A Note on Essential Self-adjointness of Dirac Operator with a Monopole (Spectral and Scattering Theory and Related Topics)

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A Note on Essential Self-adjointness of Dirac Operator with a Monopole

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Abstract

The purpose of this paper is to analyse the essential self-adjointness of Dirac operator $H = H_0 + V = c\alpha \cdot (-i\nabla + iA) + \beta m_0 c^2 + V$, where $A$ is the vector potential induced by a monopole. The potential $V$ is assumed to be spherically symmetric and of the form $V = u(r)I_4 + v(r)\beta + iw(r)\beta(\alpha \cdot e_r)$. It is shown that $H$ is essentially self-adjoint under some conditions on the behavior of $u, v$ and $w$ in a neighbourhood of the origin.

Key words. Dirac operator, essential self-adjointness, monopole, complex line bundle, section

§1 Introduction

Since 1976 several authors have investigated the Schrödinger operator with a magnetic field induced by a magnetic monopole (simply called a monopole) [7, 8, 15, 17]. It seems worthwhile to throw light upon Dirac operator in such a case [15, 17].

Mathematically, a wave function is described as a section of a vector bundle [3] and a vector potential is represented by a connection form of the principal fibre bundle associated with the vector bundle. In this paper we construct the Hilbert space on which Dirac operator $H$ with a monopole operates and study the essential self-adjointness of $H$. In the sequel, we use the quantity $q = \frac{eg}{c\hbar}$ ($e$: electric charge, $g$: magnetic charge) as a monopole parameter on the basis of Dirac’s quantization condition $2q$ should be an integer [1].

In §2 we build up a line bundle $D^{(q)}$ over $\mathbb{R}^3 \setminus \{0\}$ and another one $E^{(q)}$ over the sphere $S^2$ with the same structure group $U(1)$. Then we make the Hilbert space $\tilde{\Gamma}(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$ on which $H$ operates and the corresponding one $\tilde{\Gamma}(S^2, E^{(q)})^4$. Subsequently we define the vector potential $A$ explicitly. Since we assume that the potential $V$ in $H$ is spherically symmetric, we rewrite the unperturbed part $H_0$ of $H$ so that it may contain radial terms and a generalized spin-orbit coupling operator $K$ (Eq.(2.11)).

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1) $c$ is the speed of light and $\hbar$ is the Planck constant.
In §3, using Wu-Yang’s monopole harmonic sections \( Y_{t,m}^{2}(\theta, \varphi) \) \[7\] which form an orthonormal basis for \( \mathcal{F}(S^2, E^{(q)}) \), we decompose \( \mathcal{F}(S^2, E^{(q)})^4 \) into the direct sum of the simultaneous eigenspaces \( \mathcal{R}^{(q)}_{j,m,k} \) of \( J^2, J_3 \) and \( K^4 \). The restriction of \( H \) to the partial wave subspace \( L^2((0, \infty), dr) \otimes \mathcal{R}^{(q)}_{j,m,k}, h_{j,m,k} \), is represented on \( L^2((0, \infty), dr)^2 \) by radial terms.

In §4 we show under what condition the total Hamiltonian \( H \) is essentially self-adjoint on \( \Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4 \) (As for \( \Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)}) \), see the lower half part of this page). Arnold-Kalf-Schneider’s theorems \[16\] are useful for the essential self-adjointness of \( h_{j,m,k} \). Then we obtain the three main results, Theorems 4.2, 4.3, 4.4 by setting some reasonable assumptions on the behavior of \( V = u(r)I_4 + v(r)\beta + iw(r)\beta(\alpha \cdot e_r) \) in a neighbourhood of the origin.

§2 Formulation of Dirac operator with a monopole

We first construct two line bundles. Let \( \{W_N, W_S\} \) be an open covering of a base space \( S^2 \) as follows:

\[
W_N = \left\{ (\theta, \varphi); 0 < \theta < \frac{\pi}{2} + \delta, 0 < \varphi < 2\pi \right\}, \quad \left( 0 < \delta < \frac{\pi}{2} \right) \tag{2.1}
\]

\[
W_S = \left\{ (\theta, \varphi); \frac{\pi}{2} - \delta < \theta < \pi, 0 < \varphi < 2\pi \right\}. \tag{2.2}
\]

