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On the spectrum of magnetic Schrödinger operators with Aharonov-Bohm field

京都大学理学部 峯 拓矢 (Takuya MINE)
Faculty of Science, Department of Mathematics, Kyoto Univ.

1 Introduction

We consider the spectral problem for the Schrödinger operators in a plane with a non-zero uniform magnetic field in addition to δ-like magnetic fields. The operator of this type is studied by Nambu ([Nam]) and Exner, Šťovíček and Vytřas ([Ex-St-Vy]).

Let $N=1,2,3,\ldots$ or $N=\infty$. Let $\{z_j\}_{j=1}^N$ be points in $\boldsymbol{R}^2$ and put $S_N=\cup_{j=1}^N\{z_j\}$. We assume that

$$R=\inf_{j\neq k}|z_j-z_k|>0. \tag{1.1}$$

This assumption is satisfied if N is finite. Define a differential operator $\mathcal{L}_N$ on $\boldsymbol{R}^2\setminus S_N$ by

$$\mathcal{L}_N=\mathbf{p}_N^2,\quad \mathbf{p}_N=\frac{1}{i}\nabla+\boldsymbol{a}_N,$$

where $i=\sqrt{-1}$ and $\nabla=(\partial_x,\partial_y)$ is the gradient vector with respect to the coordinate $z=(x,y)\in\boldsymbol{R}^2$. We assume that the magnetic vector potential $\boldsymbol{a}_N=(a_{N,x},a_{N,y})$ belongs to $C^\infty(\boldsymbol{R}^2\setminus S_N;\boldsymbol{R}^2)\cap L^1_{loc}(\boldsymbol{R}^2;\boldsymbol{R}^2)$. The function $\mathrm{rot}\,\boldsymbol{a}_N(z)=(\partial_x a_{N,y}-\partial_y a_{N,x})(z)$ represents the intensity of the magnetic field perpendicular to the plane. We assume that

$$\mathrm{rot}\,\boldsymbol{a}_N(z)=B+\sum_{j=1}^N 2\pi\alpha_j\delta(z-z_j) \tag{1.2}$$

in $\mathcal{D}'(\boldsymbol{R}^2)$ (the Schwartz distribution space), where B, α_j are constants satisfying $B>0$ and

$$0<\alpha_j<1 \text{ for every } j=1,\ldots,N. \tag{1.3}$$

The constant B represents the intensity of a uniform magnetic field. The constant $2\pi\alpha_j$ represents the magnetic flux of an infinitesimally thin solenoid placed at z_j. We can show that the difference of integer magnetic fluxes can be gauged out by a suitable unitary gauge transform. Since we consider only the spectral problem, the assumption (1.3) loses no generality. We find a proof of the existence of the vector potential with δ-like singularities in Arai's paper (see [Ar1] and [Ar2]). When $N = 1$ and $\alpha_1 = \alpha$, we always assume that $z_1 = 0$ and take the circular gauge, that is,

$$\boldsymbol{a}_1(z) = \left(-\frac{B}{2}y - \frac{\alpha}{|z|^2}y, \frac{B}{2}x + \frac{\alpha}{|z|^2}x\right). \tag{1.4}$$

When we need to indicate the value α explicitly, we denote $\mathcal{L}_1^\alpha$ for $\mathcal{L}_1$ (this notation is used for the operator L_1 defined below).

Define a linear operator L_N on $L^2(\boldsymbol{R}^2)$ by

$$L_N u = \mathcal{L}_N u, \ u \in D(L_N) = C_0^\infty(\boldsymbol{R}^2 \setminus S_N),$$

where $D(L)$ is the operator domain of a linear operator L and $C_0^\infty(U)$ is the space of compactly supported smooth functions in an open set U. The operator L_N is symmetric, positive and has the deficiency indices $(2N, 2N)$ (see (i) of Lemma 3.3 below). Thus the operator L_N has self-adjoint extensions parameterized by $(2N \times 2N)$-unitary matrices (see [Re-Si, Theorem X.2]). We denote one of self-adjoint extensions of L_N by H_N. In particular, we denote the Friedrichs extension of L_N (the self-adjoint operator associated with the form closure of $D(L_N)$, see [Re-Si, Theorem X.23]) by H_N^{AB}, which is called the standard Aharonov-Bohm Hamiltonian (this name is used in [Ex-St-Vy], when $N = 1$).

The Schrödinger operator with constant magnetic field is given by

$$\mathcal{L}_0 = \left(\frac{1}{i}\nabla + \boldsymbol{a}_0\right)^2, \ \boldsymbol{a}_0 = \left(-\frac{B}{2}y, \frac{B}{2}x\right). \tag{1.5}$$

It is well-known that the linear operator defined by

$$L_0 u = \mathcal{L}_0 u, \ D(L_0) = C_0^\infty(\boldsymbol{R}^2) \tag{1.6}$$

is essentially self-adjoint and the spectrum of the unique self-adjoint extension H_0 of L_0 satisfies

$$\sigma(H_0) = \{(2n-1)B; n = 1, 2, \ldots\}.$$

The set $\sigma(H_0)$ is called the Landau levels.

When solenoids exist, the spectrum in a gap of the Landau levels appears. Our aim is to give an estimate for the number of eigenvalues between two Landau levels or below the lowest Landau level.

We recall known results in the case $N = 1$. Nambu ([Nam]) treats the standard Aharonov-Bohm Hamiltonian H_1^{AB} and gives an explicit representation of all eigenvalues and eigenfunctions using complex integration (he treats also the case $B = 0$). Exner, Šťovíček and Vytřas ([Ex-St-Vy]) give a detailed analysis for every self-adjoint extension H_1. We summarize a part of their results as follows.

