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WKB Method applied to Spectral Asymptotics

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0 Introduction

We consider Schrödinger operators $P=-h^2\Delta+V(x)$ on $\mathbf{R}^n$, where V is a real (bounded) potential and h is a small parameter, and study the semiclassical distribution (i.e. asymptotic distribution as h tends to 0) of resonances in a small complex neighborhood of a fixed real energy E_0.

The semiclassical distribution of resonances (and also eignevalues) near E_0 is closely related with the geometry of the corresponding classical mechanics with Hamiltonian $p(x,\xi)=\xi^2+V(x)$, more precisely, with the set of trapped orbits $K(E_0)$ of the Hamilton flow $\exp tH_p$ on $p^{-1}(E_0)$ where $H_p=\nabla_\xi p\cdot\nabla_x-\nabla_x p\cdot\nabla_\xi$ is the Hamilton vector field.

In certain cases where $K(E_0)$ has a simple geometrical structure, such as a non-degenerate stationary point or a periodic orbit, it is possible to specify the semiclassical distribution of resonances near E_0. For example in the periodic case, we can construct the resonant state as an exact WKB solution associated with the outgoing Lagrangian submanifold near the periodic orbit microlocally in the phase space. It can be continued along the orbit and the quantization condition of resonances is then obtained as the condition that this WKB solution is single-valued on this periodic orbit. This idea is based on the fact that the resonant state is, after a complex scaling, supported microlocally on the periodic orbit as h tends to 0.

In the case where $K(E_0)$ is a homoclinic orbit, which consists of a non-degenerate stationary point and an orbit which tends to this point as t tends to $\pm\infty$, this strategy can also be applied. However, because of the singularity at the stationary point of the classical orbits, we cannot continue the microlocal WKB solution through this point by solving the transport equation. In the one-dimensional case, this corresponds to the connection problem at a double turning point.

In this report, after reviewing some known results about the semiclassical distribution of resonances in simpler cases, we discuss the above connection problem in the multidimensional case, which is a collaboration in progress with J.-F.Bony, T.Ramond and M.Zerzeri [3] and partially with E.Amar [1], [2].

1 Fundamental Elements

In this section, we recall some fundamental elements of the semiclassical microlocal analysis.

1.1 Trapped orbits

Recall that the value of $p(x,\xi)$ is invariant under the Hamilton flow $\exp tH_p$. Let E be a real energy and $\Gamma_\pm(E)$ the outgoing and incoming tails:

$$\Gamma_\pm(E) = \{(x,\xi) \in p^{-1}(E_0); \exp tH_p(x,\xi) \not\to \infty \quad \text{as} \quad t \to \mp\infty\}.$$

The set

$$K(E) = \Gamma_+(E) \cap \Gamma_-(E).$$

is compact and is the union of completely trapped set.

1.2 Bargman transform

For $u \in L^2(\mathbb{R}^n)$, the Bargman transform (or global FBI transform) is defined by

$$Tu(x,\xi;h) = c_n \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi/h-(x-y)^2/2h} u(y;h)dy.$$

$Tu(x,\xi;h)$ belongs to $L^2(\mathbb{R}^{2n}_{x,\xi})$ and c_n is taken so that T be an isometry from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$. It is seen that by this transform, the function u is localized in x by a Gaussian up to $O(\sqrt{h})$. Moreover, it is localized also in ξ up to $O(\sqrt{h})$. Indeed we have an identity

$$Tu(x,\xi;h) = e^{ix\cdot\xi/h} T\hat{u}(\xi,-x;h),$$

where $\hat{u}$ is the semiclassical Fourier transform

$$\hat{u}(\xi) = (2\pi h)^{-n/2} \int_{\mathbb{R}^n} e^{-x\cdot\xi/h} u(x)dx.$$

1.3 Microsupport

A (h-dependent) function $u \in L^2$ is said to be *microlocally exponentially small* at a point (x_0, ξ_0) in the phase space iff there exists a neighborhood U of (x_0, ξ_0) and a positive number ϵ such that

$$Tu(x, \xi; h) = O(e^{-\epsilon/h})$$

as $h \to 0$ uniformly in U. The complement of such points is called *microsupport* of u and denoted by $MS(u)$.