A transition function \( \tau_{NS} \) of \( W_N \cap W_S \) into the unitary group \( U(1) \) is defined by

\[
\tau_{NS}(\theta, \varphi) := e^{2iq\varphi}. \tag{2.3}
\]

Using these quantities, we build up a complex line bundle \( E^{(q)} \). Subsequently, let \( D^{(q)} \) be the pull-back of \( E^{(q)} \) by the smooth mapping \( f \) of \( \mathbb{R}^3 \setminus \{0\} \) onto \( S^2 \) defined as \( f(x) := \frac{x}{||x||} \). The open covering \( \{\{r; r > 0\} \times W_N, \{\{r; r > 0\} \times W_S\} \) of \( \mathbb{R}^3 \setminus \{0\} \) is chosen and the transition function \( \tau_{NS}(r, \theta, \varphi) \) of \( D^{(q)} \) is essentially the same as that of \( E^{(q)} \): \( \tau_{NS}(r, \theta, \varphi) = e^{2iq\varphi} \).

Furthermore, let \( \Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)}) \) denote the set of all \( C^\infty \)-class global sections of \( D^{(q)} \) with compact support and \( \Gamma^\infty(S^2, E^{(q)}) \) the set of all \( C^\infty \)-class global sections of \( E^{(q)} \). They are complex linear spaces. We equip \( \Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)}) \) and \( \Gamma^\infty(S^2, E^{(q)}) \) with an inner product as follows:

\[
\langle \eta, \xi \rangle = \int_{\mathbb{R}^3 \setminus \{0\}} \eta(r, \theta, \varphi)\ast \xi(r, \theta, \varphi)r^2 \sin \theta dr d\theta d\varphi, \tag{2.4}
\]

\[
\langle \Xi, \Psi \rangle = \int_{S^2} \Xi(\theta, \varphi)\ast \Psi(\theta, \varphi) \sin \theta d\theta d\varphi. \tag{2.5}
\]

\[1\) \( J \): total angular momentum operator. See (2.11).
Then we obtain the two Hilbert spaces by completing $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})$ and $\Gamma^\infty(S^2, E^{(q)})$. We denote them by $\tilde{\Gamma}(\mathbb{R}^3 \setminus \{0\}, D^{(q)})$ and $\tilde{\Gamma}(S^2, E^{(q)})$, respectively.

Obviously we get

$$\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)}) \cong C_0^\infty(0, \infty) \otimes \Gamma^\infty(S^2, E^{(q)})$$

(2.6)

and

$$\tilde{\Gamma}(\mathbb{R}^3 \setminus \{0\}, D^{(q)}) \cong L^2((0, \infty), dr) \otimes \tilde{\Gamma}(S^2, E^{(q)}).$$

(2.7)

Since any wave function satisfying Dirac equation has 4 components, the next decomposition provides a starting point

$$\tilde{\Gamma}(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4 \cong L^2((0, \infty), dr) \otimes \tilde{\Gamma}(S^2, E^{(q)})^4.$$

(2.8)

We have now reached the stage of construction of the vector potential in a free Hamiltonian $H_0$. It must be described with a connection form of the principal fibre bundle associated to $D^{(q)}$. Since the magnetic field induced by a monopole $q$ is a curvature form of the connection form, we choose Wu-Yang's connection form $A^{[13]}$ and take the vector potential $A$ to be the dual of $A$:

$$\{A_N = \frac{iq(1 - \cos \theta)}{r \sin \theta} e_\varphi\}$$ on $\{r; r > 0\} \times W_N,$

$$\{A_S = \frac{-iq(1 + \cos \theta)}{r \sin \theta} e_\varphi\}$$ on $\{r; r > 0\} \times W_S.$

(2.9)

With the help of $A$ we can define $H_0$ as

$$H_0 = c \alpha \cdot (-i \nabla + i A) + \beta m_0 c^2 \mathbf{1}.$$ (2.10)

We shall here assume that the perturbed potential $V$ is spherically symmetric and that $V(r)$ is $4 \times 4$ Hermitian matrix composed of continuous functions on $(0,\infty)$. The total Hamiltonian $H = H_0 + V$ operates on $\tilde{\Gamma}(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$. We take the domain $\text{Dom}(H)$ to be $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$ for the present.

