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On the spectrum of magnetic Schrödinger operator on the hyperbolic plane

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1 Introduction and Result

Let $\mathbf{H} = \{z = (x, y) | x \in \mathbf{R}, y > 0\}$ be the hyperbolic plane. The Riemannian measure on $\mathbf{H}$ is given by $dxdy/y^2$ and the hyperbolic distance $d_{\mathbf{H}}(z, z_0)$ on $\mathbf{H}$ is given by $\cosh(d(z, z_0)) = (|x - x_0|^2 + y^2 + y_0^2)/(2yy_0)$ for any $z = (x, y), z_0 = (x_0, y_0) \in \mathbf{H}$.

We consider the Schrödinger operator

$$H(\boldsymbol{a}; V) = y^2 \left(\frac{1}{\sqrt{-1}} \frac{\partial}{\partial x} - a_1(z) \right)^2 + y^2 \left(\frac{1}{\sqrt{-1}} \frac{\partial}{\partial y} - a_2(z) \right)^2 + V(z) \tag{1.1}$$

acting in $L^2(\mathbf{H})$, where $\boldsymbol{a} = a_1 dx + a_2 dy$ is magnetic vector potential and V is scalar potential on $\mathbf{H}$. Note that, when $\boldsymbol{a} = \mathbf{0}$ and $V = 0$, the operator $H(\mathbf{0}; 0)$ coincides with the Laplace-Beltrami operator on $\mathbf{H}$.

We recall some results concerning the spectral property of $H(\boldsymbol{a}; V)$. The essential self-adjointness of $H(\boldsymbol{a}; V)$ obeys under rather weaker conditions on $\boldsymbol{a}$ and V; For example, it is sufficient if $\boldsymbol{a}$ is a smooth, $\mathbf{R}^2$-valued function and V is real-valued, locally bounded, measurable and bounded from below. (See, [Shu] and the references therein).

Inahama and the author [I-S] studied the essential spectrum of the Schrödinger operator $H(\boldsymbol{a}; 0)$ with magnetic field $d\boldsymbol{a}(z) = B(z) dx \wedge dy/y^2$, where B is real-valued, smooth function on $\mathbf{H}$, and $B - B_0$ tends to zero at infinity for some real constant B_0. In that case, the essential spectrum of $H(\boldsymbol{a}; 0)$ coincides with that of the operator $H(\boldsymbol{a}_0; 0)$. Here, the vector potential $\boldsymbol{a}_0 = (B_0/y)dx$ gives the 'constant' magnetic field $d\boldsymbol{a}_0 = B_0 dx \wedge dy/y^2$. (We give a precise description of the essential spectrum of $H(\boldsymbol{a}_0; 0)$ below.) Similar results hold for the Dirac and Pauli operators on (the trivial bundle over) $\mathbf{H}$ ([I-S2]).

In the case where V diverges (e.g., like $C \exp(\varepsilon d(z, \sqrt{-1}))$) at infinity and the magnetic field is absent, $H(\mathbf{0}; V)$ has compact resolvent. Under some additional conditions on V, Inahama and the author [I-S3] studied the large eigenvalue asymptotics for the Schrödinger operator $H(\mathbf{0}; V)$:

$$N(H(\mathbf{0}; V) < \lambda) = (2\pi)^{-2} |\{(z, \xi) \in \mathrm{T}^*\mathbf{H} |\ y^2|\xi|^2 + V(z) < \lambda\}| (1 + o(1))$$

as $\lambda \to \infty$ ([I-S3]). Here, $N(H(\mathbf{0}; V) < \lambda)$ stands for the number of eigenvalues of $H(\mathbf{0}; V)$ (counting multiplicity) less than λ, and $|\cdot|$ is the four dimensional Lebesgue measure (the Liouville measure). Similar results hold for the case of the real, complex and quaternion hyperbolic spaces ([I-S4], Inahama, Kuwada and the author [IKS]). In the magnetic field case, we can derive the same kind of asymptotic relations as above for $H(\boldsymbol{a}; V)$ if the growth of the magnetic

field B is weaker than that of the scalar potential V in an appropriate sense ([I-S5]). To the author's knowledge, there is no result for the large eigenvalue asymptotics in the case of general electro-magnetic fields $\boldsymbol{a}$, V.

In this article we consider the Maass Hamiltonian $H(\boldsymbol{a}_0; V)$, where $\boldsymbol{a}_0 = (B_0/y)dx$ as above, and study the eigenvalue asymptotics near the essential spectrum when V decays at infinity, i.e., for any $\varepsilon > 0$ there exists a compact subset K of $\mathbf{H}$ such that $|V(x,y)| < \varepsilon$ outside K. In what follows, for notational simplicity, we denote $H(\boldsymbol{a}_0; V)$ and B_0 by $H(V)$ and B, respectively.

The spectral properties of the Maass Hamiltonian has been investigated by many authors ([Roe], [Els], [Fay], [Gro], [C-H], [Com], [A-P] and references therein). We recall some basic results. The Maass Hamiltonian $H(0)$ is essentially self-adjoint on $C_0^\infty(\mathbf{H})$, the set of all complex-valued, smooth functions with compact support on $\mathbf{H}$ ([Roe], Satz 3.2). (In what follows we use the same notation for an operator and its operator closure if there is no fear of confusion.) The spectrum of $H(0)$ consists of the absolutely continuous part $[B^2+1/4, \infty)$ and the discrete Landau levels $\{E_n\}_{n=0}^{N(|B|-1/2)}$, where $E_n = (2n+1)|B| - n(n+1)$ and $N(x)$ denotes the largest integer less than x. In case $|B| \leq 1/2$, the set of discrete Landau levels is empty. If $|B| > 1/2$, each of E_n's is an eigenvalue of infinite multiplicity. In what follows, we may restrict ourselves to the case $B > 1/2$, provided we are concerned with the discrete Landau levels, since the Maass Hamiltonian $H(0)$ with B is unitarily equivalent to the one with $-B$ via the transform $(x,y) \mapsto (-x,y)$.

Any bounded, measurable function V decaying at infinity is relatively compact with respect to $H(0)$ ([I-S], Lemma 3.10), so the operator

$$H(V) = H(0) + V$$

is a well-defined self-adjoint operator when V is real-valued, and the essential spectrum of $H(V)$ coincides with that of $H(0)$ ([R-S], Vol. IV). (Note that, examining the proof, one can easily find that Lemma 3.10 in [I-S] is still valid if we drop the continuity condition of V.) Then the perturbed operator $H(V)$ may have the discrete spectrum (i.e., discrete eigenvalues of finite multiplicity) in the spectral gaps.

The purpose of this paper is to obtain the asymptotic distribution of the number of the discrete spectrum near E_n's.

To formulate our results, we make the following condition on the perturbation V:

(V)$_\varepsilon$ The perturbation V is a real-valued, bounded, measurable and non-negative function on $\mathbf{H}$. Moreover, there exist $z_0 \in \mathbf{H}$ and positive constants ε and C_V such that the asymptotic relation

$$\lim_{d(z,z_0)\to\infty} \exp(\varepsilon d(z,z_0))V(z) = C_V \tag{1.2}$$

holds, where d is the hyperbolic distance introduced at the beginning of this section.