Microsupport has the following properties: Let u be a solution of $Pu = Eu$ in a domain $\Omega \subset \mathbb{R}^n$, and assume that $||u||_{L^2(\Omega)} \leq 1$.

- The microsupport of u is included in the energy surface $p^{-1}(E)$.
- The microsupport of u propagates along a simple Hamilton flow.
- The microsupport of a WKB solution $u = e^{i\psi(x)/h} b(x, h)$, $b(x, h) = O(1)$ as h tends to 0, is included in the Lagrangian submanifold $\{(x, \xi); \xi = \partial_x \psi(x)\}$.

1.4 Resonance

Assume the *non-trapping* condition near the infinity: There exist a real function $G(x, \xi)$, a compact set $K \subset \mathbb{R}^{2n}$ and a constant $C > 0$ such that on $p^{-1}(E_0) \backslash K$ one has

$$H_p G \geq C \xi^2.$$

Such a function G is called an *escape function.*

On the I-Lagrangian manifold

$$\Lambda_{tG} = \{(z, \zeta) = (x + iy, \xi + i\eta) \in \mathbb{C}^{2n}; y = t\partial G/\partial \xi, \eta = -t\partial G/\partial x\},$$

the Hamiltonian $p(x, \xi)$ is developed in Taylor series with respect to the small parameter t on Λ_{tG}:

$$p(z, \zeta) = p(x, \xi) - itH_p G(x, \xi) + O(t^2).$$

Hence the non-trapping condition implies the ellipticity of p near the infinity on Λ_{tG} for small enough t. For E in a small but h-independent complex neighborhood of E_0, $P - E$ considered as operator on a Sobolev space on Λ_{tG} with an appropriate weight is bijective except for a discrete set. The elements of this discrete set are called *resonances* ([8]).

- If u is a resonant state, the microsupport of u is included in the outgoing tail Γ_+.

2 Known Results

We suppose in what follows $E - E_0 = O(h)$.

2.1 The case where $K(E_0) = \emptyset$

In this case, there is no resonance in an h-independent neighborhood of E_0. In fact we can construct a global escape function $G(x,\xi)$, and $P-E$ becomes globally elliptic on Λ_{tG}, that is, invertible.

2.2 The case where $K(E_0)$ is a non-degenerate stationary point

Let the stationary point be the origin $(0,0)$ of the phase space. After a canonical change of coordinates, $p(x,\xi)$ is written in the form

$$p(x,\xi) - E_0 = \xi^2 + \sum_{j=1}^{d} \frac{\lambda_j^2}{4} x_j^2 - \sum_{j=d+1}^{n} \frac{\lambda_j^2}{4} x_j^2 + O(|(x,\xi)|^3)$$

as $(x,\xi) \to (0,0)$. The eigenvalues of the fundamental matrix of p at $(0,0)$

$$\begin{pmatrix} \frac{\partial^2 p}{\partial x \partial \xi} & \frac{\partial^2 p}{\partial \xi^2} \\ -\frac{\partial^2 p}{\partial x^2} & -\frac{\partial^2 p}{\partial x \partial \xi} \end{pmatrix} |_{(0,0)} = \begin{pmatrix} 0 & 2I \\ -V''(0) & 0 \end{pmatrix}$$

are $\pm i\lambda_1, \dots, \pm i\lambda_d$ and $\pm\lambda_{d+1}, \dots, \pm\lambda_n$ $(\lambda_1, \dots, \lambda_n > 0)$. SJöstrand [15], Biret, Combes, Duclos [4] independently and Kaidi, Kerdelhué [12] in a complex neighborhood of E_0 of size $O(h^\delta)$, $\delta > 0$ arbitrary, showed that the resonances (or eventually eigenvalues if $d = n$) are found near the lattice points $\{E_\alpha\}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ where

$$E_\alpha = E_0 + \sum_{j=1}^{d} (\alpha_j + 1/2)\lambda_j h - i \sum_{j=d+1}^{n} (\alpha_j + 1/2)\lambda_j h,$$

which are the sum of eigenvalues of d harmonic oscillators $-\partial^2/\partial x_j^2 + \lambda_j^2 x_j^2/4$ $(j = 1, \dots, d)$ and resonances of $n-d$ anti-harmonic oscillators $-\partial^2/\partial x_j^2 - \lambda_j^2 x_j^2/4$ $(j = d+1, \dots, n)$.