Theorem 1.1 (Nambu, Exner-Šťovíček-Vytřas) *(i) The spectrum of the standard Aharonov-Bohm Hamiltonian H_1^{AB} is given by*

$$\sigma(H_1^{AB}) = \{(2n-1)B; n = 1, 2, \ldots\} \cup \{(2n+2\alpha-1)B; n = 1, 2, \ldots\}.$$

The multiplicity of each eigenvalue is given by

$$\begin{aligned} \mathrm{mult}((2n-1)B; H_1^{AB}) &= \infty, \ n = 1, 2, \ldots, \\ \mathrm{mult}((2n+2\alpha-1)B; H_1^{AB}) &= n, \ n = 1, 2, \ldots, \end{aligned}$$

where $\mathrm{mult}(\lambda; H)$ *is the multiplicity of an eigenvalue λ of a self-adjoint operator H.*

(ii) $L^2(\boldsymbol{R}^2)$ is decomposed into the direct sum of two closed subspaces $\mathcal{H}_s$ and $\mathcal{H}_c$, called the stable subspace and critical subspace, respectively. The spaces $\mathcal{H}_s$ and $\mathcal{H}_c$ are invariant subspaces for any self-adjoint extension H_1 of L_1. The restricted operator $H_1|_{\mathcal{H}_s}$ is independent of the choice of H_1 and the spectrum of $H_1|_{\mathcal{H}_s}$ is given by

$$\sigma(H_1|_{\mathcal{H}_s}) = \{(2n-1)B; n = 1, 2, \ldots\} \cup \{(2n+2\alpha-1)B; n = 2, 3, \ldots\}.$$

The multiplicity of each eigenvalue is given by

$$\begin{aligned} \mathrm{mult}((2n-1)B; H_1|_{\mathcal{H}_s}) &= \infty, \ n = 1, 2, \ldots, \\ \mathrm{mult}((2n+2\alpha-1)B; H_1|_{\mathcal{H}_s}) &= n-1, \ n = 2, 3, \ldots. \end{aligned}$$

The restricted operator $H_1|_{\mathcal{H}_c}$ depends on the choice of self-adjoint extension H_1. However, the following estimates hold independently of the choice of H_1.

$$\begin{aligned} \dim \mathrm{Ran}\, P_{(-\infty,(2\alpha-1)B)}(H_1|_{\mathcal{H}_c}) &\leq 2, \\ \dim \mathrm{Ran}\, P_{((2n+2\alpha-1)B,(2n+1)B)}(H_1|_{\mathcal{H}_c}) &\leq 2, \ n = 0, 1, 2, \ldots \\ \dim \mathrm{Ran}\, P_{((2n-1)B,(2n+2\alpha-1)B)}(H_1|_{\mathcal{H}_c}) &\leq 2, \ n = 1, 2, \ldots \end{aligned}$$

where $P_I(H)$ denotes the spectral projection of a self-adjoint operator H corresponding to an interval I. The left-hand side of each of three inequalities above takes the values $0, 1, 2$ if we take an appropriate self-adjoint extension H_1.

From Theorem 1.1, it follows that

$$n - 1 \leq \dim \operatorname{Ran} P_{((2n-1)B,(2n+1)B)}(H_1) \leq n + 3 \tag{1.7}$$

for $n = 1, 2, \ldots$, if $(2n+2\alpha-1)B$ is not an eigenvalue of $H_1|_{\mathcal{H}_c}$ (this condition holds for generic self-adjoint extension H_1). Later we show that the upper bound can be sharpened (see (1.11) below).

According to (i) of Theorem 1.1, there are n eigenstates of the Hamiltonian H_1^{AB} with the energy between n th Landau level and the $(n+1)$ st Landau level. We shall try to give a physical interpretation of this phenomenon.

In classical mechanics, an electron in a uniform magnetic field moves along a circle (cyclotron motion). The energy of an electron is quantized by the condition that the phase variation of the electron wave in one cyclotron rotation is 2π times an integer. Thus the energy of an electron takes one of the values in the Landau levels.

If some solenoids are contained in the circle of the cyclotron motion, then the phase of the electron wave is shifted by $e/\hbar$ times the magnetic flux of solenoids in the circle (the Aharonov-Bohm phase shift). Thus the energy of the electron is obliged to change, to correct the phase shift caused by the magnetic flux of solenoids. Hence the spectrum between Landau levels appears.

For this reason, the number of eigenstates with an energy between n th and $(n+1)$ st Landau level is roughly estimated by the possible number of electrons with the n th Landau level energy, in the circle of the Larmor radius centered at the position of solenoid. This number is calculated as follows. If we normalize physical constants as the mass $m = 1/2$, the Planck constant (divided by 2π) $\hbar = 1$ and the charge of an electron $e = 1$, then the cyclotron radius r of an electron with n th Landau level energy $(2n-1)B$ equals to $\sqrt{(2n-1)/B}$. It is known that the density of states (the number of eigenstates per unit area) for each Landau level is $B/2\pi$ (see [Nak, Proposition 15]). Thus, the number of possible eigenstates in the circle is

$$\pi r^2 \times \frac{B}{2\pi} = n - \frac{1}{2}.$$

The difference between this estimate and the rigorous result ((i) of Theorem 1.1) is only 1/2.

When $N \geq 2$, Nambu ([Nam]) gives a representation of eigenfunctions for the Landau levels by the multiple integral in the complex plane. But no information about the eigenvalues between the Landau levels are known. However, the physical explanation above gives us a conjecture about the number of eigenvalues in a gap of Landau levels, when $N \geq 2$. This number is roughly estimated by the number of eigenstates with the n th Landau energy in the union set, with respect to $j = 1, \ldots, N$, of the disks of Larmor radius centered at z_j. Each disk contains n eigenstates with n th Landau energy. These disks may intersect in general, but they are disjoint if solenoids are far from each other. Thus we reach the following conjecture.

Conjecture *(I) The number of eigenvalues between n th and $(n+1)$ st Landau levels is bounded by nN.*

(II) If solenoids are far from each other compared with the cyclotron radius, the number of eigenvalues between n th and $(n+1)$ st Landau levels equals to nN.

Our aim is to give an answer to these conjectures. Our answer to the conjecture (I) is the following.

Theorem 1.2 *Let $1 \leq N < \infty$. Then, the following holds.*

(i) For any self-adjoint extension H_N of L_N, we have that $(2n-1)B$ is an infinitely degenerated eigenvalue for every $n = 1, 2, 3, \ldots$.

