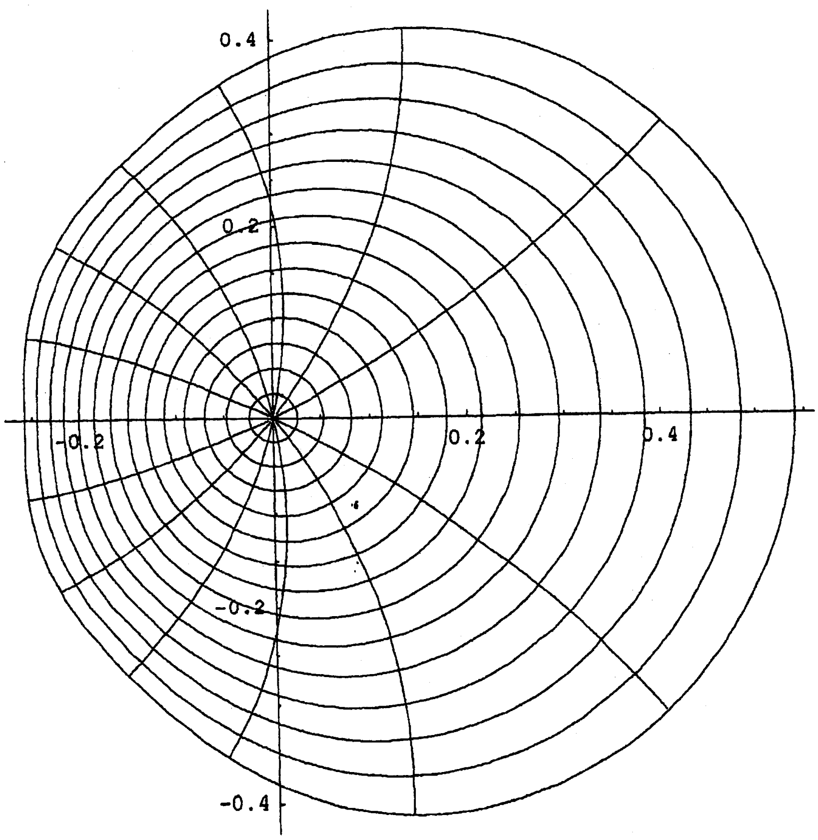

Fig.4.6

Example 4.7. Considering the partial sum $f_4(z) = z + 2z^2 + 3z^3 + 4z^4$ for $|z| = r$ with $0 \leqq r \leqq r_1 < 1$ and

$$r_1 = \frac{9}{64} + \frac{\sqrt[3]{4257 + 64\sqrt{18681}}}{64\sqrt[3]{9}} - \frac{269}{64\sqrt[3]{3(4257 + 64\sqrt{18681})}} = 0.1933\ldots,$$

in Theorem 3.3, we see the image domain of $f_4(z)$ in Fig.4.7.

```
<< Graphics`ComplexMap`
f4[z_] = z + 2 z^2 + 3 z^3 + 4 z^4
Pmf4 := PolarMap[f4, {0, r1}, {0, 2 π}];
```

$$\texttt{r1} = \frac{9}{64} + \frac{\sqrt[3]{4257 + 64\sqrt{18681}}}{64\sqrt[3]{9}} - \frac{269}{64\sqrt[3]{3\,(4257 + 64\sqrt{18681})}}$$

```
Show[Pmf4]
```

$$z + 2 z^2 + 3 z^3 + 4 z^4$$

$$\frac{9}{64} + \frac{(4257 + 64\sqrt{18681})^{1/3}}{64\ 3^{2/3}} - \frac{269}{64\,(3\,(4257 + 64\sqrt{18681}))^{1/3}}$$

