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<td>Owa, Shigeyoshi; Srivastava, H.M.; Saito, Nobuyuki</td>
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Partial Sums of Certain Analytic Functions

Shigeyoshi Owa, H.M. Srivastava and Nobuyuki Saito

Abstract

It is well-known that Koebe function $f(z) = \frac{z}{(1-z)^2}$ is the extremal function for the class $\mathcal{S}^*$ of starlike functions in the open unit disk $\mathbb{U}$, and that the function $g(z) = \frac{z}{1-z}$ is the extremal function for the class $\mathcal{K}$ of convex functions in the open unit disk $\mathbb{U}$. But the partial sum $f_n(z)$ of $f(z)$ is not starlike in $\mathbb{U}$, and the partial sum $g_n(z)$ of $g(z)$ is not convex in $\mathbb{U}$. The object of the present paper is to discuss for starlikeness and convexity of the partial sums $f_n(z)$ and $g_n(z)$.

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\mathbb{U}$. Let $\mathcal{S}^*(\alpha)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in \mathbb{U})$$

for some $\alpha (0 \leq \alpha < 1)$. A function $f(z)$ in $\mathcal{S}^*(\alpha)$ is said to be starlike of order $\alpha$ in $\mathbb{U}$. Furthermore, let $\mathcal{K}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

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Key Words and Phrases: Univalent function, starlike function, convex function, partial sum.
for some $\alpha(0 \leq \alpha < 1)$. A function $f(z)$ belonging to $\mathcal{K}(\alpha)$ is said to be convex of order $\alpha$ in $\mathbb{U}$.

By the definitions for the classes $S^*(\alpha)$ and $\mathcal{K}(\alpha)$, we note that $f(z) \in S^*(\alpha)$ if and only if $zf'(z) \in \mathcal{K}(\alpha)$ and denote by $S^*(0) \equiv S^*$ and $\mathcal{K}(0) \equiv \mathcal{K}$.

It is well-known that Koebe function $f(z)$ given by

$$f(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} kz^k$$

is the extremal function for the class $S^*$, and that the function

$$g(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k$$

is the extremal function for the class $\mathcal{K}$.

For a function $f(z)$ given by (1.1), we introduce the partial sum of $f(z)$ by

$$f_n(z) = z + \sum_{k=2}^{n} a_k z^k.$$  

For the partial sums $f_n(z)$ of $f(z) \in S^*$, Szegö [4] showed the following result.

**Theorem 1.1.** If $f(z) \in S^*$, then $f_n(z) \in S^*$ for $|z| < \frac{1}{4}$, and $f_n(z) \in \mathcal{K}$ for $|z| < \frac{1}{8}$.

Further, Padmanabhan [3] proved the following theorem.

**Theorem 1.2.** If $f(z)$ is 2-valently starlike in $\mathbb{U}$, then $f_n(z)$ is 2-valently starlike for $|z| < \frac{1}{6}$.

Recently, Li and Owa [1] derived some interesting results for partial sums of the Libera integral operator which is defined by

$$L(f)(z) = \frac{2}{z} \int_{0}^{z} f(t)dt.$$  

for $f(z) \in A$, and Owa [2] considered partial sums for the extremal functions of the classes $S^*$ and $\mathcal{K}$. 
Remark 1.1. If \( f(z) \in S \), then \( f_n(z) \notin S \) for \( |a_n| \geq \frac{1}{n} \).

Proof. Note that

\[
f'_n(z) = 1 + \sum_{k=2}^{\infty} k a_k z^{k-1} = n a_n \left\{ z^{n-1} + \frac{(n-1)a_{n-1}}{n a_n} z^{n-2} + \cdots + \frac{1}{n a_n} \right\} = 0
\]

for \( z = z_j \) \((j = 1, 2, 3, \cdots, n - 1)\)

Therefore, we have

\[
\prod_{j=1}^{n-1} |z_j| = \left| (-1)^{n-1} \frac{1}{n a_n} \right| \leq 1.
\]

This shows that there exists a point \( z_j \in U \) such that \( |z_j| < 1 \). Thus we say that \( f_n(z) \notin S \) for \( |a_n| \geq \frac{1}{n} \). \(\square\)

Noting that \( K \subset S^* \subset S \), we also have

(i) \( f_n(z) \notin S^* \) for \( f(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} k z^k \in S^* \).