(ii) For the standard Aharonov-Bohm Hamiltonian H_N^{AB}, we have

$$\begin{aligned} \dim \operatorname{Ran} P_{(-\infty,B)}(H_N^{AB}) &= 0, \\ \dim \operatorname{Ran} P_{((2n-1)B,(2n+1)B)}(H_N^{AB}) &\leq nN, \text{ for } n = 1, 2, 3, \ldots. \end{aligned} \tag{1.8}$$

(iii) For any self-adjoint extension H_N of L_N, we have

$$\dim \operatorname{Ran} P_{(-\infty,B)}(H_N) \leq 2N, \tag{1.9}$$

$$\dim \operatorname{Ran} P_{((2n-1)B,(2n+1)B)}(H_N) \leq (n+1)N, \text{ for } n = 1, 2, 3, \ldots \tag{1.10}$$

In the case $N = 1$, our result and (1.7) imply that

$$n - 1 \leq \dim \operatorname{Ran} P_{((2n-1)B,(2n+1)B)}(H_1) \leq n + 1 \tag{1.11}$$

for $n = 1, 2, \ldots$. The upper bound of (1.11) is sharper than that of (1.7) (however, [Ex-St-Vy, Fig 1,2] seems to indicate that there are at most two eigenvalues of $H_1|_{\mathcal{H}_c}$ in each gap of Landau levels).

Next, we shall exhibit our answer to the conjecture (II). We shall consider the special case where the physical situations around every z_j are the same. To represent this situation rigorously, we shall prepare an operator which intertwines two magnetic Schrödinger operators.

Definition 1.1 *Let $w \in \boldsymbol{R}^2$. Let U be a simply connected open set, and $V = U + w = \{z + w; z \in U\}$. Let S be an at most countable subset of U with no accumulation points in U and $T = S + w$. Let $\boldsymbol{a} \in C^\infty(U \setminus S; \boldsymbol{R}^2) \cap L^1_{loc}(U; \boldsymbol{R}^2)$ and $\boldsymbol{b} \in C^\infty(V \setminus T; \boldsymbol{R}^2) \cap L^1_{loc}(V; \boldsymbol{R}^2)$ be two vector potentials satisfying*

$$\operatorname{rot} \boldsymbol{a}(z) = \operatorname{rot} \boldsymbol{b}(z + w)$$

in $\mathcal{D}'(U)$. Then, there exists an operator t_{-w} from $\mathcal{D}'(V \setminus T)$ to $\mathcal{D}'(U \setminus S)$ satisfying the following (i) and (ii):

(i) There exists a complex-valued smooth function $\Phi(z) \in C^\infty(U \setminus S)$ with $|\Phi(z)| = 1$ for every $z \in U \setminus S$, such that

$$t_{-w} v(z) = \Phi(z) v(z + w), \quad v \in \mathcal{D}'(V \setminus T).$$

(ii) The following distributional equality holds:

$$\mathbf{p}(\boldsymbol{a}) t_{-w} v = t_{-w} \mathbf{p}(\boldsymbol{b}) v, \quad \mathcal{L}(\boldsymbol{a}) t_{-w} v = t_{-w} \mathcal{L}(\boldsymbol{b}) v \tag{1.12}$$

for $v \in \mathcal{D}'(V \setminus T)$, where

$$\mathbf{p}(\boldsymbol{a}) = \frac{1}{i}\nabla + \boldsymbol{a}, \qquad \mathbf{p}(\boldsymbol{b}) = \frac{1}{i}\nabla + \boldsymbol{b},$$
$$\mathcal{L}(\boldsymbol{a}) = \mathbf{p}(\boldsymbol{a})^2, \qquad \mathcal{L}(\boldsymbol{b}) = \mathbf{p}(\boldsymbol{a})^2.$$

We call the operator t_{-w} the magnetic translation operator from V to U intertwining $\mathcal{L}(\boldsymbol{b})$ with $\mathcal{L}(\boldsymbol{a})$. We denote the inverse operator of t_{-w} by t_w, that is,

$$t_w u(z) = \Phi(z - w) u(z - w)$$

for $u \in \mathcal{D}'(U \setminus S)$.

We call the equality (1.12) the intertwining property of t_{-w}. The existence of the function Φ can be proved by a little modified form of the Poincaré lemma.

Definition 1.2 *Let H_N be a self-adjoint extension of L_N. We say the operator H_N has the same boundary condition at every z_j, if the following two conditions hold:*

(i) There exists a constant α with $0 < \alpha < 1$ such that $\alpha_j = \alpha$ for every $j = 1, \ldots, N$.

(ii) Let t_{-z_j} be the magnetic translation operator from $\{|z - z_j| < \frac{R}{2}\}$ to $\{|z| < \frac{R}{2}\}$ intertwining $\mathcal{L}_N$ with $\mathcal{L}_1^{\alpha_j}$. Let $\chi \in C_0^\infty(\boldsymbol{R}^2)$ be a function satisfying $0 \le \chi \le 1$ on $\boldsymbol{R}^2$, $\chi = 0$ in $|z| > \frac{R}{2}$ and $\chi = 1$ in $|z| < \frac{R}{3}$. Put $\chi_j(z) = \chi(z - z_j)$. There exists a self-adjoint extension H_1 of L_1 independent of j such that

$$D(H_N) = \left\{u \in D(L_N^*); t_{-z_j}(\chi_j u) \in D(H_1) \text{ for every } j = 1, \ldots, N\right\}. \quad (1.13)$$

Here, L_N^* is the adjoint operator of L_N.

Remark 1. The right hand side of (1.13) is independent of the choice of the function χ; the condition $t_{-z_j}(\chi_j u) \in D(H_1)$ rules only the asymptotic behavior at z_j of the function u.

Remark 2. There exists a self-adjoint extension H_N of L_N satisfying (1.13) for any given self-adjoint extension H_1 of L_1.

Our (partial) answer to the conjecture (II) is the following.

Theorem 1.3 *Let $1 \le N < \infty$ or $N = \infty$. Let H_N be a self-adjoint extension of L_N which has the same boundary condition at every z_j. Let $I = [c, d]$ be a closed interval satisfying that $I \cap \{(2n-1)B; n = 1, 2, \ldots\} = \emptyset$, that $c, d \notin \sigma(H_1)$ and that $\sigma(H_1) \cap I = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \neq \emptyset$.*

Then, there exist constants $u > 0$ and $R_0 > 0$ dependent on B, α, I, H_1 (independent of N, R) satisfying the following:

(i) If $R \ge R_0$, we have

$$\sigma(H_N) \cap I \subset \bigcup_{l=1}^{k} [\lambda_l - \delta, \lambda_l + \delta],$$

where $\delta = e^{-uR^2}$.

(ii) If $R \ge R_0$, we have

$$\dim \operatorname{Ran} P_I(H_N) = N \dim \operatorname{Ran} P_I(H_1).$$

Note that $\sigma(H_1) \cap I$ is a finite set by (iii) of Theorem 1.2.