2.3 The case where $K(E_0)$ is a periodic orbit

If $n = 1$, this is a simple-well eigenvalue problem and it is well known that the eigenvalues satisfy the so-called Bohr-Sommerfeld quantization condition:

$$\int_{\gamma(E)} \xi dx = (2k+1)\pi h + O(h^2), \quad k = 0, 1, 2, \ldots,$$

where $\gamma(E)$ is the unique periodic orbit on $p^{-1}(E)$. The integral on the left hand side is called action integral and its derivative with respect to E is the period $T(E)$ of the orbit. Therefore the eigenvalues near E_0 is a real sequence whose interval is asymptotically equal to $2\pi h/T(E_0)$.

In case $n > 1$, we can define the Poincaré map associated with the periodic orbit. Let us assume that it is of hyperbolic type, i.e. the eigenvalues of the linearized Poincaré map are $\theta_2, \ldots, \theta_n, \theta_2^{-1}, \ldots, \theta_n^{-1}$ and $|\theta_2|, \ldots, |\theta_n| > 1$. Then Gérard and Sjöstrand showed in [6] that the resonances are found near the lattice points whose real interval is equal to $2\pi h/T(E_0)$ and whose imaginary part is given by

$$-\frac{ih}{T(E_0)} \sum_{j=2}^{n} (\alpha_j + 1/2) \log |\theta_j|, \quad \alpha = (\alpha_2, \ldots, \alpha_n) \in \mathbb{N}^{n-1}.$$

2.4 The case where $K(E_0)$ is a homoclinic orbit ($n = 1$)

A homoclinic orbit consists of a stationary point, which we assume to be the origin $(0,0)$, and an orbit tending to this point as t tends to $+\infty$ and $-\infty$. For $E < E_0$, $|E - E_0|$ small, $K(E)$ consists of a periodic orbit. The period $T(E)$ should diverge as $E \to E_0-$. In fact

$$T(E) = \frac{1}{\lambda} \log \frac{1}{E_0 - E} + O(1) \quad \text{as} \quad E \to E_0-,$$

where $\pm\lambda$ ($\lambda > 0$) are the eigenvalues of the fundamental matrix of p at the stationary point.

In this case, the resonances near E_0 make a sequence parallel to the real axis with imaginary part of $O(h/\log(1/h))$ and the interval is asymptotically equal to $2\pi h/\{\lambda^{-1} \log(1/h)\}$ ([5]).

3 General homoclinic case

Here we study the case where $K(E_0) = \{(0,0)\} \cup \gamma$, where $(0,0)$ is a non-degenerate saddle point of the symbol $p(x,\xi)$ and γ is an orbit which tends to $(0,0)$ as $t \to \pm\infty$. Let $\pm\lambda_1, \ldots, \pm\lambda_n$, $(0 < \lambda = 1 \leq \lambda_2 \leq \cdots \leq \lambda_n)$ be the eigenvalues of dH_p at $(0,0)$. After a canonical change of coordinate, we can assume that

$$p(x,\xi) - E_0 = \xi^2 - \sum_{j=1}^{n} \frac{\lambda_j^2}{4} x_j^2 + O(|(x,\xi)|^3).$$

First we assume

(A1) $\lambda_1 < \lambda_2$.