(ii) \( g_n(z) \notin K \) for \( g(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k \in K \).

2 Partial sums \( f_3(z) \) and \( g_3(z) \)

For Koebe function \( f(z) \) given by

\[
f(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} k z^k
\]

which is the extremal function for the class \( S^* \), we consider the partial sum \( f_3(z) = z + 2z^2 + 3z^3 \).

Theorem 2.1. The partial sum \( f_3(z) = z + 2z^2 + 3z^3 \) of Koebe function \( f(z) = \frac{z}{(1-z)^2} \)

satisfies

\[
\text{Re} \left( 1 + \frac{z f''_3(z)}{f'_3(z)} \right) > \alpha(r) = 3 - \frac{2(1-2r)}{1-4r+9r^2} > 2 \left( 1 - \frac{\sqrt{10}}{5} \right) = 0.7350 \ldots
\]

Where

\[
0 \leq r < \frac{7-2\sqrt{10}}{9} = 0.0750 \ldots
\]
Proof. We consider $\alpha$ such that

\[(2.2) \quad \text{Re} \left\{ 1 + \frac{zf_3'(z)}{f_3(z)} \right\} = \text{Re} \left\{ 3 - \frac{2(1+2z)}{1+4z+9z^2} \right\} > \alpha \]

for $0 \leq r < \frac{7 - 2\sqrt{10}}{9} = 0.0750\ldots$. It follows that

\[\text{(2.3)} \quad \text{Re} \left\{ \frac{1+2z}{1+4z+9z^2} \right\} = \frac{1}{2} + \frac{(1 - 9r^2)(1 + 9r^2 + 4r\cos \theta)}{2(1 - 2r^2 + 81r^4 + 8r(1 + 9r^2)\cos \theta + 36r^2\cos^2 \theta)} < \frac{3 - \alpha}{2},\]

that is, that

\[\text{(2.4)} \quad \text{Re} \left\{ \frac{(1 - 9r^2)(1 + 9r^2 + 4r\cos \theta)}{1 - 2r^2 + 81r^4 + 8r(1 + 9r^2)\cos \theta + 36r^2\cos^2 \theta} \right\} < 2 - \alpha.\]

Let the function $g(t)$ be given by

\[\text{(2.5)} \quad g(t) = \frac{(1 - 9r^2)(1 + 9r^2 + 4rt)}{1 - 2r^2 + 81r^4 + 8r(1 + 9r^2)t + 36r^2t^2} \quad (t = \cos \theta).\]

Then, we have

\[\text{(2.6)} \quad g'(t) = \frac{-4r(1 + 3r)(1 - 3r)(1 + 38r^2 - 81r^3 + 162r^4 + 18r(1 + 4r + 9r^2)t + 36r^2t^2)}{(1 - 2r^2 + 81r^4 + 8r(1 + 9r^2)t + 36r^2t^2)^2}.\]

Letting

\[\text{(2.7)} \quad h(t) = 1 + 38r^2 - 81r^3 + 162r^4 + 18r(1 + 4r + 9r^2)t + 36r^2t^2,\]

we see that

(i) $h(t) < 0 \implies g'(t) > 0$,

(ii) $h(t) > 0 \implies g'(t) < 0$,

and

(iii) $h(t) = 0$ for $t = t_1, t = t_2$ ($t_1 > t_2$).
It is easy to see that $t_2 < -1$. Since

\begin{equation}
(2.8) \quad t_1 = \frac{-3(1 + 4r + 9r^2) + \sqrt{5 - 72r + 154r^2 + 972r^3 + 81r^4}}{12r},
\end{equation}

our condition $0 \leq r < \frac{7 - 2\sqrt{10}}{9}$ implies that $t_1 \leq -1$, so that, $h(t) \geq 0$. Consequently, we conclude that

\begin{equation}
(2.9) \quad g(t) \leq g(-1) = \frac{1 - 9r^2}{1 - 4r + 9r^2} < \frac{2\sqrt{10}}{5} \leq 2 - \alpha,
\end{equation}

that is,

\[ \alpha = 2 - \frac{1 - 9r^2}{1 - 4r + 9r^2} = 3 - \frac{2(1 - 2r)}{1 - 4r + 9r^2}. \]