Combining (ii) of Theorem 1.3 with Theorem 1.1, we have the following corollary.

Corollary 1.4 *Let $1 \leq N < \infty$ and let $\alpha_1 = \alpha_2 = \cdots = \alpha_N = \alpha$. Then, for every $n_0 = 1, 2, \ldots$, there exists a constant $R_0 > 0$ dependent on B, α, n_0 (independent of N, R) satisfying the following: If $R \geq R_0$, then there exist self-adjoint extensions $H_N^0, H_N^1, \ldots, H_N^{n_0}$ of L_N such that*

$$\begin{aligned}
\dim \operatorname{Ran} P_{((2n-1)B,(2n+1)B)}(H_N^{AB}) &= nN, \\
\dim \operatorname{Ran} P_{(-\infty,B)}(H_N^0) &= 2N, \\
\dim \operatorname{Ran} P_{((2n-1)B,(2n+1)B)}(H_N^n) &= (n+1)N
\end{aligned}$$

for $n = 1, 2, \ldots, n_0$.

We make some remarks about the proofs of our results.

In the proof of Theorem 1.2, the canonical commutation relation (CCR) of the annihilation operator A_N and the creation operator $A_N^\dagger$ plays a crucial role (the definitions of the operators A_N and $A_N^\dagger$ are given in section 2 below). It is well-known that the spectrum of the Schrödinger operators with constant magnetic fields are completely determined by CCR. In our case, CCR holds with a perturbation by δ-like magnetic fields. This perturbation makes two self-adjoint operators $(A_N^\dagger)^* \overline{A_N^\dagger} - B$ and $\overline{A_N^\dagger}(A_N^\dagger)^* + B$ different (the overline denotes the operator closure; notice that the note * denotes the operator adjoint, while the note † denotes only the formal adjoint). Comparing the spectrum of these two operators, we can reach the conclusion of Theorem 1.2. Note that Iwatsuka ([Iw]) uses the argument of this type, to determine the essential spectrum of the Schrödinger operators on $\boldsymbol{R}^2$ with the magnetic fields converging to a non-zero constant at infinity.

Theorem 1.3 is an analogy of the result of Cornean and Nenciu ([Co-Ne, Theorem III.1, Corollary III.1]). They treat the case

$$\begin{aligned}
\operatorname{rot} \boldsymbol{a}_N(z) &= B + \sum_{j=1}^{N} \operatorname{rot} \boldsymbol{a}_0(z - z_j), \ \boldsymbol{a}_0 \in C_0^\infty(\{|z| < 1\}; \boldsymbol{R}^2), \\
V_N(z) &= \sum_{j=1}^{N} V_0(z - z_j), \ V_0 \in C_0^\infty(\{|z| < 1\}; \boldsymbol{R}),
\end{aligned}$$

and obtain the same conclusion as that of Theorem 1.3, for the operator $(\frac{1}{i}\nabla - \boldsymbol{a}_N)^2 + V_N$. The proof of Theorem 1.3 is similar to that of their result. The main difference is that our operators H_N ($N < \infty$) and H_∞ do not have the same core in general, while $C_0^\infty(\boldsymbol{R}^2)$ is the common core for the Schrödinger operators with smooth vector potentials (see [Ik-Ka, Theorem

1] or [Le-Si, Theorem 2]). Thus we do not use the approximating argument $N \to \infty$, which is used in their paper. We prove the statement of Theorem 1.3 directly even when $N = \infty$, using the argument of approximating eigenfunctions.

In the sequel, we shall exhibit the outline of the proof of Theorem 1.2, which contains our main new ideas. For the proof of Theorem 1.3, see our preprint ([Mi]).

2 Outline of the Proof of Theorem 1.2

Define differential operators $\mathcal{A}_N$, $\mathcal{A}_N^\dagger$ by

$$\mathcal{A}_N = i\Pi_{N,x} + \Pi_{N,y},\ \mathcal{A}_N^\dagger = -i\Pi_{N,x} + \Pi_{N,y},$$

where $\Pi_{N,x} = \frac{1}{i}\partial_x + a_{N,x}$ and $\Pi_{N,y} = \frac{1}{i}\partial_y + a_{N,y}$. When $N = 1$ and $\alpha_1 = \alpha$, we can describe the operators $\mathcal{A}_1$, $\mathcal{A}_1^\dagger$ explicitly as

$$\mathcal{A}_1 = \mathcal{A}_1^\alpha = 2\partial_z + \frac{B}{2}\bar{z} + \frac{\alpha}{z}, \tag{2.1}$$

$$\mathcal{A}_1^\dagger = \mathcal{A}_1^{\dagger,\alpha} = -2\partial_{\bar{z}} + \frac{B}{2}z + \frac{\alpha}{\bar{z}}, \tag{2.2}$$

where

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y),\ \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

In the above, we identify an element $z = (x, y)$ in $\boldsymbol{R}^2$ with the element $z = x + iy$ in $\boldsymbol{C}$. A formal computation shows that

$$\begin{aligned} \mathcal{A}_N\mathcal{A}_N^\dagger + \mathcal{A}_N^\dagger\mathcal{A}_N &= 2\mathcal{L}_N, \\ \mathcal{A}_N\mathcal{A}_N^\dagger - \mathcal{A}_N^\dagger\mathcal{A}_N &= 2\left(B + \sum_{j=1}^{N} 2\pi\alpha_j\delta(z - z_j)\right). \end{aligned}$$

Define linear operators A_N, $A_N^\dagger$ on $L^2(\boldsymbol{R}^2)$ by

$$\begin{aligned} A_N u &= \mathcal{A}_N u, \quad D(A_N) = C_0^\infty(\boldsymbol{R}^2 \setminus S_N), \\ A_N^\dagger u &= \mathcal{A}_N^\dagger u, \quad D(A_N^\dagger) = C_0^\infty(\boldsymbol{R}^2 \setminus S_N). \end{aligned}$$

Then, the following holds in the operator sense:

$$\overline{A_N^\dagger}(A_N^\dagger)^* \supset A_N^\dagger A_N = L_N - B, \tag{2.3}$$

$$(A_N^\dagger)^*\overline{A_N^\dagger} \supset A_N A_N^\dagger = L_N + B, \tag{2.4}$$

where the overline denotes the operator closure.

It is known that the following lemma holds.