To decompose $H$ into the direct sum of radial terms on the basis of (2.8), we rewrite $H_0$ by four new operators $L, S, J$ and $K$.

$$L = M - q e_r, \quad S = \frac{1}{2} \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix},$$

$$J = LI_4 + S, \quad K = \beta (2S \cdot M + I_4).$$

(2.11)

\footnote{\(e_r = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), e_\theta = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta),\)}

\footnote{\(e_\varphi = (-\sin \varphi, \cos \varphi, 0),\)}

\footnote{\(\alpha_j (j = 1, 2, 3), \beta = \alpha_0\) are $4 \times 4$ constant Hermitian matrices satisfying the anti-commutation relations $\alpha_j \alpha_k + \alpha_k \alpha_j = 2 \delta_{jk} I_4$.}
where \( M \) is the auxiliary operator in \( \Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)}) \) given by
\[
M := x \wedge (-i\nabla + i A).
\] (2.12)

Then \( L \) is a symmetric operator defined on \( \Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)}) \) and \( S, J, K \) are symmetric operators defined on \( \Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4 \). The operators \( J \) and \( K \) are called the total angular momentum operator and the generalized spin-orbit coupling one, respectively. These operators enable us to deduce
\[
H_0 = -i\alpha \cdot \mathbf{e}_r \left( \frac{\partial}{\partial r} + \frac{1}{r} - \frac{1}{r} \beta K \right) + \beta m_0 c^2.
\] (2.13)

§3 Decomposition of Dirac operator

We first decompose \( \tilde{\Gamma}(S^2, E^{(q)})^4 \) into the direct sum of simultaneous eigenspaces of \( J^2 \), \( J_3 \), and \( K \). We here put
\[
\Xi_q := \left\{ |q| - \frac{1}{2}, |q| + \frac{1}{2}, |q| + \frac{3}{2}, \ldots \right\}, \quad \left( q = \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \ldots \right), \quad \kappa_{j}^{(q)} := \sqrt{\left( j + \frac{1}{2} \right)^2 - q^2} \quad (j \in \Xi_q) . \tag{3.2}
\]

There exists an orthonormal basis
\[
\{ \Phi_{j,m,k}^\pm | j \in \Xi_q, m = -j, -j + 1, \ldots, j - 1, j, k = \pm \kappa_j^{(q)} \} \tag{3.3}
\]
of \( \tilde{\Gamma}(S^2, E^{(q)})^4 \) whose elements satisfy the following simultaneous eigenvalue equations of \( J^2 \), \( J_3 \), and \( K \), according to Y. Kazama et al [8].
\[
\begin{align*}
J^2 \Phi_{j,m,k}^\pm &= j(j+1) \Phi_{j,m,k}^\pm, \\
J_3 \Phi_{j,m,k}^\pm &= m \Phi_{j,m,k}^\pm, \quad m = -j, -j + 1, \ldots, j - 1, j, \\
K \Phi_{j,m,k}^\pm &= -k \Phi_{j,m,k}^\pm, \quad k = -\kappa_j^{(q)}, \kappa_j^{(q)}. \tag{3.4}
\end{align*}
\]

All \( \Phi_{j,m,k}^\pm \) are constructed with Wu-Yang's monopole harmonic sections \( Y_{l,m}^q \) [7].

The above consideration leads us to the following decomposition theorem.

**Theorem 3.1.** When setting \( \mathcal{R}_{j,m,k}^{(q)} := \text{span}\{ \Phi_{j,m,k}^+, \Phi_{j,m,k}^- \} \) we obtain
\[
\tilde{\Gamma}(S^2, E^{(q)})^4 \cong \bigoplus_{j \in \Xi_q} \bigoplus_{m = -j}^{j} \bigoplus_{k = \pm \kappa_j^{(q)}} \mathcal{R}_{j,m,k}^{(q)}. \tag{3.5}
\]

owing to [7] and [8].
Combination of Eqs. (2.8) and (3.5) yields the relation

$$
\tilde{\Gamma}(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4 \cong \bigoplus_{j \in \mathbb{Z}_q} \bigoplus_{m=-j}^{j} \bigoplus_{k=\pm \kappa_{j}^{(q)}} (L^2((0, \infty), dr) \otimes R_{j,m,k}^{(q)}) \wedge (3.6)
$$

Each subspace $L^2((0, \infty), dr) \otimes R_{j,m,k}^{(q)}$ is called a partial wave subspace and isomorphic to $L^2((0, \infty), dr)^2$.