Thus, we have

\[ \text{Re}\left\{ 1 + \frac{zf_3'(z)}{f_3(z)} \right\} > \alpha(r) \]

and

\begin{equation}
(2.10) \quad \alpha(r) = 3 - \frac{2(1 - 2r)}{1 - 4r + 9r^2}
\end{equation}

for $0 \leq r \leq \frac{7 - 2\sqrt{10}}{9}$. \qed

Next, for the function

\[ g(z) = \frac{z}{1 - z} = z + \sum_{k=2}^{\infty} z^k \]

which is the extremal function for the class $\mathcal{K}$, we consider the partial sum

\[ g_3(z) = z + z^2 + z^3. \]

\textbf{Theorem 2.2.} The partial sum $g_3(z) = z + z^2 + z^3$ of the function $g(z) = \frac{z}{1 - z}$ satisfies

\begin{equation}
(2.11) \quad \text{Re}\left( \frac{zg_3'(z)}{g_3(z)} \right) > \alpha(r) = 3 - \frac{2 - r}{1 - r + r^2} > \frac{4 - \sqrt{5}}{2} = 0.9919\ldots.
\end{equation}

Where

\[ 0 \leq r < \frac{7 - 3\sqrt{5}}{2} = 0.1458\ldots. \]
Proof. We consider $\alpha$ such that

\begin{equation}
\text{Re} \left\{ \frac{zg_3'(z)}{g_{\alpha}(z)} \right\} = \text{Re} \left\{ 3 - \frac{2 + z}{1 + z + z^2} \right\} > \alpha
\end{equation}

for $0 \leq r < \frac{7 - 3\sqrt{5}}{2} = 0.1458 \cdots$.

This implies that

\begin{equation}
\text{Re} \left\{ \frac{2 + z}{1 + z + z^2} \right\} = 1 + \frac{(1 - r^2)(1 + r^2 + r\cos \theta)}{1 - r^2 + r^4 + 4r^2 \cos^2 \theta + 2r(1 + r^2)\cos \theta}
\end{equation}

\begin{equation}
< 3 - \alpha,
\end{equation}

that is,

\begin{equation}
\text{Re} \left\{ \frac{(1 - r^2)(1 + r^2 + r\cos \theta)}{1 - r^2 + r^4 + 4r^2 \cos^2 \theta + 2r(1 + r^2)\cos \theta} \right\} < 2 - \alpha.
\end{equation}

Let the function $g(t)$ be given by

\begin{equation}
g(t) = \frac{(1 - r^2)(1 + r^2 + rt)}{1 - r^2 + r^4 + 4r^2 t^2 + 2r(1 + r^2) t}. \quad (t = \cos \theta).
\end{equation}

Then, we have

\begin{equation}
g'(t) = \frac{r(r + 1)(r - 1)(1 + 5r^2 + r^4 + 4r^2 t^2 + 8r(1 + r^2) t)}{(1 - r^2 + r^4 + 4r^2 t^2 + 2r(1 + r^2) t)^2}
\end{equation}

Defining the function $h(t)$ by

\begin{equation}
h(t) = 1 + 5r^2 + r^4 + 4r^2 t^2 + 8r(1 + r^2) t,
\end{equation}

we see that

(i) $h(t) < 0 \implies g'(t) > 0,$

(ii) $h(t) > 0 \implies g'(t) < 0,$

and

(iii) $h(t) = 0$ for $t = t_1, t = t_2 \quad (t_1 > t_2)$.

Note that $t_2 < -1$. Since

\begin{equation}
t_1 = \frac{-2(1 + r^2) + \sqrt{3(1 + r^2 + r^4)}}{2r} < 0,
\end{equation}
our condition $0 \leq r < \frac{7 - 3\sqrt{5}}{2}$ of the theorem implies that $t_1 \leq -1$, so that, $h(t) \geq 0$.