Lemma 2.1 *Let X be a densely defined closed operator on a Hilbert space $\mathcal{H}$. Then, the following holds.*

(i) The operators X^*X and XX^* are self-adjoint.

(ii) The operator $(XX^*)|_{(\mathrm{Ker}\, XX^*)^\perp}$ is unitarily equivalent to the operator $(X^*X)|_{(\mathrm{Ker}\, X^*X)^\perp}$.

Proof. (i) See [Re-Si, Theorem X.25].

(ii) See [De, Theorem 3]. □

By (2.3), (2.4) and (i) of Lemma 2.1, we have that there exist self-adjoint extension H_N^-, H_N^0 such that

$$\overline{A_N^\dagger}(A_N^\dagger)^* = H_N^- - B, \ (A_N^\dagger)^*\overline{A_N^\dagger} = H_N^0 + B.$$

In section 3, we shall prove the following lemma.

Lemma 2.2 *The following assertions hold.*

(i) $H_N^0 = H_N^{AB}$.

(ii) $H_N^{AB} \geq B$ *in the form sense.*

(iii) $\dim D(H_N^-)/(D(H_N^{AB}) \cap D(H_N^-)) = N$.

As a result, we have the following.

Lemma 2.3 *The following holds.*

(i) *The operator $H_N^-|_{(\mathrm{Ker}(H_N^- - B))^\perp}$ is unitarily equivalent to the operator $H_N^{AB} + 2B$.*

(ii) *For any $n = 0, 1, 2, \ldots$, we have*

$$\dim \mathrm{Ran}\, P_{((2n-1)B,(2n+1)B)}(H_N^{AB}) = \dim \mathrm{Ran}\, P_{((2n+1)B,(2n+3)B)}(H_N^-).$$

Proof. (i) By (ii) of Lemma 2.1 and (i) of Lemma 2.2, we have that the operator $(H_N^- - B)|_{(\mathrm{Ker}(H_N^- - B))^\perp}$ and the operator $(H_N^{AB} + B)|_{(\mathrm{Ker}(H_N^{AB}+B))^\perp}$ are unitarily equivalent. Moreover, (ii) of Lemma 2.2 implies that $\mathrm{Ker}(H_N^{AB} + B) = \{0\}$. Thus the assertion holds.

(ii) By (i), we have that the spectral projection operators $P_I(H_N^{AB})$ and $P_{I+2B}(H_N^-)$ are unitarily equivalent for any interval I in $\boldsymbol{R}$ which does not contain B. Putting $I = ((2n-1)B, (2n+1)B)$ and taking the trace of the operators $P_I(H_N^{AB})$ and $P_{I+2B}(H_N^-)$, we obtain the assertion. □

The following lemma enables us to compare the spectrum of two self-adjoint extensions.

Lemma 2.4 *Let L be a symmetric operators on a Hilbert space $\mathcal{H}$. Suppose that the deficiency indices of L are (n,n) and n is finite. Let X and Y be two self-adjoint extensions of L. Then, the following holds.*

(i) We have $\sigma_{ess}(X) = \sigma_{ess}(Y)$.

(ii) For any open interval I in $\boldsymbol{R}$ satisfying $\dim \operatorname{Ran} P_I(X) < \infty$, we have $\dim \operatorname{Ran} P_I(Y) < \infty$ and

$$|\dim \operatorname{Ran} P_I(X) - \dim \operatorname{Ran} P_I(Y)| \leq d,$$

where

$$d = \dim D(X)/\left(D(X) \cap D(Y)\right) = \dim D(Y)/\left(D(X) \cap D(Y)\right).$$

Proof. (i) See [We, Theorem 8.17].

(ii) This assertion is an immediate corollary of [We, Exercise 8.8]. □

Proof of Theorem 1.2. First we prove

$$\sigma_{\mathrm{ess}}(H_N) = \{(2n-1)B \ ; \ n = 1, 2, \ldots\} \tag{2.5}$$

for any self-adjoint extension H_N of L_N, by an argument similar to the argument used in the paper of Iwatsuka ([Iw]). Since the deficiency indices of L_N are $(2N, 2N)$ and N is finite, we see by (i) of Lemma 2.4 that the set $S = \sigma_{\mathrm{ess}}(H_N)$ is independent of the choice of the self-adjoint extension H_N. This fact and (i) of Lemma 2.3 imply that

$$S \setminus \{B\} = S + 2B. \tag{2.6}$$

Moreover, we can show that S contains a real number B, by constructing a Weyl sequence for the value B. In particular, S is not empty. We can easily prove that a non-empty set satisfying (2.6) coincides with the right hand side of (2.5).

Put

$$\begin{aligned} a_0 &= \dim \operatorname{Ran} P_{(-\infty,B)}(H_N^{AB}), \\ b_0 &= \dim \operatorname{Ran} P_{(-\infty,B)}(H_N^{-}) \end{aligned}$$

and

$$\begin{aligned} a_n &= \dim \operatorname{Ran} P_{((2n-1)B,(2n+1)B)}(H_N^{AB}), \\ b_n &= \dim \operatorname{Ran} P_{((2n-1)B,(2n+1)B)}(H_N^{-}) \end{aligned}$$

for $n = 1, 2, \ldots$. By (ii) of Lemma 2.2 and (i) of Lemma 2.3, we have

$$a_0 = b_0 = 0. \tag{2.7}$$

By (ii) of lemma 2.3, we have

$$a_{n-1} = b_n \tag{2.8}$$

for any $n = 1, 2, \ldots$. By (iii) of Lemma 2.2 and (ii) of Lemma 2.4, we have

$$a_n \leq b_n + N \tag{2.9}$$

for any $n = 1, 2, \ldots$. By (2.7), (2.8), (2.9) and an inductive argument, we have

$$\begin{aligned} a_n \leq nN\ , &\qquad n = 0, 1, 2, \ldots, \\ b_0 = b_1 = 0,\ b_n \leq (n-1)N\ , &\qquad n = 2, 3, 4, \ldots. \end{aligned} \tag{2.10}$$

Thus (ii) of Theorem 1.2 holds.