Let n be any non-negative integer n satisfying $0 \leq n \leq N(B-1/2)$ and let $\varepsilon > 0$. We introduce the notations

$$\beta_n = 2B - 2n - 1 \ (> 0)$$

and

$$\Theta_n(\varepsilon) = \frac{\Gamma(\beta_n+\varepsilon)\Gamma(\beta_n+n+1)}{\Gamma(\beta_n)\Gamma(n+1)\Gamma(\beta_n+1)} F_2(\beta_n+\varepsilon; -n, -n; \beta_n+1, \beta_n+1; 1, 1),$$

Here, $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$ is the gamma function and

$$F_2(a; b, b'; c, c'; x, y) = \sum_{l,m=0}^{\infty} \frac{(a)_{l+m}(b)_l(b')_m}{(c)_l(c')_m} \frac{x^l y^m}{l!m!}$$

is the Appell hypergeometric series (See [G-R], Section 9.18, [Sla], Section 8) and $(x)_0 = 1$ and $(x)_m = x(x+1)\cdots(x+m-1)$ if $m \geq 1$. We note that, because of the parameter $-n$, the Appell series in the expressions of $\Theta_n(\varepsilon)$ terminates and it turns out that $\Theta_n(\varepsilon)$ is positive (See Lemma 2.2 below).

For any real numbers a, b and for any self-adjoint operator T acting in a Hilbert space, we set

$$N(a < T < b) \quad = \quad \dim \mathrm{ran}(P_T((a,b))),$$

where $P_T(I)$ denotes the the spectral projection for T on an open interval I.

The main results of this paper are the following two theorems:

Theorem 1.1 *Assume that $|B| > 1/2$. Let E' be any point between E_n and E_{n+1}, where we set $E_{n+1} = B^2 + 1/4$ for $n = N(B - 1/2)$. Then the condition $(V)_\varepsilon$ implies that*

$$N(E_n + E < H(V) < E') = \frac{1}{4\pi}(\Theta_n(\varepsilon))^{1/\varepsilon}\mathrm{Vol}_{\mathbf{H}}\{z \in \mathbf{H} | V(z) > E\}(1 + o(1)) \tag{1.3}$$

as $E \searrow 0$, where $\mathrm{Vol}_{\mathbf{H}}$ is the Riemannian volume on $\mathbf{H}$.

For any $z_0 \in \mathbf{H}$, we denote by F_{T,t,z_0} the characteristic function on the set $\{z \in \mathbf{H} | t \leq d(z_0, z) \leq T\}$.

Theorem 1.2 *Assume that $|B| > 1/2$ and V is bounded, measurable, non-negative on $\mathbf{H}$ and decays at infinity. Let E' be any point between E_n and E_{n+1}, where we set $E_{n+1} = B^2 + 1/4$ for $n = N(B - 1/2)$. Let $z_0 \in \mathbf{H}$ and $0 \leq t < T$. Then the following assertions hold:*

(i) If there exists a positive constant c such that $0 \leq V(z) \leq cF_{T,t,z_0}(z)$ holds for all $z \in \mathbf{H}$, then we have

$$|\log \tanh^2(T/2)| \limsup_{E \searrow 0} N(E_n + E < H(V) < E')/|\log E| \leq 1.$$

(ii) If there exists a positive constant c such that $cF_{T,t,z_0}(z) \leq V(z)$ holds for all $z \in \mathbf{H}$. then we have

$$|\log \tanh^2(T/2)| \liminf_{E \searrow 0} N(E_n + E < H(V) < E')/|\log E| \geq 1.$$

(iii) In particular, if there exist positive constants c, c' such that $cF_{T,t,z_0}(z) \leq V(z) \leq c'F_{T,t,z_0}(z)$ holds for all $z \in \mathbf{H}$, then we have

$$|\log \tanh^2(T/2)| \lim_{E \searrow 0} N(E_n + E < H(V) < E')/|\log E| = 1.$$

Let $\mathrm{SL}(2,\mathbf{R})$ be the special linear group of 2×2 real matrices, which acts on $\mathbf{H}$ transitively and isometrically as the linear fractional transform $z \mapsto \gamma z = (az+b)/(cz+d)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Fix $z_0 = x_0 + y_0\sqrt{-1} \in \mathbf{H}$ and set $\lambda = \sqrt{y_0}$, $a = x_0/y_0$ and $\gamma = \begin{pmatrix} \lambda & \lambda a \\ 0 & \lambda^{-1} \end{pmatrix}$. One can observe that $\gamma \in \mathrm{SL}(2,\mathbf{R})$ and $\gamma\sqrt{-1} = \lambda^2 a + \lambda^2\sqrt{-1} = x_0 + y_0\sqrt{-1} = z_0$. If we define the unitary operator S acting on on $L^2(\mathbf{H})$ by $(Sf)(z) = f(\gamma z) = f(\lambda^2(x+a), \lambda^2 y)$, we can find that $(S^{-1}f)(z) = f(x/\lambda^2 - a, y/\lambda^2)$, $S^{-1}\frac{\partial}{\partial x}S = \lambda^2\frac{\partial}{\partial x}$, $S^{-1}\frac{\partial}{\partial y}S = \lambda^2\frac{\partial}{\partial y}$ hold and the multiplication operator g transforms as $S^{-1}gS = (S^{-1}g) = g(\gamma^{-1}\cdot)$ on $C_0^\infty(\mathbf{H})$, from which we can deduce that the operator $H(0)$ commutes with S. Then we have the unitary equivalence $S^{-1}H(V)S = H(0) + V(\gamma^{-1}\cdot) = H(V(\gamma^{-1}\cdot))$. Hence it is enough to prove Theorem 1.1 and Theorem 1.2 in the case of $z_0 = \sqrt{-1}$.

We canonically identify any point $z = (x,y) \in \mathbf{H}$ with $z = x + \sqrt{-1}y$ in the upper-half complex plane. Let $\mathbf{D}$ be the Poincaré disk $\{w = re^{i\theta} | 0 \le r < 1,\ 0 \le \theta < 2\pi\}$ equipped with the standard measure $4r(1-r^2)^{-2}drd\theta$. The Cayley transform A is defined by $Az = (z-i)/(z+i)$ for each $z \in \mathbf{H}$, and A defines an isometric diffeomorphism between $\mathbf{H}$ and $\mathbf{D}$, so it induces the unitary transform A_* from $L^2(\mathbf{H})$ to $L^2(\mathbf{D})$ by $f(z) \mapsto f(A^{-1}w)$. For any $w = re^{\sqrt{-1}\theta} \in \mathbf{D}$, the distance $d_{\mathbf{D}}(w,0)$ on $\mathbf{D}$ is given by $\log[(1+r)/(1-r)]$, which coincides with $d(A^{-1}w, \sqrt{-1})$ on $\mathbf{H}$. In the sequel, we shall idenfity $\mathbf{H}$ and $\mathbf{D}$ via A.

We note that, in the case of $z_0 = \sqrt{-1}$, the asymptotic relation (1.2) is equivalent to the condition that

$$\lim_{r\nearrow 1} \frac{V(A^{-1}w)}{(1-r^2)^\varepsilon} = 4^{-\varepsilon}C_V$$

holds on $\mathbf{D}$, because of the relation $1 - r^2 = \cosh^{-2}(d_{\mathbf{D}}(w,0)/2)$ for any $w = re^{\sqrt{-1}\theta} \in \mathbf{D}$.

Remark 1.3 *Let V satisfy $(V)_\varepsilon$ for some $\varepsilon > 0$ and let F_{T,t,z_0} be the function as in Theorem 1.2. Then a simple calculation shows that*

$$\begin{aligned} \lim_{E\searrow 0} E^{1/\varepsilon}\mathrm{Vol}_{\mathbf{H}}\{z \in \mathbf{H} |\ V(z) > E\} &= \pi C_V^{1/\varepsilon}, \\ \lim_{E\searrow 0} \mathrm{Vol}_{\mathbf{H}}\{z \in \mathbf{H} |\ F_{T,t,z_0}(z) > E\} &= 4\pi(\cosh^2 T - \cosh^2 t). \end{aligned}$$

Remark 1.4 *Our results are concerned with the asymptotic distribution of the discrete spectrum accumulating to each discrete Landau level E_n from the right. Analogous results hold if we consider the eigenvalues of $H(-V)$ accumulating to E_n from the left.*

Unfortunately, the author have not obtained the result at the lower edge of the continuous spectrum of $H(0)$.