Then the asymptotic behavior of the orbit $\gamma(t)$ as $t \to \infty$ is

$$\gamma(t) = c\, {}^t(1,0,\ldots,0,-\lambda_1/2,0,\ldots,0)e^{-\lambda_1 t} + O(e^{-\min(\lambda_2,2\lambda_1)t}).$$

We make a generic assumption:

(A2) $c \neq 0$.

This means that the natural projection of $\gamma(t)$ to the x-space tends to the saddle point tangentially to the x_1-axis as $t \to +\infty$.

Finally we make a global assumption. Let Λ_- and Λ_+ be the stable and unstable manifold associated with the saddle point $(0,0)$ on which all the Hamilton flows tend to $(0,0)$ as t tends to $+\infty$ and $-\infty$ respectively.

(A3) The extension of $\Lambda_\pm$ by the flow of H_p intersects transversally on γ.

Under these assumptions, very roughly speaking, the Schrödinger operator P has as model the sum of the one-dimentional operator with respect to x_1 variable associated with a unique homoclinic orbit and the $n-1$-dimensional operator with respect to $(x_2, \ldots, x_n)$ associated with a unique non-degenerate saddle point. Hence we expect from the previous results in §2.4 and §2.2 that the resonances are found near the lattice points whose real interval is equal to $2\pi h/\{\lambda^{-1}\log(1/h)\}$ and the imaginary part is given by

$$-i\sum_{j=2}^{n}(\alpha_j + 1/2)\lambda_j h, \quad \alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{N}^{n-1} \tag{1}$$

4 WKB Method and Monodromy Operator

The method consists in constructing a WKB solution microlocally near the trapped set $K(E_0)$.

The resonant state is, after a normalization, concentrated on the outgoing tail $\Gamma_+(E)$ (see §1.4). If Γ_+ is a Lagrangian manifold of the form $\{(x,\xi);\xi = \partial_x\psi(x)\}$, there corresponds locally a WKB solution

$$u(x,h) = b(x,h)e^{i\psi(x)/h},$$

where the symbol

$$b(x,h) \sim \sum_{l=0}^{\infty} b_l(x)h^l,$$

satisfies the transport equation

$$2\partial_x\psi\cdot\partial_x b_l + (\Delta\phi - iE_l)b_l = i\Delta b_{l-1} \quad (l = 0,1,\ldots),$$

with $E \sim \sum E_l h^l$. This is a first order differential equation along the Hamilton vector field H_p.

In the periodic case, where $K(E_0) = \{\gamma(E_0)\}$, it is seen that for E sufficiently colse to E_0, $K(E) = \{\gamma(E)\}$, $\gamma(E)$ is a periodic orbit. $\Gamma_+(E)$ and $\Gamma_-(E)$ are Lagrangian submanifold intersecting transversally along $\gamma(E)$. Then the quantization condition of resonances is equivalent to the condition that the WKB solution is single-valued on $\Gamma_+(E)$, i.e. a WKB solution microlocally defined in a neighborhood of a point on $\gamma(E)$ coincides with the one obtained after one tour near $\gamma(E)$.

In the homoclinic case, the outgoing (incoming) tail $\Gamma_+(E)$ ($\Gamma_-(E)$) is the extension by the Hamilton flow of the unstable (stable) manifold at the stationary point $(0,0)$.

By (A3), the structure of Γ_+ and Γ_- is the same as the periodic case except near $(0,0)$. The problem is thus reduced to the continuation of the WKB solution on Γ_+ through $(0,0)$.

Let Ω be a small neighborhood of $(0,0)$ and γ_+, γ_- the outgoing and incoming part of $\gamma\cap\Omega$. The problem is to continue a microlocal solution near γ_- to a microlocal solution on γ_+ under the condition that the solution is a resonant state.

4.1 Uniqueness

If u is a resonant state, it is supported microlocally on the outgoing tail, i.e. Λ_+ and its extension. On $\Lambda_- \cap \Omega$, therefore, it is supported only on γ_- by (A3). If u is given on γ_-, then it is uniquely determined on γ_+. In fact we have the following result, which can be proved by the propagation of microsupport (see §1.3):

Proposition 4.1 *There exist a discrete set $\Gamma(h)$ and a neighborhood Ω' of $(0,0)$ such that if $E \notin \Gamma(h)$ is a resonance with resonant state u, and if*

$$MS(u) \cap \Lambda_- \cap \Omega \backslash (0,0) = \emptyset$$

then $MS(u) \cap \Omega' = \emptyset$.