Since the deficiency indices of L_N are $(2N, 2N)$, we have

$$\dim D(H_N)/(D(H_N) \cap D(H_N^{-})) \leq 2N$$

for any self-adjoint extension H_N of L_N. By (ii) of Lemma 2.4, we have

$$\dim \operatorname{Ran} P_I(H_N) \leq \dim \operatorname{Ran} P_I(H_N^{-}) + 2N$$

for any open interval I which does not intersect with the set $\{(2n-1)B\ ;\ n = 1, 2, \ldots\}$. Applying this inequality with $I = (-\infty, B)$ or $I = ((2n-1)B, (2n+1)B)$, we have that (iii) of Theorem 1.2 holds.

The equality (2.5) and (iii) of Theorem 1.2 imply the assertion (i) of Theorem 1.2. □

3 Operator domain of the self-adjoint extensions

Define four functions $\phi_{-1}^{\alpha}, \psi_1^{\alpha}, \phi_0^{\alpha}, \psi_0^{\alpha}$ by

$$\phi_{-1}^{\alpha}(z) = |z|^{\alpha} z^{-1} e^{-\frac{B}{4}|z|^2}, \qquad \psi_1^{\alpha}(z) = |z|^{-\alpha} \bar{z} e^{-\frac{B}{4}|z|^2},$$
$$\phi_0^{\alpha}(z) = |z|^{\alpha} e^{-\frac{B}{4}|z|^2}, \qquad \psi_0^{\alpha}(z) = |z|^{-\alpha} e^{-\frac{B}{4}|z|^2}.$$

In the definition above, we identify an element $z = (x, y)$ in $\boldsymbol{R}^2$ with the element $z = x + iy$ in $\boldsymbol{C}$. The functions above have the following asymptotics as $z \to 0$:

$$\phi_{-1}^{\alpha}(z) \sim r^{\alpha-1} e^{-i\theta}, \qquad \psi_1^{\alpha}(z) \sim r^{1-\alpha} e^{-i\theta},$$
$$\phi_0^{\alpha}(z) \sim r^{\alpha}, \qquad \psi_0^{\alpha}(z) \sim r^{-\alpha},$$

where (r, θ) is the polar coordinate given by $z = re^{i\theta}$, $r \geq 0$ and $\theta \in \boldsymbol{R}$. The result of Exner, Št'ovícek and Vytřas ([Ex-St-Vy]) implies that

$$D((L_1^{\alpha})^*) = D(\overline{L_1^{\alpha}}) \oplus \text{L.h.}\{\phi_{-1}^{\alpha}, \psi_1^{\alpha}, \phi_0^{\alpha}, \psi_0^{\alpha}\}. \tag{3.1}$$

We shall determine the operator domain $D(L_N^*)$ when $N \geq 2$. The following lemma gives fundamental properties of $D(L_N^*)$ and $D(\overline{L_N})$.

Lemma 3.1 *Let $N = 1, 2, \ldots,$ or $N = \infty$. Then, the following holds.*
(i) The operator domain of $D(L_N^*)$ is given by

$$D(L_N^*) = \{u \in L^2(\boldsymbol{R}^2) \cap H^2_{loc}(\boldsymbol{R}^2 \setminus S_N);\ \mathcal{L}_N u \in L^2(\boldsymbol{R}^2)\}.$$

(ii) Let $u \in D(L_N^*)$. Suppose that there exists a constant R_1 with $0 < R_1 < R$ such that $\operatorname{supp} u \subset \boldsymbol{R}^2 \setminus U(R_1)$, *where*

$$U(r) = \bigcup_{j=1}^{N} \{z \in \boldsymbol{R}^2; |z - z_j| < r\}.$$

Then, $u \in D(\overline{L_N})$.

Proof. (i) This assertion follows from the definition of the adjoint operator.

(ii) Take a function $u \in D(L_N^*)$ which satisfies the assumption. Then the function $\boldsymbol{a}_N$ is smooth on the support of u. Since the magnetic Schrödinger operators on $\boldsymbol{R}^2$ with smooth magnetic potentials are essentially self-adjoint on $C_0^{\infty}(\boldsymbol{R}^2)$ (see [Ik-Ka]), we can approximate u with respect to the graph norm of L_N^* by smooth functions supported on a neighborhood of $\operatorname{supp} u$. This implies that $u \in D(\overline{L_N})$. □

Lemma 3.2 *Let $1 \leq N \leq \infty$. Let χ be an element of $C_0^\infty(\{|z| < \frac{R}{2}\})$ satisfying $\chi(z) = 1$ in $\{|z| < \frac{R}{3}\}$. Let t_{-z_j} be the magnetic translation from $\{|z - z_j| < \frac{R}{2}\}$ to $\{|z| < \frac{R}{2}\}$ intertwining $\mathcal{L}_N$ with $\mathcal{L}_1^{\alpha_j}$. Put $\chi_j(z) = \chi(z - z_j)$. Let T be a linear operator, from the quotient Hilbert space $D(L_N^*)/D(\overline{L_N})$ to the direct sum of the quotient Hilbert spaces $\oplus_{j=1}^N D((L_1^{\alpha_j})^*)/D(\overline{L_1^{\alpha_j}})$, defined by*

$$T[u] = ([t_{-z_1}(\chi_1 u)], \dots, [t_{-z_N}(\chi_N u)]),$$

where the bracket denotes the equivalence class in a quotient space. Then, T is a bijective bicontinuous linear operator.

Proof. We show this lemma only in the case $N < \infty$, for simplicity. In this case, the vector space $\oplus_{j=1}^N D((L_1^{\alpha_j})^*)/D(\overline{L_1^{\alpha_j}})$ is finite dimensional. Thus the continuity statement automatically holds.

Define a linear operator $\widetilde{T}$ from $D(L_N^*)$ to $\oplus_{j=1}^N D((L_1^{\alpha_j})^*)/D(\overline{L_1^{\alpha_j}})$ by

$$\widetilde{T}u = ([t_{-z_1}(\chi_1 u)], \dots, [t_{-z_N}(\chi_N u)]).$$

The well-definedness of the operator $\widetilde{T}$ follows from the intertwining property

$$t_{-z_j}\mathcal{L}_N u = \mathcal{L}_1^{\alpha_j} t_{-z_j} u$$

and (i) of Lemma 3.1. We see that $\widetilde{T}$ is surjective by the equality

$$[u] = [t_{-z_j}\chi_j t_{z_j}\chi u], \quad \text{for } u \in D((L_1^{\alpha_j})^*),$$

which follows from (ii) of Lemma 3.1.