Assume that $V$ has the form of

$$
V(r) = u(r)I_4 + v(r)\beta + iw(r)\beta(\alpha \cdot \mathbf{e}_r),
$$

where $u$, $v$, and $w$ are real-valued $C^1$-class functions on $(0, \infty)$. Since $\beta \Phi_{j,m,k}^{\pm} = \pm \Phi_{j,m,k}^{\pm}$ and $-i(\alpha \cdot \mathbf{e}_r)\Phi_{j,m,k}^{\pm} = \pm \Phi_{j,m,k}^{\mp}$, we obtain the following fundamental theorem.

**Theorem 3.2.** Let $h_{j,m,k}$ denote the restriction of the total Hamiltonian $H$ to the partial wave subspace. Then we have

$$
H \cong \bigoplus_{j \in \mathbb{Z}_q} \bigoplus_{m=-j}^{j} \bigoplus_{k=\pm \kappa_{j}^{(q)}} h_{j,m,k}
$$

and $h_{j,m,k}$ is represented by

$$
h_{j,m,k} = \begin{pmatrix}
m_0 c^2 + u(r) + v(r) & c \left\{ \frac{d}{dr} + \frac{k}{r} \right\} + w(r) \\
c \left\{ \frac{d}{dr} + \frac{k}{r} \right\} + w(r) & -m_0 c^2 + u(r) - v(r)
\end{pmatrix} (k = \pm \kappa_{j}^{(q)})
$$

on $C_0^\infty(0, \infty)^2$.

The operator $h_{j,m,k}$ is called a radial Dirac operator.

**§4 Essential self-adjointness of Dirac operator**

We are now in a position to state a sufficient condition that Dirac operator be essentially self-adjoint. The following theorem serves well for the purpose.

**Theorem 4.1.** Let $u, v \in C^1(0, \infty)$ and $f_\pm = u \pm v$. Suppose $\lim_{r \to 0} rf_\pm(r)$ exist. Put $l_\pm = \frac{1}{c} \lim_{r \to 0} rf_\pm(r)$. If $l_+ l_- < (\kappa_j^{(q)})^2 - \frac{1}{4}$, then $h_{j,m,k}$ is essentially self-adjoint on $C_0^\infty(0, \infty)^2$ for all $j \in \mathbb{Z}_q$.

The proof is easily given owing to V. Arnold, H. Kalf, and A. Schneider [16].
Theorem 4.2. Let $g \in C^1(0, \infty)$. If $\lim_{r \to 0} g(r)$ exists and $|g(+0)| > \frac{1}{2}$, then the total Hamiltonian $H = H_0 + \frac{cg(r)}{r} \beta$ is essentially self-adjoint on $\Gamma^{\infty}_0(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$ for all $|q| \geq \frac{1}{2}$. $(u = w = 0, v = \frac{cg(r)}{r})$

Proof. It is sufficient to prove the essential self-adjointness of the radial Dirac operator $h_{j,m,k}$ for each $j \in \Xi_q$. The constants $\pm m_0c^2$ in the diagonal part of $h_{j,m,k}$ may be omitted in discussion of essential self-adjointness. Then we have

$$-g(+0)^2 < (\kappa_j^{(q)}) - \frac{1}{4}$$

for all $j \in \Xi_q$. Hence $h_{j,m,k}$ is essentially self-adjoint on $C^\infty_0(0, \infty)^2$ for all $j \in \Xi_q$. This implies that $H$ is essentially self-adjoint on $\Gamma^{\infty}_0(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$. \hfill \qed

Theorem 4.3. Let $|q| \geq \frac{1}{2}$. If the inequalities $\frac{1}{2} < |b| < \sqrt{2|q| + 1} - \frac{1}{2}$ hold, then the total Hamiltonian $H = H_0 + i \frac{cb}{r} \beta(\alpha \cdot e_r)$ is essentially self-adjoint on $\Gamma^{\infty}_0(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$. $(u = v = 0, w = \frac{cb}{r})$

Proof. The constants $\pm m_0c^2$ in the diagonal part may be omitted in argument on the essential self-adjointness of $h_{j,m,k}$.