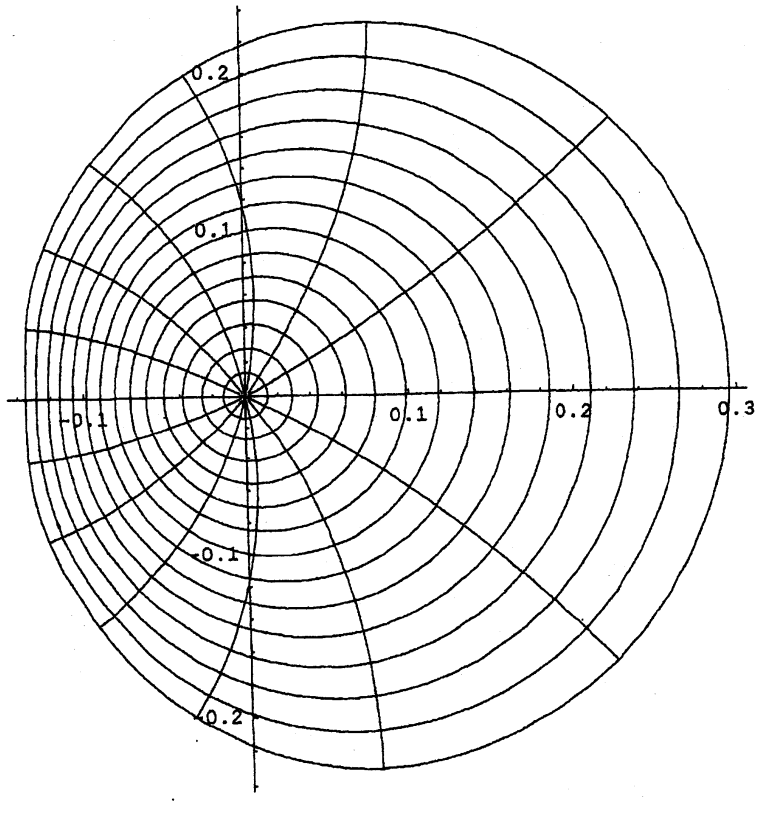

Fig.4.7

Example 4.8. Taking the partial sum $g_4(z) = z + z^2 + z^3 + z^4$ for $|z| = r$ with $0 \leqq r \leqq r_2 < 1$ and

$$r_2 = \frac{1}{4} + \frac{\sqrt[3]{5(9+4\sqrt{6})}}{4\sqrt[3]{9}} - \frac{\sqrt[3]{25}}{4\sqrt[3]{3(9+4\sqrt{6})}} = 0.6058\ldots,$$

we have the image domain of $g_4(z)$ in Fig.4.8.

```
<< Graphics`ComplexMap`
g4[z_] = z + z^2 + z^3 + z^4
Pmg4 := PolarMap[g4, {0, r2}, {0, 2 π}];
```

$$\text{r2} = \frac{1}{4} + \frac{\sqrt[3]{5\,(9+4\sqrt{6})}}{4\,\sqrt[3]{9}} - \frac{\sqrt[3]{25}}{4\,\sqrt[3]{3\,(9+4\sqrt{6})}}$$

```
Show[Pmg4]
```

$$z + z^2 + z^3 + z^4$$

$$\frac{1}{4} - \frac{5^{2/3}}{4\,(3\,(9+4\sqrt{6}))^{1/3}} + \frac{(5\,(9+4\sqrt{6}))^{1/3}}{4\;3^{2/3}}$$

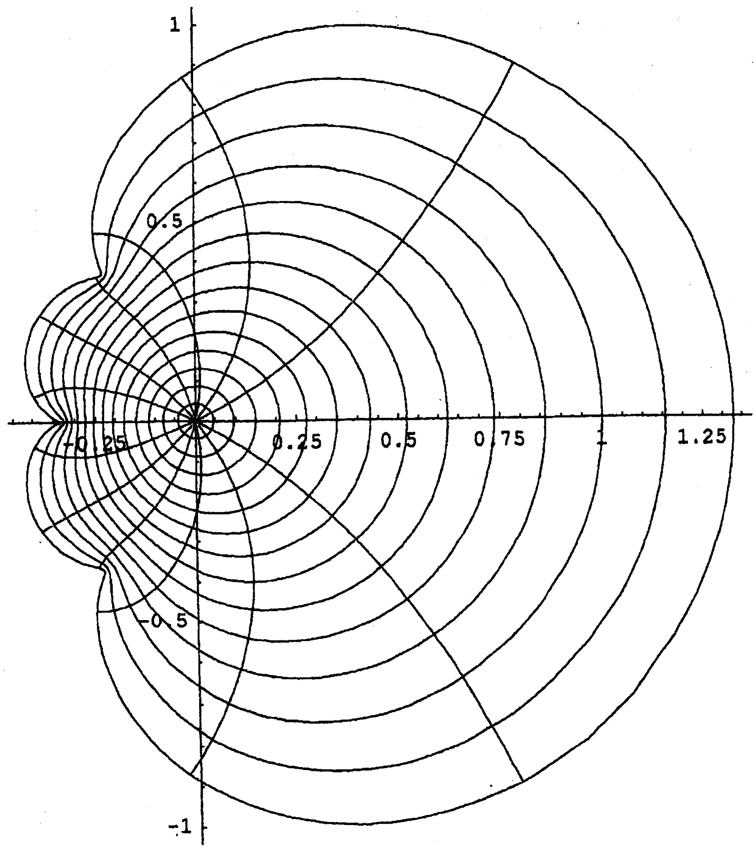

Fig.4.8

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