In the Euclidean case, Raikov ([Rai], [Rai2]) has obtained the asymptotic distribution of the number of the discrete spectrum near the boundary of the essential spectrum of the Schrödinger operators with constant magnetic fields and power-like decreasing electric potentials. In the two

dimensional case, the leading asymptotics are independent of the level-number n, and behaves quasi-classically, i.e., behaves like $(B/2\pi)\mathrm{Vol}_{\mathbf{R}^2}\{x \in \mathbf{R}^2 | V(x) > E\}$ as $E \searrow 0$ (See, e.g., [R-W], Remark 2.5). Here B is the strength of the constant magnetic field and $B/(2\pi)$ is the density of states for the n-th Landau level of the Landau Hamiltonian.

Recently, several authors ([R-W], [M-R]) investigated the asymptotics for the case where the decay of the electric potentials V is Gaussian or faster. They showed that the asymptotics are non-classical if the decay of V is faster than Gaussian (in an appropriate sense) or support of V is compact. The leading asymptotics are independent of n, and in the case of compact support, it does not depend on V.

On the other hand, our results shows that the asymptotic behaviour of $N(E_n + E < H(V) < E')$ has the form (1.3) as $E \searrow 0$. The density of states of the Maass Hamiltonian can be found in [Com], Eq.(5.14)–(5.16), Eq.(B.19). In particular the density of states for the n-th discrete Landau level is given by $\beta_n/(4\pi)$, which depends on n. The quantity $\beta_n/(4\pi)$ does not coincide with the leading coefficient $\Theta_n(\varepsilon)^{1/\varepsilon}/(4\pi)$ in (1.3). So this is different from the flat case.

Remark 1.5 *The asymptotic coefficient $(\Theta_n(\varepsilon))^{1/\varepsilon}$ in (1.3) depends on both n and ε. For example, we calculate*

$$\begin{aligned}
\Theta_0(\varepsilon) &= \beta_0 \frac{\Gamma(\beta_0+\varepsilon)}{\Gamma(\beta_0+1)}, \\
\Theta_1(\varepsilon) &= \beta_1 \frac{\Gamma(\beta_1+\varepsilon)}{\Gamma(\beta_1+1)} \left(1 + \frac{\varepsilon(\varepsilon-1)}{\beta_1+1}\right), \\
\Theta_2(\varepsilon) &= \beta_2 \frac{\Gamma(\beta_2+\varepsilon)}{\Gamma(\beta_2+1)} \left(1 + \frac{\varepsilon(\varepsilon-1)}{\beta_2+1} + \frac{\varepsilon(\varepsilon-1)}{\beta_2+2} + \frac{\varepsilon^2(\varepsilon-1)^2}{2(\beta_2+1)(\beta_2+2)}\right),
\end{aligned}$$

etc. An integral representation of $\Theta_n(\varepsilon)$, from which the positivity of $\Theta_n(\varepsilon)$ obeys, is given in Lemma 2.2 in Section 2. By using some hypergeometric identities (See, e.g, [A-A-R], [Sla]), we can also express $\Theta_n(\varepsilon)$ as

$$\frac{\Gamma(\beta_n+\varepsilon)}{\Gamma(\beta_n)} {}_3F_2\left(\begin{matrix} -n, 1-\varepsilon, \varepsilon \\ \beta_n+1, 1 \end{matrix}; 1\right),$$

where ${}_3F_2$ is the (generalized) Gauss hypergeometric function (See Section 2 below). However, the last expression is not used in this paper.

The organization of this paper is as follows: In Section 2, we recall some elementary results for the gamma function and the hypergeometric functions. In Section 3, we derive an integral representation of $\Theta_n(\varepsilon)$, from which the positivity of $\Theta_n(\varepsilon)$ obeys. In Section 4, following [R-W], we reduce the problem for $H(V)$ to the one for the associated compact operator P_nVP_n. Here P_n denotes the spectral projection of $H(0)$ corresponding to E_n. In Section 5 and Section 6, we obtain the asymptotic distribution of the eigenvalues of P_nVP_n when V is functions as in Theorems 1.1 and 1.2, respectively. In Section 7 and Section 8, we give proofs for Theorem 1.1 and Theorem 1.2, respectively.

2 Preliminaries

For later use, we prepare some elementary formulae for special functions. However, all results in this section are well-known in special function theory (See, e.g., [A-A-R], [Sla], [Leb] and [G-R]). We also show the positivity of the coefficient $\Theta_n(\varepsilon)$.

The hypergeometric function ${}_pF_q$ is given by

$$(2.1) \qquad {}_pF_q\left(\begin{matrix} x_1 & x_2 & \cdots & x_p \\ y_1 & y_2 & \cdots & y_q \end{matrix}; z\right) = \sum_{m=0}^{\infty} \frac{(x_1)_m (x_2)_m \cdots (x_p)_m}{(y_1)_m (y_2)_m \cdots (y_q)_m} \frac{z^m}{m!}.$$

Lemma 2.1 *Let $\Gamma(z)$ be the gamma function and let $(a)_m$ as in Section 1. Then we have the following assertions:*

(i) For any real numbers α, β, we have $\lim_{k\to\infty} k^{\beta-\alpha}\Gamma(k+\alpha)/\Gamma(k+\beta) = 1$.

(ii) If x is not a non-positive integer, we have $(x)_m = \Gamma(x+m)/\Gamma(x)$ and $(-x)_m = (-1)^m\Gamma(x+1)/\Gamma(x-m+1)$. For any non-negative integer n, we have

$$(-n)_m = \begin{cases} (-1)^m \Gamma(n+1)/\Gamma(n-m+1) & \text{if } 0 \le m \le n, \\ 0 & \text{if } m \ge n+1. \end{cases}$$

(iii) Let $\Re\gamma > \Re\beta > 0$ and $|\arg(1-z)| < \pi$. Then we have

$${}_2F_1\left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; z\right) = (1-z)^{-\alpha} {}_2F_1\left(\begin{matrix} \alpha, \gamma-\beta \\ \gamma \end{matrix}; \frac{z}{z-1}\right).$$

Here, $\Re$ and arg *stand for the real part and the argument of a complex number, respectively.*

Proof. The assertion (i) follows from the Stirling asymptotic formula (e.g., [Leb], Section 1.2, Eq. 1.2.2 and Section 1.4, Eq. 1.4.23). The assertion (ii) is obvious by definition. and the assertion (iii) is well-known (See, e.g., [Leb], Section 9.5, Eq.9.5.1). ∎

In the rest of this section we show the positivity of the asymptotic coefficient $\Theta_n(\varepsilon)$ as we stated in Section 1.

The Laguerre polynomial is given by

$$(2.2) \qquad \begin{aligned} L_n^{\alpha}(x) &= \frac{1}{n!} e^x x^{-\alpha} \left(\frac{d}{dx}\right)^n (e^{-x} x^{n+\alpha}) \\ &= \sum_{m=0}^{n} (-1)^m \binom{n+\alpha}{n-m} \frac{x^m}{m!} \\ &= \binom{n+\alpha}{n} {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix}; x\right) \end{aligned}$$

(See [G-R], Section 8.97).