Case I. $j \geq |q| + \frac{1}{2}$: Assume the inequalities $\frac{1}{2} < b < \sqrt{2|q| + 1} - \frac{1}{2}$ hold. In the case of $k = \kappa_j^{(q)}$, we have

$$(b + k)^2 - \frac{1}{4} \geq b^2 - \frac{1}{4} > 0.$$

In the case of $k = -\kappa_j^{(q)}$, we have

$$(b + k)^2 - \frac{1}{4} = \left(b - \kappa_j^{(q)} + \frac{1}{2}\right) \left(b - \kappa_j^{(q)} - \frac{1}{2}\right). \quad (*)$$

Since $\kappa_j^{(q)} \geq \sqrt{2|q| + 1}$, we get $b - \kappa_j^{(q)} \leq -\frac{1}{2}$ and the right-hand side of Eq.$(*)$ is non-negative. Hence it follows from Theorem 4.1 that $h_{j,m,k}$ is essentially self-adjoint on $C^\infty_0(0, \infty)^2$. Likewise in the case of $b < 0$, we can obtain the assertion.

Case II. $j = |q| - \frac{1}{2}$: In this case, we have $0 < b^2 - \frac{1}{4}$ ($\kappa_j^{(q)} = 0$), and so $h_{j,m,k}$ is essentially self-adjoint.

As a consequence, $h_{j,m,k}$ is essentially self-adjoint on $C^\infty_0(0, \infty)^2$ for all $j \in \Xi_q$. This means that $H$ is essentially self-adjoint on $\Gamma^{\infty}_0(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$. \hfill \qed
Theorem 4.4. Let $|q| \geq \frac{1}{2}$. Assume that $u$ is a $C^1$-class function on $(0, \infty)$ and
\[ p_0 := \lim_{r \to 0} ru'(r) \] exists. If the inequalities $\frac{1}{2} < |p_0 \lambda| < \sqrt{2|q| + 1} - \frac{1}{2}$ hold, then the total Hamiltonian $H = H_0 + u(r)I_4 + i\lambda u'(r)\beta(\alpha \cdot e_r)$ is essentially self-adjoint on $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$. ($v = 0, w = \lambda u'(r)$)

Proof. The constants $\pm m_0 c^2$ in the diagonal part may be omitted.

Case I. $j \geq |q| + \frac{1}{2}$: Assume the inequalities $\frac{1}{2} < |p_0 \lambda| < \sqrt{2|q| + 1} - \frac{1}{2}$ hold. In a similar way to the proof of Theorem 4.3 we get
\[ (k + p_0 \lambda)^2 - \frac{1}{4} > 0 \]
for $k = \pm \kappa_j^{(q)}$. Likewise in the case of $p_0 \lambda < 0$ we obtain the above inequality. Hence it follows from Corollary 2 of Theorem 3 of Ref.[16] that $h_{j,m,k}$ is in the limit-point case at the origin. Consequently, $h_{j,m,k}$ is essentially self-adjoint.

Case II. $j = |q| - \frac{1}{2}$: In this case, we have $0 < (p_0 \lambda)^2 - \frac{1}{4}$ ($\kappa_j^{(q)} = 0$).

The both cases imply that $H$ is essentially self-adjoint on $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$. $\square$

§5 Discussion

In §4 we have proved the essential self-adjointness of $H$ by the limit-point case at the origin of every radial Dirac operator $h_{j,m,k}$ (Theorem 4.1, [16]) and the decomposition theorem (Theorem 3.1 and 3.2). In our case (a monopole exists), it is an interesting fact that although the unperturbed operator
\[ h_{j,m,k}^{(0)} = \begin{pmatrix} m_0 c^2 & -c \frac{d}{dr} \\ c \frac{d}{dr} & -m_0 c^2 \end{pmatrix} \]
for $j = |q| - 1/2 (\kappa_j^{(q)} = 0)$ is not essentially self-adjoint, $h_{j,m,k}$ becomes essentially self-adjoint if $H$ has a special-type potential.

The investigation of the essential self-adjointness the usual $n$-dimensional Dirac operator was treated by Kalf and Yamada [19]. Under the assumption that $m$ and $V$ are spherically symmetric, they reduced the problem to that of every radial Dirac operator $h$ with $k \in \pm\{N_0 + (n-1)/2\}$. Their method** is the same as ours. But since $k = \pm \sqrt{(j + 1/2)^2 - q^2}$ and $j \in \Xi_q$ in our case, it is more difficult to study the essential self-adjointness of $h_{j,m,k}$.

**Behncke and Thaller already discussed this case for the usual Dirac operator (No monopole) [10, 14]. cf. Corollaries 2 and 3 of Theorem 3 in [16].
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