We shall show that $\operatorname{Ker}\widetilde{T} = D(\overline{L_N})$. The inclusion $\operatorname{Ker}\widetilde{T} \supset D(\overline{L_N})$ follows from the inclusion relation

$$t_{-z_j}\chi_j C_0^\infty(\mathbf{R}^2 \setminus S_N) \subset C_0^\infty(\mathbf{R}^2 \setminus \{0\}) = D(L_1^{\alpha_j})$$

and an approximating argument. We shall show the contrary inclusion. Take $u \in \operatorname{Ker}\widetilde{T}$. Then $t_{-z_j}\chi_j u \in D(\overline{L_1^{\alpha_j}})$ for $j = 1, \dots, N$. Decompose the function u as

$$u = (u - \sum_{j=1}^N \chi_j u) + \sum_{j=1}^N t_{z_j}(t_{-z_j}\chi_j u).$$

Since $u - \sum_{j=1}^N \chi_j u \in D(\overline{L_N})$ by (ii) of Lemma 3.1, it is sufficient to show that

$$t_{z_j} v \in D(\overline{L_N}) \quad \text{for } v \in D(\overline{L_1^{\alpha_j}}), \ \operatorname{supp} v \subset \{|z| < \frac{R}{2}\}.$$

This assertion follows from the inclusion

$$t_{z_j} C_0^\infty(\{0 < |z| < \frac{R}{2}\}) \subset C_0^\infty(\boldsymbol{R}^2 \setminus S_N)$$

and an approximating argument.

Thus the assertion of this lemma follows from the homomorphism theorem. □

Remark. When $N = \infty$, we need to prove the convergence of the sum $\sum_{j=1}^\infty \chi_j u$. For the detailed argument, see our preprint ([Mi]).

By the previous lemma, we can determine the structure of the operator domain of L_N^*.

Lemma 3.3 *Assume that all the assumption of Lemma 3.2 hold. Then, the following assertions hold.*

(i) *The deficiency indices of L_N are $(2N, 2N)$.*

(ii) *Assume moreover that there exist constants α_-, α_+ such that*

$$0 < \alpha_- \leq \alpha_j \leq \alpha_+ < 1 \tag{3.2}$$

for every $j = 1, \ldots, N$. Put

$$\phi_{-1}^{(j)} = t_{z_j}(\chi \phi_{-1}^{\alpha_j}), \qquad \psi_1^{(j)} = t_{z_j}(\chi \psi_1^{\alpha_j}),$$
$$\phi_0^{(j)} = t_{z_j}(\chi \phi_0^{\alpha_j}), \qquad \psi_0^{(j)} = t_{z_j}(\chi \psi_0^{\alpha_j}),$$

for $j = 1, \ldots, N$. Then, the operator domain $D(L_N^*)$ is decomposed into a direct sum

$$D(L_N^*) = D(\overline{L_N}) \oplus_{alg} \bigoplus_{j=1}^N \text{L.h.}\{\phi_{-1}^{(j)}, \psi_1^{(j)}, \phi_0^{(j)}, \psi_0^{(j)}\},$$

where $\oplus_{alg}$ denotes the algebraic direct sum and $\oplus_{j=1}^N$ denotes the orthogonal direct sum of mutually orthogonal closed subspaces.

Remark. The assumption of (ii) holds if N is finite.

Proof. (i) Since the operator L_N is symmetric and positive, the deficiency indices $m_\pm = \dim \text{Ker}(L_N^* \mp i)$ are equal (see [Re-Si, Corollary of Theorem X.1]). Since $D(L_N^*) = D(\overline{L_N}) \oplus \text{Ker}(L_N^* - i) \oplus \text{Ker}(L_N^* + i)$ (see [Re-Si, (b) of Lemma in page 138]), it is sufficient to show that

$$\dim D(L_N^*)/D(\overline{L_N}) = 4N.$$

This equality follows from from Lemma 3.2 and (3.1).

(ii) It is easy to see that the operator T^{-1} defined by

$$T^{-1}([u_1], \ldots, [u_N]) = \left[\sum_{j=1}^{N} t_{z_j} \chi u_j\right]$$

is the inverse operator of the operator T defined in Lemma 3.2. By (3.1), we see that the functions

$$\bigcup_{j=1}^{N} \{([0], \ldots, \underbrace{[\phi]}_{j-\mathrm{th}}, \ldots, [0]) \; ; \; \phi = \phi_{-1}^{\alpha_j}, \psi_1^{\alpha_j}, \phi_0^{\alpha_j}, \psi_0^{\alpha_j}\}$$

form a basis of $\oplus_{j=1}^{N} D((L_1^{\alpha_j})^*)/D(\overline{L_1^{\alpha_j}})$. When N is finite, we have that the image of the above basis by the operator T^{-1} is a basis of $D(L_N^*)/D(\overline{L_N})$. This implies the assertion.

When $N = \infty$, we have to show the convergence of the sum

$$u = u_0 + \sum_{j=1}^{\infty} (c_{4j-3}\phi_{-1}^{(j)} + c_{4j-2}\psi_1^{(j)} + c_{4j-1}\phi_0^{(j)} + c_{4j}\psi_0^{(j)}),$$

where $u_0 \in D(\overline{L_N})$ and the coefficients $c_{4j-3}, c_{4j-2}, c_{4j-1}, c_{4j}$ are determined by the asymptotic behavior as $z \to z_j$ of the function u. The assumption (3.2) guarantees the convergence of the sum (for the detail, see our preprint [Mi]). □

By using above basis, we can describe $D(H_N^{AB})$ and $D(H_N^-)$ as follows:

Lemma 3.4 *The following equalities hold.*
(i) $D(H_N^{AB}) = D(\overline{L_N}) \oplus_{alg} \oplus_{j=1}^{N} \mathrm{L.h.}\{\psi_1^{(j)}, \phi_0^{(j)}\}$.
(ii) $D(H_N^-) = D(\overline{L_N}) \oplus_{alg} \oplus_{j=1}^{N} \mathrm{L.h.}\{\psi_1^{(j)}, \psi_0^{(j)}\}$.