Lemma 2.2 *Let n be a non-negative integer and let $\varepsilon > 0$. Then we have*

$$\begin{aligned}(2.3)\qquad \Theta_n(\varepsilon) &= \frac{\Gamma(\beta_n+\varepsilon)\Gamma(\beta_n+n+1)}{\Gamma(\beta_n)\Gamma(n+1)\Gamma(\beta_n+1)} F_2(\beta_n+\varepsilon;-n,-n;\beta_n+1,\beta_n+1;1,1)\\ &= \frac{\beta_n\Gamma(n+1)}{\Gamma(\beta_n+n+1)}\int_0^\infty t^{\beta_n+\varepsilon-1}e^{-t}L_n^{\beta_n}(t)^2dt.\end{aligned}$$

In particular, the integral ensures the positivity of $\Theta_n(\varepsilon)$.

Proof. We show the equality (2.3) in the same way as in the proof of Lemma 1 in [S-H]. We note that the Appell series in (2.3) converges because of the parameter $-n$. It follows from (2.2) that

$$\begin{aligned}&\int_0^\infty t^{\beta_n+\varepsilon-1}e^{-t}L_n^{\beta_n}(t)^2dt\\ &= \binom{n+\beta_n}{n}^2\int_0^\infty t^{\beta_n+\varepsilon-1}e^{-t}{}_1F_1\left(\begin{matrix}-n\\ \beta_n+1\end{matrix};t\right)^2dt\\ &= \binom{n+\beta_n}{n}^2\sum_{l,m=0}^n\frac{(-n)_l(-n)_m}{(\beta_n+1)_l(\beta_n+1)_m}\frac{1}{l!m!}\int_0^\infty t^{\beta_n+\varepsilon+l+m-1}e^{-t}dt\\ &= \binom{n+\beta_n}{n}^2\sum_{l,m=0}^n\frac{(-n)_l(-n)_m}{(\beta_n+1)_l(\beta_n+1)_m}\frac{1}{l!m!}\Gamma(\beta_n+\varepsilon+l+m)\\ &= \binom{n+\beta_n}{n}^2\Gamma(\beta_n+\varepsilon)\sum_{l,m=0}^n\frac{(-n)_l(-n)_m(\beta_n+\varepsilon)_{l+m}}{(\beta_n+1)_l(\beta_n+1)_m}\frac{1}{l!m!}\\ &= \binom{n+\beta_n}{n}^2\Gamma(\beta_n+\varepsilon)F_2(\beta_n+\varepsilon;-n,-n;\beta_n+1,\beta_n+1;1,1),\end{aligned}$$

where we used Lemma 2.1 in the fourth equality. Then the result follows since $\Gamma(\beta_n+1) = \beta_n\Gamma(\beta_n)$ and

$$\binom{n+\beta_n}{n}^2\Gamma(\beta_n+\varepsilon) = \frac{\Gamma(\beta_n+\varepsilon)\Gamma(\beta_n+n+1)^2}{\Gamma(n+1)^2\Gamma(\beta_n+1)^2}.$$

∎

3 Reduction to a single Landau-level eigenspace

In this section, following the argument as in [R-W], Section 3, we reduce the eigenvalue asymptotics for $H(V)$ near E_n to that for the compact operator P_nVP_n near 0. Here P_n denotes the spectral projection of $H(0)$ corresponding to E_n.

For any real numbers a, b and for any selfadjoint operator T, we denote

$$\begin{aligned} N(a < T) &= \dim \mathrm{ran}(P_T((a,\infty))), \\ N(T < b) &= \dim \mathrm{ran}(P_T((-\infty,b))). \end{aligned}$$

Here P_T is the spectral projection for T.

The following result can be found in Chapter 11 in [B-S]:

Lemma 3.1 *Let T_1 and T_2 be compact operators acting on a Hilbert space. Then for any $s > 0$ and for any $\delta > 0$ with $0 < \delta < 1$, we have*

$$\begin{aligned} & N(\pm T_1 > s(1+\delta)) - N(\mp T_2 > s\delta) \\ \leq\ & N(\pm(T_1+T_2) > s) \\ \leq\ & N(\pm T_1 > s(1-\delta)) + N(\pm T_2 > s\delta), \end{aligned} \tag{3.1}$$

respectively.

Lemma 3.2 *Let T be a self-adjoint operator acting in a Hilbert space and assume that the resolvent set of T contains an interval $[\alpha, \beta]$. Assume that V is non-negative, bounded and relatively compact with respect to T. Then we have*

$$\begin{aligned} N(\alpha < T+V < \beta) &= N(V^{1/2}(\alpha - T)^{-1}V^{1/2} > 1) \\ &\quad - N(V^{1/2}(\beta - T)^{-1}V^{1/2} > 1) - \dim \ker(T+V-\beta). \end{aligned}$$

Proof. This is an easy consequence of the (generalized) Birman-Schwinger principle (e.g., [A-D-H], Theorem 1.3, [Bir], Proposition 1.5), however we give a proof for the sake of completeness.

Let $E \in [\alpha, \beta]$. The Birman-Schwinger kernel is given by $X(E) = V^{1/2}(E-T)^{-1}V^{1/2}$. Then the B-S principle says that an eigenvalue E of $T + \lambda V$ $(\lambda > 0)$ of multiplicity m corresponds to an eigenvalue $1/\lambda$ of $X(E)$ of multiplicity m. Thus we have

$$\begin{aligned} \sum_{0<\lambda<1} \dim \ker(T+\lambda V - E) &= \sum_{0<\lambda<1} \dim \ker(X(E) - 1/\lambda) \\ &= N(X(E) > 1). \end{aligned} \tag{3.2}$$

On the other hand, we can deduce that each eigenvalue of $X(E)$ is monotonically decreasing in E, since the non-negativity of V implies that

$$\frac{\partial}{\partial E} V^{1/2}(E-T)^{-1}V^{1/2} = -V^{1/2}(E-T)^{-2}V^{1/2} \leq 0.$$

Then it follows from the B-S principle and the analytic perturbation theory (e.g., [R-S], vol. IV) that each eigenvalue of $T + \lambda V$ is monotonically increasing in λ (See [A-D-H], Theorem 1.5, and

see also the argument after Proposition 1.5 in [Bir]). Then we have

$$(3.3)\qquad N(\alpha < T+V < \beta) = \sum_{0<\lambda<1} \dim \ker(T+\lambda V-\alpha) - \sum_{0<\lambda<1} \dim \ker(T+\lambda V-\beta) - \dim \ker(T+V-\beta).$$

Then the result follows from (3.3) and (3.2) with $E=\alpha,\beta$. ∎

Lemma 3.3 *The operator P_nVP_n is compact and, for any $\delta > 0$ small enough, we have, as $E \searrow 0$,*

$$\begin{aligned} & N((1-\delta)P_nVP_n > E) + O(1) \le N(E_n + E < H(V) < E') \\ \le\ & N((1+\delta)P_nVP_n > E) + O(1). \end{aligned}$$

Proof. The proof is similar to the one of Proposition 4.2 in [R-W], however, we give a proof for the sake of completeness.