Consequently, we conclude that

\begin{equation}
(2.19) \quad g(t) \leq g(-1) = \frac{1 - r^2}{1 - r + r^2} < \frac{4 - \sqrt{5}}{2} \leq 2 - \alpha,
\end{equation}

that is,

$$\alpha = 2 - \frac{1 - r^2}{1 - r + r^2} = 3 - \frac{2 - r}{1 - r + r^2}.$$

Thus, we have

$$\Re \left\{ \frac{zg_3'(z)}{g_3(z)} \right\} > \alpha(r),$$

and

\begin{equation}
(2.20) \quad \alpha(r) = 3 - \frac{2 - r}{1 - r + r^2}
\end{equation}

for $0 \leq r < \frac{7 - 3\sqrt{5}}{2} = 0.1458\ldots$. \hfill \square

Using the same method in the above, we also derive the following result.

**Theorem 2.3.** The partial sum $f_3(z) = z + 2z^2 + 3z^3$ of Koebe function $f(z) = \frac{z}{(1-z)^2}$ satisfies

\begin{equation}
(2.21) \quad \Re \left\{ \frac{zf_3'(z)}{f_3(z)} \right\} > \alpha(r) = 3 - \frac{2(1 - r)}{1 - 2r + 3r^2} > \frac{3(89 - 16\sqrt{22})}{137} = 0.3055\ldots
\end{equation}

Where

$$0 \leq r < \frac{5 - \sqrt{22}}{3} = 0.1031\ldots.$$

**Theorem 2.4.** The partial sum $g_3(z) = z + z^2 + z^3$ of the function $g(z) = \frac{z}{1 - z}$ satisfies

\begin{equation}
(2.22) \quad \Re \left\{ \frac{zg_3''(z)}{g_3'(z)} \right\} > \alpha(r) = 3 - \frac{2(1 - r)}{1 - 2r + 3r^2} > \frac{3(89 - 16\sqrt{22})}{137} = 0.3055\ldots
\end{equation}

Where

$$0 \leq r < \frac{5 - \sqrt{22}}{3} = 0.1031\ldots.$$
3 Partial sums $f_4(z)$ and $g_4(z)$

For the Koebe function

$$f(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} k z^k$$

which is the extremal function for the class $S^*$, we consider the partial sum $f_4(z) = z + 2z^2 + 3z^3 + 4z^4$.

**Theorem 3.1.** The partial sum $f_4(z) = z + 2z^2 + 3z^3 + 4z^4$ of Koebe function $f(z) = \frac{z}{(1-z)^2}$ satisfies

\[
\text{Re}\left\{\frac{zf_4(z)}{f_4(z)}\right\} > \alpha(r) = 4 - \frac{3 - 4r + 3r^2}{1 - 2r + 3r^2 - 4r^3}
\]

Where $0 \leq r \leq r_0 < 1$ and

\[
r_0 = \frac{3}{16} + \frac{\sqrt[3]{531 + 16\sqrt{1695}}}{16\sqrt[3]{9}} - \frac{37}{16\sqrt[3]{3(531 + 16\sqrt{1695})}} = 0.3545\ldots
\]

**Proof.** For $f_4(z) = z + 2z^2 + 3z^3 + 4z^4$, we have

\[
\text{Re}\left\{\frac{zf_4'(z)}{f_4(z)}\right\} = \text{Re}\left\{\frac{1 + 4z + 9z^2 + 16z^3}{1 + 2z + 3z^2 + 4z^3}\right\}
\]

\[
= 4 - \text{Re}\left\{\frac{3 + 4z + 3z^2}{1 + 2z + 3z^2 + 4z^3}\right\}
\]

\[
= 4 - \text{Re}\left\{\frac{3 + 4re^{i\theta} + 3r^2 e^{i2\theta}}{1 + 2re^{i\theta} + 3r^2 e^{i2\theta} + 4r^3 e^{i3\theta}}\right\} (z = re^{i\theta}).
\]

By using Mathematica, we know that the value of (3.2) takes its minimum value for $\theta = \pi$. This gives that

\[
\text{Re}\left\{\frac{zf_4(z)}{f_4(z)}\right\} \geq 4 - \frac{3 - 4r + 3r^2}{1 - 2r + 3r^2 - 4r^3} (0 \leq r \leq r_0).
\]