Proof. (i) Let D_1 be the right hand side of the equality (i). Since $D(H_N^{AB})$ is included in the form domain $\overline{C_0^\infty(\boldsymbol{R}^2 \setminus S_N)}$, we have that any element $u \in D(H_N^{AB})$ satisfies

$$\mathcal{A}_N u \in L^2(\boldsymbol{R}^2), \; \mathcal{A}_N^\dagger u \in L^2(\boldsymbol{R}^2). \tag{3.3}$$

By Lemma 3.3, we have that an element $u \in D(L_N^*)$ is written as the sum of a vector u_0 in $D(\overline{L_N})$ and a linear combination of $4N$ vectors

$\{\phi_{-1}^{(j)}, \psi_1^{(j)}, \phi_0^{(j)}, \psi_0^{(j)}\}_{j=1}^N$. An explicit calculation using (2.1) and (2.2) shows that

$$\begin{aligned} \mathcal{A}_1^\alpha \phi_0^\alpha &= 2\alpha|z|^\alpha z^{-1} e^{-\frac{B}{4}|z|^2} \in L^2(\boldsymbol{R}^2), \\ \mathcal{A}_1^{\dagger,\alpha} \phi_0^\alpha &= B|z|^\alpha z e^{-\frac{B}{4}|z|^2} \in L^2(\boldsymbol{R}^2), \\ \mathcal{A}_1^\alpha \psi_1^\alpha &= 0 \in L^2(\boldsymbol{R}^2), \\ \mathcal{A}_1^{\dagger,\alpha} \psi_1^\alpha &= |z|^{-\alpha}(2(\alpha-1)+B|z|^2) e^{-\frac{B}{4}|z|^2} \in L^2(\boldsymbol{R}^2), \end{aligned}$$

and that

$$\begin{aligned} \mathcal{A}_1^\alpha \phi_{-1}^\alpha &= 2(\alpha-1)|z|^\alpha z^{-2} e^{-\frac{B}{4}|z|^2} \notin L^2(\boldsymbol{R}^2), \\ \mathcal{A}_1^{\dagger,\alpha} \psi_0^\alpha &= |z|^{-\alpha}(2\alpha\bar{z}^{-1} + Bz) e^{-\frac{B}{4}|z|^2} \notin L^2(\boldsymbol{R}^2). \end{aligned}$$

By the intertwining property of t_{-z_j}, we have that the vectors $\mathcal{A}_N \phi_0^{(j)}$, $\mathcal{A}_N^\dagger \phi_0^{(j)}$, $\mathcal{A}_N \psi_1^{(j)}$, $\mathcal{A}_N^\dagger \psi_1^{(j)}$ belong to $L^2(\boldsymbol{R}^2)$ and that the vectors $\mathcal{A}_N \phi_{-1}^{(j)}$, $\mathcal{A}_N^\dagger \psi_0^{(j)}$ do not belong to $L^2(\boldsymbol{R}^2)$, for $j = 1, \ldots, N$. Thus, an element u in $D(L_N^*)$ satisfying (3.3) is contained in D_1. Therefore we have $D(H_N^{AB}) \subset D_1$. Moreover, we can prove that the operator $\mathcal{L}_N|_{D_1}$ is self-adjoint. Thus we have $D(H_N^{AB}) = D_1$.

(ii) By definition, an element u in $D(H_N^-) = D(\overline{A_N^\dagger}(A_N^\dagger)^*)$ satisfies

$$\mathcal{A}_N u \in D(\overline{A_N^\dagger}) = \overline{C_0^\infty(\boldsymbol{R}^2 \setminus S_N)}, \tag{3.4}$$

where the overline denotes the closure with respect to the graph norm of A_N. By the operator equality

$$A_N^\dagger A_N = A_N A_N^\dagger - 2B,$$

we have that the graph norm of A_N and that of $A_N^\dagger$ are equivlent. Thus we have $D(\overline{A_N}) = D(\overline{A_N^\dagger})$. By (3.4), we have

$$\mathcal{A}_N u \in L^2(\boldsymbol{R}^2),\ \mathcal{A}_N^\dagger \mathcal{A}_N u \in L^2(\boldsymbol{R}^2),\ \mathcal{A}_N \mathcal{A}_N u \in L^2(\boldsymbol{R}^2). \tag{3.5}$$

Again an explicit computation using (2.1) and (2.2) shows that

$$\begin{aligned} \mathcal{A}_1^\alpha \psi_0^\alpha &= \mathcal{A}_1^{\dagger,\alpha} \mathcal{A}_1^\alpha \psi_0^\alpha = \mathcal{A}_1^\alpha \mathcal{A}_1^\alpha \psi_0^\alpha = 0 \in L^2(\boldsymbol{R}^2), \\ \mathcal{A}_1^\alpha \psi_1^\alpha &= \mathcal{A}_1^{\dagger,\alpha} \mathcal{A}_1^\alpha \psi_1^\alpha = \mathcal{A}_1^\alpha \mathcal{A}_1^\alpha \psi_1^\alpha = 0 \in L^2(\boldsymbol{R}^2), \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_1^\alpha \phi_{-1}^\alpha &= 2(\alpha-1)|z|^\alpha z^{-2} e^{-\frac{B}{4}|z|^2} \notin L^2(\boldsymbol{R}^2), \\ \mathcal{A}_1^\alpha \mathcal{A}_1^\alpha \phi_0^\alpha &= 4\alpha(\alpha-1)|z|^\alpha z^{-2} e^{-\frac{B}{4}|z|^2} \notin L^2(\boldsymbol{R}^2). \end{aligned}$$

The rest of the proof is similar to the last part of the proof of (i). □

We shall give a proof of Lemma 2.2 in section 2.

Proof of Lemma 2.2. (i) By the definition of the self-adjoint operator $H_N^0 = (A_N^\dagger)^* \overline{A_N^\dagger} - B$ and the Friedrichs extension H_N^{AB}, we can show that H_N^0 and H_N^{AB} have the same form core $C_0^\infty(\boldsymbol{R}^2 \setminus S_N)$. Moreover, the values of the form $(H_N^0 u, u)$ and $(H_N^{AB} u, u)$ coincide for u in the form core $C_0^\infty(\boldsymbol{R}^2 \setminus S_N)$. These facts imply that two self-adjoint operators H_N^0 and H_N^{AB} coincide.

(ii) For u in the form core $C_0^\infty(\boldsymbol{R}^2 \setminus S_N)$, we have

$$\begin{aligned} (H_N^{AB} u, u) &= ((\mathcal{A}_N^\dagger \mathcal{A}_N + B)u, u) \\ &= ||\mathcal{A}_N u||^2 + B||u||^2 \geq B||u||^2. \end{aligned}$$

Thus the assertion holds.

(iii) This assertion immediately follows from Lemma 3.4. □

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