The compactness of P_nVP_n follows easily from the fact that $V(H(0)-z)^{-1}$ is compact ([I-S], Lemma 3.10). By Lemma 3.2, we have

$$\begin{aligned} (3.4)\qquad & N(E_n+E<H(V)<E') \\ =\ & N(V^{1/2}(E_n+E-H(0))^{-1}V^{1/2}>1) - N(V^{1/2}(E'-H(0))^{-1}V^{1/2}>1) \\ & -\dim\ker(H(V)-E') \\ =\ & N(V^{1/2}(E_n+E-H(0))^{-1}V^{1/2}>1)+O(1) \end{aligned}$$

as $E \searrow 0$. Let $Q_n = I - P_n$. We apply Lemma 3.1 with $T_1 = V^{1/2}(E_n+E-H(0))^{-1}P_nV^{1/2}$, $T_2 = V^{1/2}(E_n+E-H(0))^{-1}Q_nV^{1/2}$ and $s=1$. Then (3.1) with upper sign yields

$$\begin{aligned} (3.5)\qquad & N(V^{1/2}(E_n+E-H(0))^{-1}P_nV^{1/2}>1+\delta) \\ & \quad -N(V^{1/2}(E_n+E-H(0))^{-1}Q_nV^{1/2}<-\delta) \\ \le\ & N(V^{1/2}(E_n+E-H(0))^{-1}V^{1/2}>1) \\ \le\ & N(V^{1/2}(E_n+E-H(0))^{-1}P_nV^{1/2}>1-\delta) \\ & \quad +N(V^{1/2}(E_n+E-H(0))^{-1}Q_nV^{1/2}>\delta). \end{aligned}$$

Since $H(0) \ge 1/2$ and the distance between the point E_n and the rest of the spectrum of $H(0)$ is positive, we have, for small $E>0$,

$$\inf\{|(E_n+E)-x|/x \mid x \in \sigma(H(0)) \setminus \{E_n\}\} \ge C_n > 0$$

for some constant C_n, where $\sigma(\cdot)$ stands for the spectrum. Hence, we have

$$\begin{aligned} |E_n + E - H(0)|^{-1}Q_n &= \sum_{j\neq n} |E_n + E - E_j|^{-1}P_j + \int_{B^2+1/4}^{\infty} |E_n + E - \lambda|^{-1} dP_{H(0)}(\lambda) \\ &\leq C_n \left(\sum_{j\neq n} E_j^{-1} P_j + \int_{B^2+1/4}^{\infty} \lambda^{-1} dP_{H(0)}(\lambda) \right) \\ &\leq C_n H(0)^{-1}. \end{aligned}$$

Then, for each $\delta > 0$ small enough, we have, as $E \searrow 0$,

$$\begin{aligned} & N(\pm V^{1/2}(E_n + E - H(0))^{-1} Q_n V^{1/2} > \delta) \\ \leq\ & N(V^{1/2}|E_n + E - H(0)|^{-1} Q_n V^{1/2} > \delta) \\ \leq\ & N(V^{1/2} C_n H(0)^{-1} V^{1/2} > \delta) \\ =\ & O(1). \end{aligned} \tag{3.6}$$

The result follows from (3.4)–(3.6). ∎

We now introduce the angular-momentum eigenfunctions (i.e., eigenfunctions of the form $e^{ik\theta}G_k(r)$) for $H(V)$ and show that the eigenvalues of P_nVP_n can be described in terms of these eigenfunctions.

Let A be the Cayley transform. We define a unitary operator U_B from $L^2(\mathbf{H})$ to $L^2(\mathbf{D})$ by $(U_B f)(w) = \left(\frac{1-\bar{w}}{1-w}\right)^B f(A^{-1}w)$ for any $f \in L^2(\mathbf{H})$, where $1^B = 1$. Then we have

$$U_B H(0) U_B^{-1} = -\frac{1}{4}(1-r^2)^2 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) + i(1-r^2)\frac{\partial}{\partial \theta} - (1-r^2)B^2 + B^2$$

([Els], Satz 2.1 and see also [Fay], Theorem 1.1). Moreover, a complete set of orthogonal angular-momentum eigenfunctions $\{\varphi_{nk}\}_{k\geq -n}^{\infty}$ corresponding to the eigenvalue E_n is known (See Satz 3.2 in [Els], Theorem 1.4 in [Fay], Eq.13 in [Gro2] and see also Eq.4.47 in [K-L]). Especially, the eigenfunctin is given by

$$\varphi_{nk} = \sqrt{C_{nk}} e^{ik\theta} r^k (1-r^2)^{B-n} {}_2F_1 \left(\begin{matrix} -n, k+\beta_n+n+1 \\ k+1 \end{matrix} ; r^2 \right) \tag{3.7}$$

in the case of $k \geq 0$, where

$$C_{nk} = \frac{\beta_n \Gamma(k+\beta_n+n+1)\Gamma(k+n+1)}{4\pi \Gamma(n+1)\Gamma(k+1)^2 \Gamma(\beta_n+n+1)}. \tag{3.8}$$

Note that, because of the parameter $-n$, the hypergeometric function above is a polynomial with respect to r^2, in fact, we can find that

$$P_n^{(k,\beta_n)}(1-r^2) = \left(\begin{matrix} n+k \\ n \end{matrix} \right) {}_2F_1 \left(\begin{matrix} -n, k+\beta_n+n+1 \\ k+1 \end{matrix} ; r^2 \right),$$

where the Jacobi polynomial $P_n^{(\alpha,\beta)}$ is given by

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\left(\frac{d}{dx}\right)^n\left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right)$$

and we set $\begin{pmatrix} n \\ m \end{pmatrix} = \frac{\Gamma(n+1)}{\Gamma(n-m+1)\Gamma(m+1)}$ ([Leb], Section 4, p.96, and for the relation between the Jacobi polynomials and the hypergeometric function, see also [G-R], Section 8.96, p.1059).

In what follows we identify the operator $U_B H(0) U_B^{-1}$, the associated spectral projections and the function $A_* V = V(A^{-1}\cdot)$ with $H(0)$, P_n and V, respectively.

Lemma 3.4 *Let V be any bounded, measurable and spherically symmetric function on $\mathbf{D}$. The set of eigenvalues of the compact operator P_nVP_n (acting on the range of P_n) is given by $\{(\varphi_{nk},\ V\varphi_{nk})\}_{k=-n}^{\infty}$, where φ_{nk} is the eigenfunction of $H(0)$ as in (3.7) and $(\cdot,\cdot)$ denotes the inner product on $L^2(\mathbf{D})$.*

Proof. Because of the orthogonality with respect to the angular momentum and the symmetry of V, we have $(\varphi_{nk},\ V\varphi_{nk'}) = 0$ if $k \neq k'$. Then it follows that $P_nVP_n\varphi_{nk} = (\varphi_{nk},\ V\varphi_{nk})\varphi_{nk}$. The result follows from the completeness of $\{\varphi_{nk}\}_{k=-n}^{\infty}$ in the range of P_n. ∎

4 Eigenvalue asymptotics for $P_nV_\varepsilon P_n$

In what follows we set $V_\varepsilon(w) = (1-|w|^2)^\varepsilon$ for any $\varepsilon > 0$, and we set

$$\gamma_{nk}(V) = (\varphi_{nk},\ V\varphi_{nk}) \tag{4.1}$$

for any function V on $\mathbf{D}$ and for $k \geq -n$. In the sequel, we investigate the asymptotic behaviour of the eigenvalues $\gamma_{nk}(V_\varepsilon)$ as $k \to \infty$, so we may assume that $k > 0$ and φ_{nk} is of the form (3.7).