Let the function $h(r)$ be given by

\[
h(r) = 4 - \frac{3 - 4r + 3r^2}{1 - 2r + 3r^2 - 4r^3} (0 \leq r \leq r_0).
\]

Since $0 = h(r_0) \leq h(r) \leq 1$ for

\[
r_0 = \frac{3}{16} + \frac{\sqrt[3]{531 + 16\sqrt{1695}}}{16\sqrt[3]{9}} - \frac{37}{16\sqrt[3]{3(531 + 16\sqrt{1695})}} = 0.3545\cdots
\]
we have

$$\text{(3.4)} \quad \text{Re} \left\{ \frac{zf_4'(z)}{f_4(z)} \right\} > \alpha(r) = 4 - \frac{3 - 4r + 3r^2}{1 - 2r + 3r^2 - 4r^3}$$

which completes the proof of the theorem.

Next, we show

**Theorem 3.2.** The partial sum $g_4(z) = z + z^2 + z^3 + z^4$ of the function $g(z) = \frac{z}{1-z}$ satisfies

$$\text{(3.5)} \quad \text{Re} \left\{ 1 + \frac{zg_4''(z)}{g_4'(z)} \right\} > \alpha(r) = 4 - \frac{3 - 4r + 3r^2}{1 - 2r + 3r^2 - 4r^3}.$$  

Where $0 \leq r \leq r_0 < 1$

$$r_0 = \frac{3}{16} + \frac{\sqrt{531 + 16\sqrt{1695}}}{16\sqrt{9}} - \frac{37}{16\sqrt{3(531 + 16\sqrt{1695})}} = 0.3545\ldots$$

**Proof.** For $g_4(z) = z + z^2 + z^3 + z^4$, we have

$$\text{(3.6)} \quad \text{Re} \left\{ 1 + \frac{zg_4''(z)}{g_4'(z)} \right\} = \text{Re} \left\{ 1 + \frac{2z + 6z^2 + 12z^3}{1 + 2z + 3z^2 + 4z^3} \right\}$$

$$= 4 - \text{Re} \left\{ \frac{3 + 4z + 3z^2}{1 + 2z + 3z^2 + 4z^3} \right\}$$

$$= 4 - \text{Re} \left\{ \frac{3 + 4re^{i\theta} + 3r^2e^{2i\theta}}{1 + 2re^{i\theta} + 3r^2e^{2i\theta} + 4r^3e^{3i\theta}} \right\} \quad (0 \leq r \leq r_0)$$

Further, an application of Mathematica shows that

$$\text{(3.7)} \quad \text{Re} \left\{ 1 + \frac{zg_4''(z)}{g_4'(z)} \right\} \geq 4 - \frac{3 - 4r + 3r^2}{1 - 2r + 3r^2 - 4r^3} \quad (0 \leq r \leq r_0).$$

Defining the function $h(r)$ by

$$h(r) = 4 - \frac{3 - 4r + 3r^2}{1 - 2r + 3r^2 - 4r^3} \quad (0 \leq r \leq r_0),$$

we see that

$$0 \leq h(r_0) \leq h(r) \leq 1$$

for
\[ r_0 = \frac{3}{16} + \frac{\sqrt[3]{531 + 16\sqrt{1695}}}{16\sqrt[3]{9}} - \frac{37}{16\sqrt[3]{3(531 + 16\sqrt{1695})}} = 0.3545 \cdots, \]

that is, that

\[ \text{Re} \left\{ 1 + \frac{zg_4'(z)}{g_4'(z)} \right\} > \alpha(r) = 4 - \frac{3 - 4r + 3r^2}{1 - 2r + 3r^2 - 4r^3} \]

for \( 0 \leq r \leq r_0 \).