The exceptional set $\Gamma(h)$ is the set of resonances associated to the stationary point $(0,0)$, therefore close to

$$E_\alpha = E_0 - i\sum_{j=1}^{n}(\alpha_j + 1/2)\lambda_j h$$

(see §2.2). Remark that the imaginary part of these points cannot coincide with (1) if we impose a non-resonant condition on $\lambda_1, \dots \lambda_n$, i.e. $\lambda_1, \dots \lambda_n$ are linearly independent over $\mathbb{N}$.

4.2 Integral representation of the resonant state

The connection problem can be achieved by composing the following two ideas: One is to express the map associating a microlocal solution on γ_- to one on γ_+ as a semiclassical Fourier integral operator, which was introduced by Sjöstrand and Zworski in [18] and was called *monodromy operator.* The other is to represent the solution as superposition of time-dependent WKB solutions, which was used by Helffer and Sjöstrand in [9].

We write the resonant state in the form

$$u(x,h) = (2\pi h)^{-(n-1)/2} \int_{\mathbb{R}^{n-1}} \int_0^\infty e^{i\phi(t,x,\eta)/h} a(t,x,\eta,h)\hat{u}_0(\eta)\,dt\,d\eta$$

Let H_0 be a small hypersurface in x-space close to 0 and transversal to the projection of γ, say $\{x_1 = \epsilon\}$. We construct ϕ and a so that the restriction of u on H_0 but microlocally near γ_- be u_0.

In particular, the phase ϕ can be constructed by evolving a suitably chosen Lagrangian manifold transversal to γ_-. For x near γ_-, there exists one and only one critical point $t(x,\eta)$ of ϕ. The stationary phase method at this point gives the microlocal solution on γ_-. In order that it be equal to u_0 on H_0 near γ_-, it suffices to impose the initial conditions

$$\begin{cases} \phi(t(x,\eta),x,\eta)|_{H_0} = x'\cdot\eta, \\ a(t(x,\eta),x,\eta,h)|_{H_0} = 1. \end{cases} \tag{2}$$

On the other hand, it turns out from the geometry near $(0,0)$ that the Lagrangian manifold converges to Λ_+ as $t\to+\infty$, more precisely, there exist $\tilde{\psi}(\eta)$ independent of t, x and $\phi_1(x,\eta))$ independent of t, which are determined up to constant by (2), such that

$$\phi(t,x,\eta)\sim\phi_+(x)+\tilde{\psi}(\eta)+e^{-\lambda_1 t}\phi_1(x,\eta), \tag{3}$$

where $\phi_+(x)$ is a generating function of Λ_+ and

$$\phi_+(x)=\sum\frac{\lambda_j}{4}x_i^2+O(|x|^3) \quad \text{as} \quad x\to 0.$$

The contribution from $t=+\infty$ in the integration with respect to t, calculated with the asymptotic formula (3) and that of the symbol a, gives a microlocal solution on γ_+.

Thus we obtain the map of microlocal solutions from γ_- to γ_+ as a semi-classical microlocal Fourier integral operator.

Remark 4.2 In the case where $n=1$, we do not need the integration with respect to η and if $p=\xi^2-x^2/4$ near $(0,0)$, the above procedure reduces to the well known asymptotic expansion of the Weber function $D_\mu(x)$, which is a solution of

$$\frac{d^2w}{dx^2}+\left(\mu+\frac{1}{2}-\frac{x^2}{4}\right)w=0,$$

and which has an integral representation

$$D_\mu(x)=\frac{e^{-x^2/4}}{\Gamma(-\mu)}\int_0^\infty e^{-xt-t^2/2}t^{-\mu-1}dt.$$

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