Lemma 4.1

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix} -n & k+\beta_n+n+1 \\ k+1 \end{matrix}; r^2\right)^2 \\
&= \sum_{l,m=0}^{n}(-1)^{l+m}\begin{pmatrix} n \\ l \end{pmatrix}\begin{pmatrix} n \\ m \end{pmatrix} \times \\
&\quad \times \frac{\Gamma(\beta_n+n+1)^2\Gamma(k+1)^2}{\Gamma(\beta_n+n-l+1)\Gamma(\beta_n+n-m+1)\Gamma(k+1+l)\Gamma(k+1+m)} \times \\
&\quad \times (1-r^2)^{2n-l-m} r^{2(l+m)}.
\end{aligned}$$

Proof. By Lemma 2.1 (ii) with $\beta = k+\beta_n+n+1$, $\gamma = k+1$, $\gamma-\beta = -(\beta_n+n)$ and $z = r^2$, we have

$$ {}_2F_1\left(\begin{matrix} -n & \beta \\ \gamma & \end{matrix}; r^2\right) = (1-r^2)^n {}_2F_1\left(\begin{matrix} -n & -(\beta_n+n) \\ k+1 & \end{matrix}; \frac{r^2}{r^2-1}\right). $$

Then the result follows from the series expression (2.1) and Lemma 2.1 (i). ∎

Lemma 4.2 *Let $V = V(r)$ be bounded, continuous and spherically symmetric. Then we have*

$$ \begin{aligned} \gamma_{nk}(V) &= 4\pi C_{nk} \sum_{m,l=0}^{n} (-1)^{m+l} \binom{n}{m}\binom{n}{l} \times \\ &\quad \times \frac{\Gamma(\beta_n+n+1)^2\Gamma(k+1)^2}{\Gamma(\beta_n+n-m+1)\Gamma(\beta_n+n-l+1)\Gamma(k+1+m)\Gamma(k+1+l)} \times \\ &\quad \times \int_0^1 t^{k+m+l}(1-t)^{\beta_n+2n-m-l-1} V(\sqrt{t})dt. \end{aligned} $$

In particular, with $V = V_\varepsilon$, we have

$$ \begin{aligned} \gamma_{nk}(V_\varepsilon) &= 4\pi C_{nk} \sum_{m,l=0}^{n} (-1)^{m+l} \binom{n}{m}\binom{n}{l} \\ &\quad \times \frac{\Gamma(\beta_n+n+1)^2\Gamma(k+1)^2}{\Gamma(\beta_n+n-m+1)\Gamma(\beta_n+n-l+1)\Gamma(k+1+m)\Gamma(k+1+l)} \\ &\quad \times \frac{\Gamma(k+m+l+1)\Gamma(\beta_n+2n-m-l+\varepsilon)}{\Gamma(\beta_n+k+2n+\varepsilon+1)}. \end{aligned} \tag{4.2} $$

Proof. By (3.7) and Lemma 4.1, we have

$$ \begin{aligned} &\gamma_{nk}(V) \\ &= 4\int_0^{2\pi} d\theta \int_0^1 \frac{rdr}{(1-r^2)^2} V(r) C_{nk} r^{2k} (1-r^2)^{2(B-n)} \times \\ &\qquad \times {}_2F_1\left(\begin{matrix} -n & k+\beta_n+n+1 \\ k+1 & \end{matrix}; r^2\right)^2 \\ &= 8\pi C_{nk} \sum_{m,l=0}^{n} (-1)^{m+l} \binom{n}{m}\binom{n}{l} \frac{\Gamma(\beta_n+n+1)^2}{\Gamma(\beta_n+n-m+1)\Gamma(\beta_n+n-l+1)} \times \\ &\qquad \times \frac{\Gamma(k+1)^2}{\Gamma(k+1+m)\Gamma(k+1+l)} \int_0^1 r^{2(k+m+l)+1}(1-r^2)^{\beta_n+2n-m-l-1} V(r)dr, \end{aligned} $$

where we used $\beta_n = 2B-2n-1$ in the last equality. Then the first assertion follows by changing the variable $t = r^2$ in the last integral.

The second assertion follows from

$$\begin{aligned}\int_0^1 t^{k+m+l}(1-t)^{\beta_n+2n-m-l-1}V_\varepsilon(\sqrt{t})dt &= \int_0^1 t^{k+m+l}(1-t)^{\beta_n+2n-m-l-1+\varepsilon}dt \\ &= B(k+m+l+1,\beta_n+2n-m-l+\varepsilon) \\ &= \frac{\Gamma(k+m+l+1)\Gamma(\beta_n+2n-m-l+\varepsilon)}{\Gamma(k+\beta_n+2n+\varepsilon+1)},\end{aligned}$$

where $B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}dt$ is the beta function. ∎

Lemma 4.3 *For any $\varepsilon>0$, we have*

$$\lim_{k\to\infty} k^\varepsilon \gamma_{nk}(V_\varepsilon) = \frac{\Gamma(\beta_n+\varepsilon)}{\Gamma(\beta_n)}\frac{\Gamma(\beta_n+n+1)}{\Gamma(n+1)\Gamma(\beta_n+1)}F_2(\beta_n+\varepsilon;-n,-n;\beta_n+1,\beta_n+1;1,1).$$

Proof. By (4.2) and (3.8), we have

$$\begin{aligned}\gamma_{nk}(V_\varepsilon) &= \sum_{m,l=0}^{n}(-1)^{m+l}\binom{n}{m}\binom{n}{l}\frac{\beta_n\Gamma(\beta_n+n+1)}{\Gamma(n+1)}\times \\ &\quad\times\frac{\Gamma(\beta_n+2n-m-l+\varepsilon)}{\Gamma(\beta_n+n-m+1)\Gamma(\beta_n+n-l+1)}\times \\ &\quad\times\frac{\Gamma(k+\beta_n+n+1)\Gamma(k+n+1)\Gamma(k+m+l+1)}{\Gamma(k+m+1)\Gamma(k+l+1)\Gamma(k+\beta_n+2n+\varepsilon+1)}.\end{aligned} \tag{4.3}$$

Using Lemma 2.1 (iii), we have

$$\lim_{k\to\infty} k^\varepsilon\frac{\Gamma(k+\beta_n+n+1)\Gamma(k+n+1)\Gamma(k+m+l+1)}{\Gamma(k+m+1)\Gamma(k+l+1)\Gamma(k+\beta_n+2n+\varepsilon+1)}=1, \tag{4.4}$$

since $(\beta_n+n+1)+(n+1)+(m+l+1)-(m+1)-(l+1)-(\beta_n+2n+\varepsilon+1)=-\varepsilon$. Then it follows from (4.3) and (4.4) that

$$\begin{aligned}&\lim_{k\to\infty} k^\varepsilon\gamma_{nk}(V_\varepsilon) \qquad (4.5)\\ &= \frac{\beta_n\Gamma(\beta_n+n+1)}{\Gamma(n+1)}\sum_{l,m=0}^{n}(-1)^{l+m}\binom{n}{l}\binom{n}{m}\frac{\Gamma(\beta_n+2n-l-m+\varepsilon)}{\Gamma(\beta_n+n-l+1)\Gamma(\beta_n+n-m+1)} \\ &= \frac{\beta_n\Gamma(\beta_n+n+1)}{\Gamma(n+1)}\sum_{i,j=0}^{n}(-1)^{i+j}\binom{n}{i}\binom{n}{j}\frac{\Gamma(\beta_n+i+j+\varepsilon)}{\Gamma(\beta_n+i+1)\Gamma(\beta_n+j+1)} \\ &= \frac{\beta_n\Gamma(\beta_n+n+1)}{\Gamma(n+1)}\frac{\Gamma(\beta_n+\varepsilon)}{\Gamma(\beta_n+1)^2}\sum_{i,j=0}^{n}\frac{(-1)^i\Gamma(n+1)}{\Gamma(n-i+1)}\frac{(-1)^j\Gamma(n+1)}{\Gamma(n-j+1)} \\ &\quad\times\frac{\Gamma(\beta_n+i+j+\varepsilon)}{\Gamma(\beta_n+\varepsilon)}\frac{\Gamma(\beta_n+1)}{\Gamma(\beta_n+i+1)}\frac{\Gamma(\beta_n+1)}{\Gamma(\beta_n+j+1)}\frac{1}{i!j!} \\ &= \frac{\beta_n\Gamma(\beta_n+n+1)}{\Gamma(n+1)}\frac{\Gamma(\beta_n+\varepsilon)}{\Gamma(\beta_n+1)^2}\sum_{i,j=0}^{n}\frac{(-n)_i(-n)_j(\beta_n+\varepsilon)_{i+j}}{(\beta_n+1)_i(\beta_n+1)_j}\frac{1}{i!j!} \\ &= \frac{\beta_n\Gamma(\beta_n+n+1)}{\Gamma(n+1)}\frac{\Gamma(\beta_n+\varepsilon)}{\Gamma(\beta_n+1)^2}F_2(\beta_n+\varepsilon;-n,-n;\beta_n+1,\beta_n+1;1,1),\end{aligned}$$

where we set $i = n - l$, $j = n - m$ in the second equality and used Lemma 2.1 in the fourth equality. This proves the lemma. ∎

5 Eigenvalue asymptotics for the potential supported in an annulus

In this section we investigate the asymptotic behaviour of the eigenvalues $\gamma_{nk}(W_{rR})$ as $k \to \infty$. Here, W_{rR} stands for the characteristic function on the set $\{w = |w|e^{i\theta} \in \mathbf{D} | r \leq |w| \leq R\}$.

Lemma 5.1 *Let β be a real number and let r, R satisfy the relation $0 \leq r < R < 1$. If we define $B_{rR}(K, \beta) = \int_r^R t^{K-1}(1-t)^{\beta-1}dt$, the estimate*

$$C_{r,R,\beta}\frac{R^K}{K} \leq B_{rR}(K,\beta) \leq C'_{r,R,\beta}\frac{R^K}{K}$$

holds for any $K > 0$ large enough. Here the constants $C_{r,R,\beta}$, $C'_{r,R,\beta}$ are independent of large K.

Proof. If $\beta > 1$, we have

$$\begin{aligned} & (1-R)^{\beta-1}\frac{R^K - r^K}{K} = (1-R)^{\beta-1}\int_r^R t^{K-1}dt \\ \leq\ & B_{rR}(K,\beta) \\ \leq\ & (1-r)^{\beta-1}\int_r^R t^{K-1}dt = (1-r)^{\beta-1}\frac{R^K - r^K}{K}, \end{aligned}$$

since $(1-R)^{\beta-1} \leq (1-t)^{\beta-1} \leq (1-r)^{\beta-1}$ holds if $r \leq t \leq R$. Similarly if $\beta \leq 1$, we have

$$(1-r)^{\beta-1}\frac{R^K - r^K}{K} \leq B_{rR}(K,\beta) \leq (1-R)^{\beta-1}\frac{R^K - r^K}{K}.$$

Thus we have

$$\begin{aligned} & \min\{(1-R)^{\beta-1}, (1-r)^{\beta-1}\}(1-(r/R)^K)\frac{R^K}{K} \\ \leq\ & B_{rR}(K,\beta) \\ \leq\ & \max\{(1-R)^{\beta-1}, (1-r)^{\beta-1}\}(1-(r/R)^K)\frac{R^K}{K}, \end{aligned}$$

from which the lemma follows since $1/2 < 1 - (r/R)^K < 1$ holds for large K. ∎

Lemma 5.2 *Let $0 \leq r < R < 1$ and let W_{rR} be the characteristic function for the set $\{w = |w|e^{i\theta} \in \mathbf{D} | r \leq |w| \leq R\}$. Then we have*

$$\lim_{k\to\infty} \frac{\log \gamma_{nk}(W_{rR})}{2k \log R} = 1.$$

Proof. By Lemma 4.2 with $V = W_{rR}$, we have

$$\gamma_{nk}(W_{rR}) = \sum_{l,m=0}^{n} C_{ml}(k) \int_0^1 t^{k+m+l}(1-t)^{\beta_n+2n-m-l-1} W_{rR}(\sqrt{t})dt \tag{5.1}$$
$$= \sum_{l,m=0}^{n} C_{ml}(k) B_{r^2R^2}(k+m+l+1, \beta_n+2n-m-l),$$

where we set

$$\begin{aligned} & C_{ml}(k) \\ = \; & 4\pi C_{nk}(-1)^{m+l} \binom{n}{m}\binom{n}{l} \frac{\Gamma(\beta_n+n+1)^2}{\Gamma(\beta_n+n-m+1)} \frac{\Gamma(k+1)^2}{\Gamma(\beta_n+n-l+1)} \times \\ & \times \frac{1}{\Gamma(k+1+m)\Gamma(k+1+l)} \\ = \; & \frac{(-1)^{m+l}\beta_n \binom{n}{m}\binom{n}{l} \Gamma(\beta_n+n+1)}{\Gamma(n+1)\Gamma(\beta_n+n-m+1)\Gamma(\beta_n+n-l+1)} \frac{\Gamma(k+\beta_n+n+1)\Gamma(k+n+1)}{\Gamma(k+1+m)\Gamma(k+l+1)}. \end{aligned}$$

In the rest of the proof, we denote by $\sum'$ the summation over l, m satisfying $0 \le l \le n, 0 \le m \le n$ and $l+m \ge 1$. It follows from (5.1) that

$$\begin{aligned} & \log \gamma_{nk}(W_{rR}) \\ = \; & \log [C_{00}(k) B_{r^2R^2}(k+1, \beta_n+2n) \times \\ & \times (1 + \sum{}' \frac{C_{ml}(k) B_{r^2R^2}(k+m+l+1, \beta_n+2n-m-l)}{C_{00}(k) B_{r^2R^2}(k+1, \beta_n+2n)})] \\ = \; & \log C_{00}(k) + \log B_{r^2R^2}(k+1, \beta_n+2n) + \\ & + \log (1 + \sum{}' \frac{C_{ml}(k) B_{r^2R^2}(k+m+l+1, \beta_n+2n-m-l)}{C_{00}(k) B_{r^2R^2}(k+1, \beta_n+2n)}). \end{aligned} \tag{5.2}$$

By Lemma 2.1 (iii), there exists $C_n > 0$, independent of k, such that

$$\lim_{k\to\infty} k^{-(\beta_n+2n)} C_{00}(k) = \frac{\beta_n}{\Gamma(n+1)\Gamma(\beta_n+n+1)}, \; |C_{ml}(k)| \le C_n k^{\beta_n+2n-m-l} \tag{5.3}$$

hold for large k. By Lemma 5.1 and (5.3), we have, for large $k > 0$,

$$\begin{aligned} & \left| \sum{}' \frac{C_{ml}(k) B_{r^2R^2}(k+m+l+1, \beta_n+2n-m-l)}{C_{00}(k) B_{r^2R^2}(k+1, \beta_n+2n)} \right| \\ \le \; & C_{r,R,\beta_n} \sum{}' \frac{k^{\beta_n+2n-m-l}}{k^{\beta_n+2n}} \frac{R^{2(k+m+l+1)}}{k+m+l+1} \frac{k+1}{R^{2(k+1)}} \\ \le \; & C'_{r,R,\beta_n} \sum{}' k^{-m-l} \\ \le \; & C''_{r,R,\beta_n} k^{-1} \end{aligned} \tag{5.4}$$

for some positive constants $C_{r,R,\beta_n}, C'_{r,R,\beta_n}, C''_{r,R,\beta_n}$ independent of k, where we used the fact that the sum is finite $(l, m \leq n)$ in the last inequality. Then it follows from (5.2) and (5.4) that

$$\text{the rhs of (5.2)} = 2k \log R + O(\log k)$$

as $k \to \infty$, since (5.3) and Lemma 5.1 imply that

$$\begin{aligned} \log C_{00}(k) &= O(\log k), \\ \log B_{r^2R^2}(k+n, \beta_n + 2n) &= 2k \log R + O(\log k) \end{aligned}$$

as $k \to \infty$, respectively. This proves the lemma. ∎

6 Proof of Theorem 1.1

Let $V_\varepsilon(w) = (1 - |w|^2)^\varepsilon$ as in Section 4 and let W_{rR} be the function as in the previous section. To the end of the paper, we identify any objects (e.g., function, point) on $\mathbf{H}$ with the corresponding ones on $\mathbf{D}$, via the Cayley transform A, stated just after Theorem 1.2 and in Section 3.

As we remarked just after Thoerem 1.2, it is enough to show in the case of $z_0 = \sqrt{-1}$, and the condition $(\mathrm{V})_\varepsilon$ implies that, for any $\delta > 0$ small enough, there exists $R > 0$ such that

$$(1-\delta)C'_V V_\varepsilon(w) \leq V(w) \leq (1+\delta)C'_V V_\varepsilon(w)$$

holds for any $w \in \mathbf{D}$ with $|w| \geq R$. Here we set $C'_V = 4^{-\varepsilon} C_V$. Thus there exists $M > 0$ such that

$$(6.1) \qquad (1-\delta)C'_V V_\varepsilon(w) - MW_{0R}(w) \leq V(w) \leq (1+\delta)C'_V V_\varepsilon(w) + MW_{0R}(w)$$

holds for all $w \in \mathbf{D}$.

By Lemma 3.3, we have

$$\begin{aligned} (6.2) \qquad & N(E_n + E < H(V) < E') \\ \geq\ & N((1-\delta)P_n V P_n > E) + O(1) \\ \geq\ & N((1-\delta)P_n((1-\delta)C'_V V_\varepsilon - MW_{0R})P_n > E) + O(1), \end{aligned}$$

as $E \searrow 0$, where we used the lower half of (6.1) in the second inequality. Similarly, we have

$$(6.3)\ N(E_n + E < H(V) < E') \leq N((1+\delta)P_n((1+\delta)C'_V V_\varepsilon + MW_{0R})P_n > E) + O(1)$$

as $E \searrow 0$. Because of the spherical symmetry of V_ε and W_{0R}, using Lemma 3.4, we have

$$\begin{aligned} (6.4) \qquad & N((1 \mp \delta)P_n((1 \mp \delta)C'_V V_\varepsilon \mp MW_{0R})P_n > E) \\ =\ & \sharp\{k | (1 \mp \delta)[((1 \mp \delta)C'_V \gamma_{nk}(V_\varepsilon) \mp M\gamma_{nk}(W_{0R}))] > E\}, \end{aligned}$$

where $\gamma_{kn}(\cdot)$ is as in (4.1) and $\sharp$ denotes the cardinality of the set. By Lemmas 4.3 and 5.2 for any $\delta > 0$, there exists $k_\delta > 0$ such that

$$(6.5)\qquad \begin{aligned}(1-\delta)[((1-\delta)C'_V\gamma_{nk}(V_\varepsilon) - M\gamma_{nk}(W_{0R}))] &\geq (1-2\delta)(1-\delta)C'_V\gamma_{nk}(V_\varepsilon)\\ &\geq (1-3\delta)(1-\delta)C'_V\Theta_n(\varepsilon)k^{-\varepsilon}\end{aligned}$$

for all $k \geq k_\delta$, if we take δ small enough. Similarly, we have, for any $\delta > 0$,

$$(6.6)\qquad \begin{aligned}(1+\delta)[((1+\delta)C'_V\gamma_{nk}(V_\varepsilon) + M\gamma_{nk}(W_{0R}))] &\leq (1+2\delta)(1+\delta)C'_V\gamma_{nk}(V_\varepsilon)\\ &\leq (1+2\delta)(1+\delta)C'_V\Theta_n(\varepsilon)k^{-\varepsilon}.\end{aligned}$$

Then it follows from (6.2)–(6.6) that

$$\begin{aligned}\left((1-2\delta)(1-\delta)C'_V\Theta_n(\varepsilon)/E\right)^{1/\varepsilon} + O(1) &\leq N(E_n + E < H(V) < E')\\ &\leq \left((1+3\delta)(1+\delta)C'_V\Theta_n(\varepsilon)/E\right)^{1/\varepsilon} + O(1)\end{aligned}$$

as $E \searrow 0$. The arbitrariness of δ completes the proof.

7 Proof of Theorem 1.2

As we remarked just after Theorem 1.2, it is enough to show in the case of $z_0 = \sqrt{-1}$.

Lemma 7.1 *Let W_{rR} be as in Lemma 5.2 and let $C > 0$. Then we have, as $E \searrow 0$,*

$$N(E_n + E < H(CW_{rR}) < E') = \left|\frac{\log E}{2\log R}\right|(1 + o(1)).$$

Proof. By Lemma 3.3, we have, for any $\delta > 0$ small enough,

$$(7.1)\qquad \begin{aligned}N((1-\delta)P_nW_{rR}P_n > E/C) + O(1) &\leq N(E_n + E < H(CW_{rR}) < E')\\ &\leq N((1+\delta)P_nW_{rR}P_n > E/C).\end{aligned}$$

By Lemma 3.4 with $V = W_{rR}$, we have

$$(7.2)\qquad \begin{aligned}N((1\pm\delta)P_nW_{rR}P_n > E/C) &= \sharp\{k|\ \gamma_{nk}(W_{rR}) > E/C(1\pm\delta)\}\\ &= \sharp\{k|\ \log\gamma_{nk}(W_{rR}) > \log[E/C(1\pm\delta)]\}.\end{aligned}$$

By Lemma 5.2, we have, for large $k > 0$,

$$(1+\delta)\log R^{2k} \leq \log\gamma_{nk}(W_{rR}) \leq (1-\delta)\log R^{2k},$$

from which we have

$$(7.3)\qquad \begin{aligned}&\sharp\{k|\ (1+\delta)\log R^{2k} > \log[E/C(1\pm\delta)]\}\\ \leq\ &\text{the rhs of (7.2)}\\ \leq\ &\sharp\{k|\ (1-\delta)\log R^{2k} > \log[E/C(1\pm\delta)]\}.\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
(7.4) \qquad & \sharp\{k|\ (1\pm\delta)\log R^{2k} > \log[E/C(1\mp\delta)]\} \\
= \ & \sharp\{k|\ k < \frac{|\log[E/C(1\mp\delta)]|}{(1\pm\delta)|\log R^2|}\} \\
= \ & \frac{1}{1\pm\delta}\left|\frac{\log E}{\log R^2}\right| + O(1)
\end{aligned}
$$

as $E \searrow 0$. The result follows from (7.1)–(7.4) since $\delta > 0$ is arbitrary. ∎

We note that

$$\{w \in \mathbf{D} | r \le |w| \le R\} \ = \ \{w \in \mathbf{D} | \log(1+r)/(1-r) \le d(0,w) \le \log(1+R)/(1-R)\}$$

and $W_{rR} = F_{T,t,\sqrt{-1}}$ with $t = \log(1+r)/(1-r)$, $T = \log(1+R)/(1-R)$ (equivalently, with $r = \tanh(t/2)$, $R = \tanh(T/2)$).

Assume that V satisfies $0 \le V \le cF_{T,0,\sqrt{-1}}$, equivalently, $0 \le V \le cW_{0,\tanh(T/2)}$. Then it follows from Lemma 7.1 and the standard min-max argument for P_nVP_n and $P_n(cW_{rR})P_n$ that

$$\limsup_{E\searrow 0} N(E_n + E < H(V) < E')/|\log E| \le 1/|\log R^2| = 1/|\log(\tanh^2(T/2))|.$$

Then the assertion (i) in Theorem 1.2 follows. The assertions (ii) and (iii) in Theorem 1.2 follow similarly in the case of $z_0 = \sqrt{-1}$. This completes the proof.

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