Using the same method in the above, we also derive

**Theorem 3.3.** The partial sum \( f_4(z) = z + 2z^2 + 3z^3 + 4z^4 \) of Koebe function 
\[ f(z) = \frac{z}{(1-z)^2} \]

satisfies

\[ \text{Re} \left\{ 1 + \frac{zf_4'(z)}{f_4'(z)} \right\} > \alpha(r) = 4 - \frac{3 - 8r + 9r^2}{1 - 4r + 9r^2 - 16r^3}. \]

Where \( 0 \leq r \leq r_1 < 1 \) and

\[ r_1 = \frac{9}{64} + \frac{\sqrt{4257 + 64\sqrt{18681}}}{64\sqrt[3]{9}} - \frac{269}{64\sqrt[3]{3(4257 + 64\sqrt{18681})}} = 0.1933 \ldots. \]

**Theorem 3.4.** The partial sum \( g_4(z) = z + z^2 + z^3 + z^4 \) of the function 
\[ g(z) = \frac{z}{1-z} \]

satisfies

\[ \text{Re} \left\{ \frac{zg_4(z)}{g_4(z)} \right\} > \alpha(r) = 4 - \frac{3 - 2r + r^2}{(1-r)(1+r^2)}. \]

Where \( 0 \leq r \leq r_2 < 1 \) and

\[ r_2 = \frac{1}{4} + \frac{\sqrt{5(9 + 4\sqrt{6})}}{4\sqrt[3]{9}} - \frac{\sqrt{25}}{4\sqrt[3]{3(9 + 4\sqrt{6})}} = 0.6058 \ldots. \]

### 4 Appendix

In this section, we try to describe the image domain of the disk by the partial sums for the theorems in Section 2 and Section 3.
Example 4.1. By Theorem 2.1, we take the partial sum $f_3(z) = z + 2z^2 + 3z^3$ for $|z| = r$ with

$$0 \leq r < \frac{7 - 2\sqrt{10}}{9} = 0.0750 \ldots$$

The image domain of $f_3(z)$ is shown in Fig.4.1.

```
<< Graphics'ComplexMap'
f3[z_] = z + 2z^2 + 3z^3
Pmf3 := PolarMap[f3, {0, \frac{7-2\sqrt{10}}{9}}, {0, 2\pi}];
Show[Pmf3]
```

\[ z + 2z^2 + 3z^3 \]
Example 4.2. By Theorem 2.2, we take the partial sum $g_3(z) = z + z^2 + z^3$ for $|z| = r$ with

$$0 \leq r < \frac{7 - 3\sqrt{5}}{2} = 0.1458\ldots$$

The image domain of $g_3(z)$ is given by Fig. 4.2.

```
<< Graphics'ComplexMap'
g3[z_] = z + z^2 + z^3
Pmg3 = PolarMap[g3, {0, 7 - 3\sqrt{5}/2}, {0, 2\pi}];
Show[Pmg3]
```

$z + z^2 + z^3$

Fig. 4.2
Example 4.3. By Theorem 2.3, we consider the partial sum \( f_3(z) = z + 2z^2 + 3z^3 \) for \(|z| = r\) with

\[
0 \leq r < \frac{5 - \sqrt{22}}{3} = 0.1031\ldots.
\]

We give the image domain of \( f_3(z) \) in Fig.4.3.

\[
\begin{align*}
\text{<< Graphics`ComplexMap}
\text{f3[Z_\_]} &= z + 2 z^2 + 3 z^3 \\
\text{Pmf3} := \text{PolarMap}[f3, \{0, \frac{5 - \sqrt{22}}{3}\}, \{0, 2\pi\}]; \\
\text{Show}[\text{Pmf3}]
\end{align*}
\]

\( z + 2 z^2 + 3 z^3 \)

Fig.4.3
Example 4.4. By Theorem 2.4, we take the partial sum $g_3(z) = z + z^2 + z^3$ for $|z| = r$ with

$$0 \leq r < \frac{7 - 3\sqrt{5}}{2} = 0.1458\ldots$$

The image domain of $g_3(z)$ is shown in Fig.4.4.

```
<< Graphics'ComplexMap'
g3[z_] = z + z^2 + z^3
Pmg3 := PolarMap[g3, {0, 5 - \sqrt{22}/3}, {0, 2 \pi}];
Show[Pmg3]
```

$z + z^2 + z^3$

Fig.4.4
Example 4.5. By Theorem 3.1, we consider the partial sum $f_4(z) = z + 2z^2 + 3z^3 + 4z^4$ for $|z| = r$ with $0 \leq r \leq r_0 < 1$ and

$$r_0 = \frac{3}{16} + \frac{\sqrt[3]{531 + 16\sqrt{1695}}}{16\sqrt{9}} - \frac{37}{16\sqrt{3(531 + 16\sqrt{1695})}} = 0.3545 \ldots$$

Then we show the image domain of $f_4(z)$ in Fig.4.5.

```
<< Graphics `ComplexMap`
f4[z_] := z + 2z^2 + 3z^3 + 4z^4
Pmf4 := PolarMap[f4, {0, r0}, {0, 2\[Pi]}];
r0 = \frac{3}{16} + \frac{\sqrt[3]{531 + 16\sqrt{1695}}}{16\sqrt{9}} - \frac{37}{16\sqrt{3(531 + 16\sqrt{1695})}}
Show[Pmf4]
```

$z + 2z^2 + 3z^3 + 4z^4$

$$\frac{3}{16} + \frac{(531 + 16\sqrt{1695})^{1/3}}{16^{2/3}} - \frac{37}{16(3(531 + 16\sqrt{1695}))^{1/3}}$$

Fig.4.5
Example 4.6. By Theorem 3.2, we consider the partial sum $g_4(z) = z + z^2 + z^3 + z^4$ for $|z| = r$ with $0 \leq r \leq r_0 < 1$ and

$$r_0 = \frac{3}{16} + \frac{\sqrt[3]{531 + 16\sqrt{1695}}}{16\sqrt{3}} - \frac{37}{16\sqrt{3}(531 + 16\sqrt{1695})} = 0.3545 \ldots$$

Then we have the image domain of $g_4(z)$ by Fig.4.6.

\[\mathfrak{g}4\epsilon_{-}\cdot \mathrm{z}^{*}\mathrm{z}^{2}\cdot \mathrm{z}^{3}\cdot \mathrm{z}^{4}\prec \cdots \}
\mathrm{C}
\mathrm{o}
\mathrm{p}
\mathrm{e}^{*}
\mathrm{M}
\mathrm{a}
\mathrm{p}^{\backslash} \\
\mathrm{r}_{0} \approx \frac{3}{16} \cdot \frac{\sqrt[3]{531 + 16\sqrt{1695}}}{16\sqrt{3}} - \frac{37}{16\sqrt{3}(531 + 16\sqrt{1695})} \\
\mathrm{z}^* + \mathrm{z}^{2} + \mathrm{z}^{3} + \mathrm{z}^{4} \\
\frac{3}{16} \cdot \frac{(531 + 16\sqrt{1695})^{1/3}}{16^{3/2}} - \frac{37}{16\sqrt{3}(531 + 16\sqrt{1695})^{1/3}} \\
\mathrm{F}
\mathrm{i}
\mathrm{g}.4.6
Example 4.7. Considering the partial sum \( f_4(z) = z + 2z^2 + 3z^3 + 4z^4 \) for \( |z| = r \) with \( 0 \leq r \leq r_1 < 1 \) and
\[
    r_1 = \frac{9}{64} + \frac{\sqrt[3]{4257 + 64\sqrt{18681}}}{64\sqrt[9]{9}} - \frac{269}{64\sqrt[3]{3(4257 + 64\sqrt{18681})}} = 0.1933\ldots,
\]
in Theorem 3.3, we see the image domain of \( f_4(z) \) in Fig.4.7.

Fig.4.7
Example 4.8. Taking the partial sum $g_4(z) = z + z^2 + z^3 + z^4$ for $|z| = r$ with $0 \leq r \leq r_2 < 1$ and

$$r_2 = \frac{1}{4} + \frac{\sqrt[3]{5(9 + 4\sqrt{6})}}{4\sqrt{9}} - \frac{\sqrt{25}}{4\sqrt{3}(9 + 4\sqrt{6})} = 0.6058\ldots,$$

we have the image domain of $g_4(z)$ in Fig.4.8.

```math
<< Graphics 'ComplexMap'
g4[z_] := z + z^2 + z^3 + z^4
Pmg4 := PolarMap[g4, {0, r2}, {0, 2\[Pi]}];
Show[Pmg4]
```

Fig.4.8
References


S. Owa:
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan
E-mail: owa@math.kindai.ac.jp

H.M. Srivastava:
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4
Canada
E-mail: hmsri@uvvm.uvic.ca

N. Saito:
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan