Study of the fractals generated by contractive mappings and their dimensions

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Abstract

Around 1970, importance of fractals was recognized by Benoit Mandelbrot and fractals have been well-studied since 1970s. One of the most important methods to construct the fractals is the application of iterated function systems (IFSs) and there are a lot of results related to the estimates of the Hausdorff dimension and the Hausdorff measure of the limit sets (fractals) generated by IFSs and many other important results. In this thesis, we consider the family of IFSs of generalized complex continued fractions as an example of (families of) infinite IFSs and we discuss the estimates of the Hausdorff dimension, the Hausdorff measure, the packing dimension and the packing measure of the limit sets generated by the IFSs. The point to estimate dimensions and measures is to construct the conformal measure since the (family of) IFSs have a "good" properties.

By the way, there is not only the study of CIFSs in recent studies. For example, there are a study of the Non-autonomous IFSs (NAIFSs) which is a generalization of the IFSs and a study of IFSs with overlaps which does not satisfy usual conditions (for example, open set condition) in the theory of IFSs. In this thesis, we discuss NAIFSs (with weights) defined on complete (separable) metric spaces, which has not been discussed before. If the NAIFS (with weights) satisfies a "good" condition, we can construct the limit set (limit measure) generated by the NAIFS (with weights) and we discuss some basic properties. The point to construct the limit sets and prove the properties of the limit sets is to apply the generalized Banach fixed point theorem to the NAIFSs (with weights) under the "good" condition. Finally, we discuss the estimate of the Hausdorff dimension of limit sets generated by the IFSs with overlaps. The point to prove the estimate of the Hausdorff dimensions is the following: we consider a minimum number of the level-l cells which cover the limit sets and we obtain the asymptotic behavior of the numbers.

Contents

1 Introduction and the main results

Iterated function systems arise in many contexts. One of the most famous applications to use the systems is to construct many kinds of fractals. Studies of these fractal sets constructed by the contractive iterated function systems (for short IFS), sometimes called limit sets, have been developed in many directions. Note that general properties of limit sets of systems with finitely many mappings have been well-studied. For example, see Hutchinson [10], Falconer [8], Barnsley [4], Bandt and Graf [2], and Schief [30] and so on.

Around 1990's, studies of the limit sets of the conformal IFSs (for short CIFS) were initiated and there are many results related to CIFSs. Especially, Mauldin and Urbański found a formula on the Hausdorff dimension of limit sets generated by finite CIFSs. For example, they found the formula on the Hausdorff dimension of the limit sets, and there exist statements which claim that the Hausdorff measure of the limit set of any finite CIFS with respect to the Hausdorff dimension is positive and finite and the packing measure of the limit set with respect to the Hausdorff dimension is also positive and finite (from this, we deduce that the Hausdorff dimension of the limit set of any finite CIFS and the packing dimension of the limit set are the same in general).

In addition, studies of limit sets of conformal iterated function systems with infinitely many mappings (for short, *infinite* CIFS) were initiated by Mauldin and Urbański ([22], [23], [24]) and there are many related results on infinite CIFSs with overlaps by Mihailescu and Urbański ([25], [26]). Note that there exist other papers of infinite IFS (for example, see [27]). Especially, Mauldin and Urbański first showed deep results to estimate the Hausdorff dimension and the Hausdorff measure of the limit sets. For example, they found a condition under which the Hausdorff measure of the limit set of the infinite CIFS with respect to the Hausdorff dimension is zero.

Moreover, Mauldin and Urbański constructed an interesting example of an infinite CIFS which is related to the complex continued fractions in the paper [22].

The construction of the example is the following. Let $X := \{z \in \mathbb{C} \mid |z - 1/2| \leq 1/2\}$. We call $\hat{S} := \{\hat{\phi}_{(m,n)}(z) \colon X \to X \mid (m,n) \in \mathbb{Z} \times \mathbb{N}\}$ the CIFS of complex continued fractions, where $\mathbb{Z}$ is the set of integers, $\mathbb{N}$ is the set of positive integers and

$$\hat{\phi}_{(m,n)}(z) := \frac{1}{z + m + ni} \quad (z \in X).$$

Let $\hat{J}$ be the limit set of $\hat{S}$ (see Definition 2.19) and $\hat{h}$ be the Hausdorff dimension of $\hat{J}$. For each $s \geq 0$, we denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure and denoted by $\mathcal{P}^s$ the s-dimensional packing measure. For this example, Mauldin and Urbański showed the following theorem.

Theorem 1.1 (D. Mauldin, M. Urbanski (1996))**.** Let $\hat{S}$ be the CIFS of complex continued fractions. Then, we have that $\mathcal{H}^{\hat{h}}(\hat{J}) = 0$ and $0 < \mathcal{P}^{\hat{h}}(\hat{J}) < \infty$.

This is an example of infinite CIFS of which the Hausdorff measure of the limit set with respect to the Hausdorff dimension is zero and the packing measure of the limit set is positive and finite. Note that this is a new phenomenon of infinite CIFSs which cannot hold in finite CIFSs.

Moreover, interests in families of CIFSs have emerged. Roy and Urbański especially studied the Hausdorff dimension functions for the families of CIFSs ([29]). They showed that the Hausdorff dimension functions for the families of CIFSs are continuous with respect to the "λ-topology" which they introduced, and if the families are analytic in some sense, then the Hausdorff dimension functions for the families of CIFSs are real-analytic and subharmonic.

There exist rich general theories of limit sets of CIFSs for given families of infinite CIFSs. However, the authors do not think we have found sufficiently many examples of families of infinite CIFSs to which we can apply the above general theories. Therefore, the first aim of this thesis is to present a new interesting family of infinite CIFSs. More precisely, we define a subset A_0 of the complex plane as a parameter space and for each point in the parameter space, we introduce a CIFS related to generalized complex continued fractions $\{S_\tau\}_{\tau \in A_0}$. Note that $\{S_\tau\}_{\tau \in A_0}$ is a family of CIFSs which has uncountably many elements.

We found Mauldin and Urbański's general theories [22], [23] and Roy and Urbański's general theory [29] can apply to this family. We also show that the Hausdorff dimension function for the family is continuous in the parameter space and is real-analytic and subharmonic in the interior of the parameter space by applying the general theories of the families of infinite CIFSs. We also show that, as a corollary for these results, the Hausdorff dimension function has a maximum point and it belongs to the boundary of the parameter space. Moreover, to find examples of infinite CIFSs with the phenomenon which cannot hold in finite CIFSs, we also show that the Hausdorff measure of the limit set with respect to the Hausdorff dimension is zero and the packing measure of the limit set is positive and finite for each $\tau \in A_0$.

The precise statement is the following. Let

$$A_0 := \{\tau = u + iv \in \mathbb{C} \mid u \geq 0 \text{ and } v \geq 1\}$$

and $X := \{z \in \mathbb{C} \mid |z - 1/2| \leq 1/2\}$. Also, we set $I_\tau := \{m + n\tau \in \mathbb{C} \mid m, n \in \mathbb{N}\}$ for each $\tau \in A_0$, where $\mathbb{N}$ is the set of the positive integers.

Definition 1.2 (The CIFS of generalized complex continued fractions)**.** For each $\tau \in A_0$, $S_\tau := \{\phi_b \colon X \to X \mid b \in I_\tau\}$ is called the CIFS of generalized complex continued fractions. Here, for each $\tau \in A_0$,

$$\phi_b(z) := \frac{1}{z + b} \quad (z \in X, b \in I_\tau).$$

The family $\{S_\tau\}_{\tau \in A_0}$ is called the family of CIFSs of generalized complex continued fractions. For each $\tau \in A_0$, let J_τ be the limit set of the CIFS S_τ (see Definitions 2.12, 2.19) and let h_τ be the Hausdorff dimension of the limit set J_τ.

We remark that this family of CIFSs is a generalization of $\hat{S}$ in some sense. The system S_τ is related to "generalized" complex continued fractions since each point of the limit set J_τ of S_τ is of the form

$$\cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}}$$

for some sequence $(b_1, b_2, b_3, \ldots)$ in I_τ (See Definition 2.19). Note that there are many kinds of general theories for continued fractions and related iterated function systems ([15], [22], [23], [26]).

We denote by $\mathrm{Int}(A_0)$ the set of interior points of A_0 with respect to the topology in $\mathbb{C}$. We now present the first five main results in this thesis.

Theorem 1.3 (**Main Theorem 1** ([11, Theorem 1.2]))**.** Let $\{S_\tau\}_{\tau \in A_0}$ be the family of CIFSs of generalized complex continued fractions. Then, the function $\tau \mapsto h_\tau$ is continuous in A_0. Moreover, for each $\tau \in A_0$, h_τ is equal to the unique zero of the pressure function of S_τ (see Definition 2.2), $1 < h_\tau < 2$ and $h_\tau \to 1$ $(\tau \in A_0, \tau \to \infty)$.

Theorem 1.4 (**Main Theorem 2** ([11, Theorem 1.3])). Let $\{S_\tau\}_{\tau \in A_0}$ be the family of CIFSs of generalized complex continued fractions. Then, we have that $\tau \mapsto h_\tau$ is real-analytic and subharmonic in $\text{Int}(A_0)$. Also, the function $\tau \mapsto h_\tau$ is not constant on any non-empty open subset of A_0.

Theorem 1.5 (**Main Theorem 3** ([11, Theorem 1.4])). Let $\{S_\tau\}_{\tau \in A_0}$ be the family of CIFSs of generalized complex continued fractions. Then, there exists a maximum value of the function $\tau \mapsto h_\tau$ ($\tau \in A_0$) and any maximum point of the function $\tau \mapsto h_\tau$ belongs to the boundary of A_0. In particular, we have that $\max\{h_\tau | \tau \in A_0\} = \max\{h_\tau | \tau \in \partial A_0\}$.

Remark 1.6. It was shown that for each $\tau \in A_0$, $\overline{J_\tau} \setminus J_\tau$ is at most countable and $h_\tau = \dim_{\mathcal{H}}(\overline{J_\tau})$ ([32, Theorem 6.11]). For the readers, we give a proof of this fact in the Appendix in section 2 of this paper. Also, for each $\tau \in A_0$, since the set of attracting fixed points of elements of the semigroup generated by S_τ is dense in J_τ, Theorem 1.1 of [31] implies that $\overline{J_\tau}$ is equal to the Julia set of the rational semigroup generated by $\{\phi_b^{-1} \mid b \in I_\tau\}$.

Theorem 1.7 (**Main Theorem 4** ([12, Theorem 1.3])). Let $\{S_\tau\}_{\tau \in A_0}$ be the family of CIFSs of generalized complex continued fractions. Then, for each $\tau \in A_0$, we have $\mathcal{H}^{h_\tau}(J_\tau) = 0$

Theorem 1.8 (**Main Theorem 5** ([12, Theorem 1.3], [13])). Let $\{S_\tau\}_{\tau \in A_0}$ be the family of CIFSs of generalized complex continued fractions. Then, for each $\tau \in A_0$, we have $0 < \mathcal{P}^{h_\tau}(J_\tau) < \infty$. In particular, for each $\tau \in A_0$, the packing dimension of the limit set J_τ equals the Hausdorff dimension h_τ of the limit set J_τ.

Remark 1.9. By the general theories of finite CIFSs, the Hausdorff measure of the limit set of each finite CIFS with respect to the Hausdorff dimension and the packing measure of the limit set with respect to the Hausdorff dimension is positive and finite. However, Theorem 1.7 and Theorem 1.8 indicates that for each S_τ of the family of CIFSs of generalized complex continued fractions, which consists of uncountably many elements, the Hausdorff measure of the limit set with respect to the Hausdorff dimension is zero and the packing measure of the limit set with respect to the Hausdorff dimension (which is equal to the packing dimension) is positive and finite. This is also a new phenomenon which cannot hold in the finite CIFSs.

As we have seen, iterated function system is one of the most famous methods to construct fractals and there are many papers of studies of the limit sets of IFSs. Especially, fractals (limit sets) generated by iterated function systems with finitely many mappings are well-studied and there exist many results related to the Hausdorff dimension and the Hausdorff measure of the limit sets, the packing dimension and the packing measure of the limit sets and so on ([10], [2], [30], [8], [4], [17], [18]). Note that the self-similarity of the limit sets of (autonomous) iterated function systems is one of the most important points to obtain these rich results. However, there exist some results related to fractals generated by non-autonomous iterated function systems ([3], [9], [29], [1], [20], [21], [7]). For example, Barlow and Hambly consider generalized Sierpiński gasket generated by the non-autonomous iterated function systems and they studied the geometric properties and analytical properties of the limit sets ([3]). In addition, there exist the general theories which assure the existence of the limit sets of non-autonomous iterated function systems and which show the estimates on the Hausdorff dimension of the limit sets ([9], [29], [1]). This indicates we can analyze not only the limit sets of (autonomous) iterated function systems but also ones of non-autonomous iterated function systems. Henceforth, we focus on the non-autonomous iterated function systems.

As we mentioned, there exist some results related to the fractals generated by the non-autonomous iterated function systems ([9], [29], [1], [20], [21], [7]). However, the authors of those papers only deal with the non-autonomous iterated function systems defined on bounded sets (in some sense) or compact sets.

On the other hand, Hutchinson's original idea to construct the fractals is now generalized as follows (for example, see [17]). We first consider a complete metic space X (which is not always bounded) and (autonomous finite) iterated function systems defined on X. We next consider the set of all non-empty compact subsets of X and the Hausdorff distance on the set. Sometimes, this metric space is denoted by $(\mathcal{K}(X), d_H)$. Note that the metric space $(\mathcal{K}(X), d_H)$ is complete since X is complete. In addition, we define a contractive mapping (which is often called the Barnsley operator, see [19]) on $\mathcal{K}(X)$ associated with the iterated function system. By the Banach fixed point theorem, we deduce that there exists the unique fixed point (the unique non-empty compact subset) of the Barnsley operator. The unique non-empty compact subset is the limit set of the iterated function system.

The author of this thesis found that we can generalize this Hutchinson's idea to the setting of non-autonomous iterated function systems on complete metric spaces which are possibly unbounded.

Therefore, the second aim of this thesis is to give the method to construct the limit sets of the non-autonomous iterated function systems on (possibly) unbounded complete metric spaces based on Hutchinson's idea. We first consider a sequence of the contractive mappings defined on a complete metric space and show a generalized Banach fixed point theorem for a sequence of the contractive mappings under a certain condition. We finally show that we can define the limit set of the system as the limit of a sequence in $\mathcal{K}(X)$ which is constructed by the non-autonomous Barnsley operators of the non-autonomous iterated function system defined on a complete metric space (which is not always bounded) under another certain condition which is easy to check. Note that if we assume a stronger condition, we also show that the convergence to the limit sets is exponentially fast. In addition, under the same condition, we can construct the projection mapping which is important to analyze geometrical properties of the limit sets.

Moreover, we consider the non-autonomous iterated function systems with (positive) weights to construct a generalization of self-similar measures (we call the measures limit measures in this thesis). By the above ideas, we also define the limit measure as the limit of a sequence in the space of all Borel probability measures in X which is constructed by non-autonomous contractive mappings (Each mapping is often called the Foias operator, see [19]) associated with the non-autonomous iterated function system with weights defined on a complete separable metric space (which is possibly unbounded) under the same condition. In addition, we also show that the support of the limit measure is compact and is equal to the limit set.

The precise statement is the following. Let I be a set and (X, ρ) be a complete metric space.

Definition 1.10. We say that $(\{f_i\}_{i\in I}, \{J_n\}_{n\in\mathbb{N}})$ satisfy the setting (NAIFS) if

(i) $\{J_n\}_{n\in\mathbb{N}}$ is a sequence in $\{J \subset I \mid J \text{ is finite}\}$, and

(ii) $\{f_i\colon X \to X\}_{i\in I}$ is a family of contractive mappings on X with the uniform contraction constant $c \in (0, 1)$, that is, there exists $c \in (0, 1)$ such that for all $i \in I$ and

$x, y \in X$,

$$\rho(f_i(x), f_i(y)) \leq c\, \rho(x, y).$$

Note that for each $i \in I$, there exists $z_i \in X$ such that z_i is the unique fixed point of f_i since X is complete and for each $i \in I$, f_i is a contractive mapping defined on X. Let d_H be the Hausdorff distance on $\mathcal{K}(X)$ which is defined by

$$d_H(A, B) := \inf\{\epsilon > 0 \mid A \subset B_\epsilon, B \subset A_\epsilon\} \quad (A, B \in \mathcal{K}(X)),$$

where for each $\epsilon > 0$ and $A \subset X$, we set $A_\epsilon := \{y \in X \mid \exists a \in A, \rho(y, a) \leq \epsilon\}$. For each $n \in \mathbb{N}$, let $F_n : \mathcal{K}(X) \to \mathcal{K}(X)$ be the mapping defined by

$$F_n(A) := \bigcup_{i \in J_n} f_i(A),$$

which is well-defined since each f_i is continuous on X and each J_n is finite. Each mapping F_n is often called the Barnsley operator. We now present the next two main results in this thesis.

Theorem 1.11 (**Main Theorem 6** (see [14])). Let $(\{f_i\}_{i\in I}, \{J_n\}_{n\in\mathbb{N}})$ satisfy the setting (NAIFS). Suppose that there exists $x_0 \in X$ such that

$$\sum_{n\in\mathbb{N}} \left\{ \max_{i\in J_n} \rho(x_0, z_i) \right\} c^n < \infty.$$

Then, there exists the unique sequence of compact subsets $\{K_m\}_{m\in\mathbb{N}}$ in $\mathcal{K}(X)$ such that for each $m \in \mathbb{N}$ and $A \in \mathcal{K}(X)$, we have

$$\lim_{n\to\infty} (F_m \circ F_{m+1} \circ \cdots \circ F_{m+n-1})(A) = K_m \quad \text{in } \mathcal{K}(X). \tag{1.1}$$

In addition, for each $m \in \mathbb{N}$, we have

$$K_m = F_m(K_{m+1}). \tag{1.2}$$

Moreover, suppose that there exists $x_0 \in X$,

$$a := \limsup_{n\longrightarrow\infty} \sqrt[n]{\max_{i\in J_n} \rho(x_0, z_i)} < \frac{1}{c}.$$

Then, for all $m \in \mathbb{N}$, $r \in \{r > 0 \mid c \leq r < 1, ac < r\}$ and $A \in \mathcal{K}(X)$, $(F_m \circ F_{m+1} \circ \cdots \circ F_{m+n-1})(A)$ converges to K_m as n tends to infinity in $\mathcal{K}(X)$ exponentially fast with the rate r.

Theorem 1.12 (**Main Theorem 7** (see [14])). Let $(\{f_i\}_{i\in I}, \{J_n\}_{n\in\mathbb{N}})$ satisfy the setting (NAIFS). Suppose that there exists $x_0 \in X$ such that

$$\sum_{n\in\mathbb{N}} \left\{ \max_{i\in J_n} \rho(x_0, z_i) \right\} c^n < \infty. \tag{1.3}$$

Then, $\mathrm{diam}(f_{w|_n}(K_{m+n}))$ converges to zero as n tends to the infinity. In addition, for each $m \in \mathbb{N}$, there exists a mapping $\pi_m : \Pi_{j=m}^\infty J_j \to K_m$ such that

$$\{\pi_m(w)\} = \bigcap_{n\in\mathbb{N}} f_{w|_n}(K_{m+n}),$$

where $w = w_m w_{m+1} \cdots \in \prod_{j=m}^{\infty} J_j$ and $f_{w|_n} = f_{w_m} \circ f_{w_{m+1}} \circ \cdots \circ f_{w_{m+n-1}}$, and π_m is surjective and uniformly continuous. Moreover, suppose that there exists $x_0 \in X$ such that

$$a := \limsup_{n\to\infty} \sqrt[n]{\max_{i \in J_n} \rho(x_0, z_i)} < \frac{1}{c}.$$

Then, for each $r \in \{r > 0 \mid c \le r < 1, ac < r\}$, $\mathrm{diam}(f_{w|_n}(K_{m+n}))$ converges to zero as n tends to the infinity exponentially fast with the rate r.

Moreover, we consider non-autonomous iterated function systems with weights. Let I be a set and (X, ρ) be a complete separable metric space.

Definition 1.13. We say that $(\{f_i\}_{i\in I}, \{J_n\}_{n\in\mathbb{N}}, \{p_n\}_{n\in\mathbb{N}})$ satisfy the setting (wNAIFS) if

(i) $\{J_n\}_{n\in\mathbb{N}}$ is a sequence in $\{J \subset I \mid J \text{ is finite}\}$,

(ii) $\{f_i \colon X \to X\}_{i\in I}$ is a family of contractive mappings on X with the uniform contraction constant $c \in (0,1)$, that is, there exists $c \in (0,1)$ such that for all $i \in I$ and $x, y \in X$,

$$\rho(f_i(x), f_i(y)) \le c\ \rho(x, y), \text{ and}$$

(iii) for each $n \in \mathbb{N}$, p_n is $[0,\infty)$-valued functions on I with $p_n(i) > 0$ if and only if $i \in J_n$, and p_n satisfies

$$\sum_{i\in J_n} p_n(i) = 1.$$

Let $\mathcal{P}_1(X)$ be the set of Borel probability measures defined on the complete separable metric space (X, ρ) for which there exists $a \in X$ such that the function $x \mapsto \rho(a, x)$ is integrable. Note that for each $b \in X$ and $P \in \mathcal{P}_1(X)$, we have $\int_X \rho(b, x) P(\mathrm{d}x) < \infty$ since

$$\int_X \rho(b,x)\ P(\mathrm{d}x) \le \int_X \rho(b,a)\ P(\mathrm{d}x) + \int_X \rho(a,x)\ P(\mathrm{d}x) = \rho(b,a) + \int_X \rho(a,x)\ P(\mathrm{d}x) < \infty.$$

Let $\mathrm{Lip}_1(X)$ be the set of $\mathbb{R}$-valued functions f on X for which $\rho(f(x), f(y)) \le \rho(x, y)$ for all $x, y \in X$. Let d_{MK} be the Monge-Kantrovich distance on $\mathcal{P}_1(X)$ which is defined by

$$d_{MK}(\mu, \nu) := \sup\left\{\int_X f\mathrm{d}\mu - \int_X f\mathrm{d}\nu \mid f \in \mathrm{Lip}_1(X)\right\} \quad (\mu, \nu \in \mathcal{P}_1(X)).$$

Let $M_n \colon \mathcal{P}_1(X) \to \mathcal{P}_1(X)$ $(n \in \mathbb{N})$ be mappings defined by

$$M_n(\mu)(B) := \sum_{i\in J_n} p_n(i)\ \mu(f_i^{-1}(B)) \quad (B \in \mathcal{B}(X)),$$

where $\mathcal{B}(X)$ is the set of all Borel sets in X. Note that for each $n \in \mathbb{N}$, M_n is well-defined since

$$\begin{aligned}
&\int_X \rho(x,a)\ \mathrm{d}M_n(\mu) = \sum_{i\in J_n} p_n(i) \int_X \rho(x,a)\ \mathrm{d}(\mu \circ f_i^{-1}) \\
&= \sum_{i\in J_n} p_n(i) \int_X \rho(f_i(x), a)\ \mathrm{d}\mu \le \sum_{i\in J_n} p_n(i) \int_X \rho(f_i(x), f_i(z_i)) + \rho(f_i(z_i), a)\ \mathrm{d}\mu \\
&= \sum_{i\in J_n} p_n(i) \left\{\int_X c\ \rho(z_i, x)\ \mathrm{d}\mu + \rho(z_i, a)\right\} < \infty.
\end{aligned}$$

Each mapping M_n is often called the Foias operator. We now present the following two more main results in this thesis.

Theorem 1.14 (**Main Theorem 8** (see [14])). Let $(\{f_i\}_{i\in I}, \{J_n\}_{n\in\mathbb{N}}, \{p_n\}_{n\in\mathbb{N}})$ satisfy the setting (wNAIFS). Suppose that there exists $x_0 \in X$ such that

$$\sum_{n\in\mathbb{N}} \left\{\max_{i\in J_n} \rho(x_0, z_i)\right\} c^n < \infty.$$

Then, there exists the unique sequence of probability measures $\{\nu_m\}_{m\in\mathbb{N}}$ in $\mathcal{P}_1(X)$ such that for each $m \in \mathbb{N}$ and $\mu \in \mathcal{P}_1(X)$,

$$\lim_{n\to\infty} (M_m \circ M_{m+1} \circ \cdots \circ M_{m+n-1})(\mu) = \nu_m \quad \text{in } \mathcal{P}_1(X). \tag{1.4}$$

In addition, for each $m \in \mathbb{N}$, we have

$$\nu_m = M_m(\nu_{m+1}). \tag{1.5}$$

Moreover, suppose that there exists $x_0 \in X$ such that

$$a := \limsup_{n\to\infty} \sqrt[n]{\max_{i\in J_n} \rho(x_0, z_i)} < \frac{1}{c}.$$

Then, for each $m \in \mathbb{N}$, $r \in \{r > 0 \mid c \le r < 1, ac < r\}$ and $\mu \in \mathcal{P}_1(X)$, $(M_m \circ M_{m+1} \circ \cdots \circ M_{m+n-1})(\mu)$ converges to ν_m as n tends to infinity in $\mathcal{P}_1(X)$ exponentially fast with the rate r.

Theorem 1.15 (**Main Theorem 9** (see [14])). Let $(\{f_i\}_{i\in I}, \{J_n\}_{n\in\mathbb{N}}, \{p_n\}_{n\in\mathbb{N}})$ satisfy the setting (wNAIFS). If $\mu \in \mathcal{P}_1(X)$ has a compact support, then $\mathrm{supp}(M_n(\mu)) = F_n(\mathrm{supp}(\mu))$ for each $n \in \mathbb{N}$. In addition, if there exists $x_0 \in X$ such that

$$\sum_{n\in\mathbb{N}} \left\{\max_{i\in J_n} \rho(x_0, z_i)\right\} c^n < \infty,$$

then for each $m \in \mathbb{N}$, we have $\mathrm{supp}(\nu_m) = K_m$ and if $\mu \in \mathcal{P}_1(X)$ has a compact support, then

$$\lim_{n\to\infty} \mathrm{supp}(M_m \circ M_{m+1} \circ \cdots \circ M_{m+n-1}(\mu)) = \mathrm{supp}(\nu_m) \quad \text{in } \mathcal{K}(X). \tag{1.6}$$

Moreover, suppose that there exists $x_0 \in X$ such that

$$a := \limsup_{n\to\infty} \sqrt[n]{\max_{i\in J_n} \rho(x_0, z_i)} < \frac{1}{c},$$

and $\mu \in \mathcal{P}_1(X)$ has a compact support, then for each $r \in \{r > 0 \mid c \le r < 1, ac < r\}$, $\mathrm{supp}(M_m \circ M_{m+1} \circ \cdots \circ M_{m+n-1}(\mu))$ converges to $\mathrm{supp}(\nu_m)$ in $\mathcal{K}(X)$ as n tends to infinity exponentially fast with the rate r.

As we have seen, it is not obvious that we can construct the limit sets of non-autonomous iterated function systems on complete metric spaces which are possibly unbounded. However, the construction of the limit sets of the systems is not the only interest in the study of fractal geometry. Another interest in the study of the fractal geometry is to estimate the dimensions of the fractals and the measures of the fractals. To estimate the Hausdorff dimension of the limit sets of iterated function systems, we usually assume some conditions (the open set condition etc.) on the iterated function systems ([8], [10], [17], [22]). Under one of the conditions, we can analyze the limit sets of the iterated function systems and we can estimate the Hausdorff dimension of them.

However, we sometimes encounter the limit sets of iterated function systems which do not satisfy the open set condition. There exist some results on the estimate of the Hausdorff dimension of the limit sets under another kind of condition for iterated function systems ([25], [26]). Also, there exist some results on the specific case of the iterated function systems (for example, see [5]). The author of this thesis and H. Sumi introduce a different type of method to estimate the Hausdorff dimension of the limit sets of iterated function systems.

Therefore, the third aim of this thesis is to find the estimate of the Hausdorff dimension of the limit sets of iterated function systems which do not satisfy the open set condition. We give another type of estimate of the Hausdorff dimension of the limit sets of iterated function systems without well-known conditions (for example, the open set condition, the transvarsality condition etc.). That is, by using the technique to define the entropy in the ergodic theory, we obtain an upper estimate on the Hausdorff dimesnion of the limit sets of iterated function systems.

The precise statement is the following. Let X be a complete metric space. Let I be a finite set with $|I| = m$ and $\{f_i\}_{i\in I}$ be a (finite) family of the contractive mappings $f_i \colon X \to X$ with contraction constant $c_i \in (0, 1)$. $\{f_i\}_{i\in I}$ is called a (finite) iterated fucntion system (for short, (finite) IFS). Note that by the above argument (original Hutchinson's idea), there exists a non-empty compact set K in X uniquely such that $K = \cup_{i\in I} f_i(K)$. K is called the limit set of an IFS $\{f_i\}_{i\in I}$. For each $l \in \mathbb{N}$ and $w = w_1 w_2 \cdots w_l \in I^l$, we set $f_w := f_{w_1} \circ f_{w_2} \circ \cdots \circ f_{w_l}$ and $|w| := l$.

Definition 1.16. Let $l \in \mathbb{N}$. The compact covering of K is defined by $\alpha_l := \{f_w(K) \mid w \in I^l\}$ and the minimun covering number of α_l is defined by

$$N(\alpha_l) := \min\{k \in \mathbb{N} \mid {}^{\exists} w_1, w_2, \dots, w_k \in I^l \text{ such that } K \subset \cup_{i=1}^{k} f_{w_i}(K)\}.$$

We set $H(\alpha_l) := \log N(\alpha_l)$.

Note that for each $l \in \mathbb{N}$, $N(\alpha_l) \leq m^l$. We set $h(\{f_i\}_{i\in I}) := \lim_{l\to\infty} H(\alpha_l)/l = \inf_{l\in\mathbb{N}} H(\alpha_l)/l$ (see Lemma 6.2). Note that $0 \leq h(\{f_i\}_{i\in I}) \leq \log m$ since for each $l \in \mathbb{N}$, $H(\alpha_l)/l \geq 0$ and $h(\{f_i\}_{i\in I}) \leq H(\alpha_1)/1 \leq \log m$. We now present the last main result in this thesis.

Theorem 1.17 (**Main Theorem 10**)**.** Let X be a complete metric space and let I be a finite set with $|I| = m$. Let $\{f_i\}_{i\in I}$ be a (finite) IFS such that each $f \colon X \to X$ is contractive mapping with contraction constant $c_i \in (0, 1)$ and let K be the limit set of the IFS $\{f_i\}_{i\in I}$. Then, we have $\dim_{\mathcal{H}}(K) \leq h(\{f_i\}_{i\in I})/{-\log}(\max_{i\in I} c_i)$, where $\dim_{\mathcal{H}}(K)$ is the Hausdorff dimension of K. In addition, if $h(\{f_i\}_{i\in I}) = \log m$, then for each $\omega, \tau \in \cup_{i\in\mathbb{N}} I^i$ with $\omega \neq \tau$, we have $f_\omega \neq f_\tau$.

The rest of the paper is organized as follows. In Section 2, we summarize the theory of CIFSs and the theory of the families of CIFSs without proofs. In addition, we give the proofs of some properties of the CIFS of the generalized complex continued fractions. In Section 3, we prove Main Theorems 1, 2, 3 of this thesis. In Section 4, we prove Main Theorems 4, 5 of this thesis. In Section 5, we give the proofs of some basic properties of non-autonomous iterated function systems defined on bounded sets and we prove Main Theorem 6 of this thesis, in which we deal with non-autonomous iterated function systems on (possibly unbounded) complete metric spaces. In addition, we give some examples of the non-autonomous iterated function systems dedined on $\mathbb{R}$. Moreover, we prove Main Theorems 7, 8, 9 of this thesis. In Section 6, we prove Main Theorem 10 of this thesis.

2 Preliminaries

2.1 Iterated function systems and their basic properties

In this subsection, we consider basic properties of infinite IFSs. Through in this subsection, (X, ρ) is a non-empty compact metric space and I is a set which have at least two elements.

Definition 2.1. We say that $S = \{\phi_i : X \to X \mid i \in I\}$ is a IFS if S satisfies the following conditions.

(i) For each $i \in I$, ϕ_i is injective.

(ii) There exists $c > 0$ such that for each $i \in I$ and $x, y \in X$, $\rho(\phi_i(x), \phi_i(y)) \leq c\rho(x, y)$.

We set $I^* := \bigcup_{n \geq 1} I^n$ and $I^\infty := I^{\mathbb{N}}$. We denote by $w_1 w_2 \cdots w_n$ $(w_1, w_2, ..., w_n) \in I^n$. If $w \in I^n$ for some $n \in \mathbb{N}$, we set $|w| := n$ and if $w \in I^\infty$, we set $|w| := \infty$. In addition, for each $w \in I^* \cup I^\infty$ and $n \in \mathbb{N}$ with $n \leq |w|$, we set $w|_n := w_1 \ldots w_n$. Besides, for each $n \in \mathbb{N}$ and $w = w_1 w_2, ..., w_n \in I^n$ for some $n \in \mathbb{N}$, we set

$$\phi_w := \phi_{w_1} \circ \cdots \circ \phi_{w_n}$$

and we set $\phi_{w|_0} :=$ id., where id. is identity map on X.

Proposition 2.2. For all $w \in I^\infty$, $\{\phi_{w|_n}(X)\}_{n \geq 1}$ is a monotone decreasing sequence of compact sets. In addition, we have $\mathrm{diam}(\phi_{w|_n}(X)) \to 0 \quad (n \to \infty)$ uniformly with respect to $w \in I^\infty$, where $\mathrm{diam}(A)$ is the diameter of A defined by $\mathrm{diam}(A) := \sup_{x, y \in A} \rho(x, y)$.

Proof. Note that since for each $i \in I$, $\phi_i \colon X \to X$ is continuous, for each $w \in I^\infty$ and $n \in \mathbb{N}$, $\phi_{w|_n}$ is continuous. Therefore, we have for each $w \in I^\infty$ and $n \in \mathbb{N}$, $\phi_{w|_n}(X)$ is a compact set and since for each $n \in \mathbb{N}$, $\phi_{w|_{n+1}}(X) = \phi_{w|_n}(\phi_{w_{n+1}}(X)) \subset \phi_{w|_n}(X)$, $\{\phi_{w|_n}(X)\}_{n \geq 1}$ is a monotone decreasing sequence. In addition, for each $w \in I^\infty$ and $n \in \mathbb{N}$, we have

$$\mathrm{diam}(\phi_{w|_n}(X)) = \sup_{x, y \in X} \rho(\phi_{w|_n}(x), \phi_{w|_n}(x)) \leq \sup_{x, y \in X} c\rho(\phi_{w|_{n-1}}(x), \phi_{w|_{n-1}}(y)). \tag{2.1}$$

By induction, we obtain that $\mathrm{diam}(\phi_{w|_n}(X)) \leq c^n \mathrm{diam}(X)$ for each $n \in \mathbb{N}$. Since $c^n \mathrm{diam}(X) \to 0$ as $n \to \infty$ uniformly with respect to $w \in I^\infty$, we have proved our lemma. □

Corollary 2.3. $\bigcap_{n=1}^\infty \phi_{w|_n}(X)$ is a single set.

Proof. If $\cap_{n=1}^\infty \phi_{w|_n}(X) = \emptyset$, we have $X \subset \cup_{n=1}^\infty \phi_{w|_n}(X)^c$. Since X is compact, there exists $N \in \mathbb{N}$ such that $X \subset \cup_{n=1}^N \phi_{w|_n}(X)^c = \phi_{w|_N}(X)^c$. Therefore, we have $\phi_{w|_N}(X) = \emptyset$. This contradicts $\phi_{w|_N}(X) \neq \emptyset$. In addition, if $x, y \in \cap_{n=1}^\infty \phi_{w|_n}(X)$, then for each $n \in \mathbb{N}$ we have $\rho(x, y) \leq \mathrm{diam}(\phi_{w|_n}(X)) \leq c^n \mathrm{diam}(X)$ by the inequality (2.1). Therefore, we have proved our corollary. □

Definition 2.4. The coding map $\pi : I^\infty \to X$ for IFS S is defined by $\{\pi(w)\} = \bigcap_{n=0}^\infty \phi_{w|_n}(X)$ for each $w \in I^\infty$.

We endow I with the discrete topology, and endow $I^\infty := I^{\mathbb{N}}$ with the product topology. Note that a basis of the topology for $I^{\mathbb{N}}$ is the set of subsets

$$V_m(\alpha) = \{\beta \in I^\infty | \beta_n = \alpha_n, n = 1, \cdots, m\} \ (\alpha \in I^\infty, m \in \mathbb{N}).$$

We set a metric d on I defined by $d(x,y)=0$ if $x=y$ and $d(x,y)=1$ if $x\neq y$. Let γ be a metric on I^∞ defined by

$$\gamma(\alpha,\beta):=\sum_{n=1}^{\infty}\frac{d(\alpha_n,\beta_n)}{2^n}\ (\alpha=(\alpha_n),\beta=(\beta_n)\in I^\infty).$$

The following lemma is deduced by the the theory of general topology.

Lemma 2.5. Let Δ_γ be the topology induced by the metric γ and Δ_I be a the product topology of I^∞. Then, we have $\Delta_\gamma=\Delta_I$.

In addition, we obtain the following property of I^∞.

Lemma 2.6. If I is a finite set, then I^∞ is a compact metric space. If I is a infinite set, I^∞ is a complete separable metric space.

Proof. If I is finite, then I^∞ is also compact by the Tychonoff theorem. We assume that I is infinite. Let $\omega\in I$. We set

$$I'_n:=\{w\in I^\infty\mid w_m=\omega\ (m\geq n+1)\}\quad\text{and}\quad I':=\bigcup_{n\in\mathbb{N}}I'_n.$$

Note that since for each $n\in\mathbb{N}$, $|I'_n|=|I^n|$ and I^n is countable, I' is countable. Let $\epsilon>0$ and $N\in\mathbb{N}$ with $\sum_{n=N+1}^{\infty}1/2^n<\epsilon$. For each $w\in I^\infty$, we set $\tau_w=w_1\cdots w_N\omega\omega\cdots\in I^\infty$. We have $\tau_w\in I'_N\subset I'$ and

$$\gamma(w,\tau_w)=\sum_{n=1}^{\infty}\frac{d(w_n,(\tau_w)_n)}{2^n}\leq\sum_{n=N+1}^{\infty}\frac{d(w_n,\omega_n)}{2^n}\leq\sum_{n=N+1}^{\infty}\frac{1}{2^n}\leq\epsilon.$$

Therefore, we have proved the separability of I^∞.

We next show that Completeness of I^∞. Let $\{w_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence in I^∞. For each $k\in\mathbb{N}$, we denote w_k by $\{w_{k,n}\}_{n\in\mathbb{N}}$. Note that for each $n\in\mathbb{N}$, $\{w_{k,n}\}_{k\in\mathbb{N}}\subset I$ is a Cauchy sequence in I since $d(w_{k,n},w_{l,n})/2^n\leq\gamma(w_k,w_l)$ for each $k,l\in\mathbb{N}$. In addition, since the metric d takes values 0 or 1, there exists $K(n)\in\mathbb{N}$ such that for each $k\geq K(n)$ we have $w_{k,n}=w_{K(n),n}$. We set $w_\infty=\{w_{K(n),n}\}_{n\in\mathbb{N}}\in I^\infty$. Let $\epsilon>0$ and $L\in\mathbb{N}$ with $\sum_{n=L+1}^{\infty}1/2^n<\epsilon$. We set $K:=\max_{n=1,2,\dots,L}K(n)$. Since for each $k\geq K$, we have $w_{\infty,n}=w_{K(n),n}=w_{k,n}$ for each $n=1,\dots,L$, for all $k\geq K$, we have

$$\gamma(w_\infty,w_k)=\sum_{n=1}^{\infty}\frac{d(w_{K(n),n},w_{k,n})}{2^n}\leq 0+\sum_{n=L+1}^{\infty}\frac{d(w_{K(n),n},w_{k,n})}{2^n}\leq\sum_{n=L+1}^{\infty}\frac{1}{2^n}\leq\epsilon.$$

Therefore, we have proved our lemma. □

Lemma 2.7. The coding map $\pi\colon I^\infty\to X$ is uniformly continuous.

Proof. Let $\epsilon>0$. Note that there exists $N\in\mathbb{N}$ such that $c^N\mathrm{diam}(X)<\epsilon$. We set $\delta:=1/2^N$. Let $w,\tau\in I^\infty$ with $\gamma(w,\tau)<\delta$. Then, we have for each $i=1,2,\dots,N$, we have $w_i=\tau_i$. Therefore, we have $\rho(\pi(w),\pi(\tau))\leq\mathrm{diam}(\phi_{w|_n}(X))\leq c^n\mathrm{diam}(X)<\epsilon$. Thus, we have proved our lemma. □

Definition 2.8. the (left) shift map $\sigma\colon I^\infty\to I^\infty$ is defined by

$$\sigma(w)=w_2w_3\cdots\ (w=w_1w_2\cdots\in I^\infty).$$

Proposition 2.9. The left shift map is Lipshitz continuous on I^∞ with Lipshitz constant 2.

Proof. Let $w = \{w_n\}_{n\in\mathbb{N}}, \tau = \{\tau_n\}_{n\in\mathbb{N}} \in I^\infty$. Then, we have

$$\gamma(\sigma(w),\sigma(\tau)) = \sum_{n=1}^{\infty} \frac{d(w_{n+1},\tau_{n+1})}{2^n} = 2\sum_{n=2}^{\infty} \frac{d(w_n,\tau_n)}{2^n} \le 2\gamma(w,\tau)$$

and if $w_1 = \tau_1$, then $\gamma(\sigma(w),\sigma(\tau)) = 2\gamma(w,\tau)$. Therefore, we have proved our lemma. □

Lemma 2.10. For each $w = w_1w_2\cdots \in I^\infty$ and $i \in I$, we have

$$\phi_i(\pi(w)) = \pi(iw) \quad \text{and} \quad \pi(w) = \phi_{w_1}(\pi(\sigma(w))).$$

Proof. Let $w = w_1w_2\cdots \in I^\infty$ and $i \in I$. Since $\phi_i(\cap_{n=1}^{\infty}\phi_{w|_n}(X)) \subset \cap_{n=1}^{\infty}\phi_i(\phi_{w|_n}(X)) = \cap_{n=1}^{\infty}\phi_{(iw)|_n}(X)$, we have $\{\phi_i(\pi(w))\} \subset \{\pi(iw)\}$. Therefore, we obtain that $\phi_i(\pi(w)) = \pi(iw)$ for each $w \in I^\infty$ and $i \in I$. Since $\phi_i(\pi(w)) = \pi(iw)$ for each $w = w_1w_2\cdots \in I^\infty$ and $i \in I$, we have

$$\phi_{w_1(\pi(\sigma(w)))} = \phi_{w_1}(\pi(w_2w_3\cdots)) = \pi(w_1w_2w_3\cdots) = \pi(w).$$

Therefore, we have proved our lemma. □

Since ϕ_i is injective, we obtain that the following corollary.

Corollary 2.11. For each $w = w_1w_2\cdots \in I^\infty$, we have $\pi\circ\sigma(w) = \phi_{w_1}^{-1}\circ\pi(w)$.

Definition 2.12. Let $S = \{\phi_i : X \to X | i \in I\}$ be a IFS. The limit set of S is defined by $J_S := \pi(I^\infty) = \bigcup_{w\in I^\infty}\bigcap_{n=1}^{\infty}\phi_{w|_n}(X)$.

Note that if I is a finite set, then J_S is compact since I^∞ is compact and π is continuous.

Lemma 2.13. Let J_S be the limit set of S. Then, we have $J_S = \bigcup_{i\in I}\phi_i(J_S)$.

Proof. Let $\pi(w) \in J_S$ $(w = w_1w_2\cdots \in I^\infty)$. By Lemma 2.10, we have

$$\pi(w) = \phi_{w_i}(\pi(\sigma(w))) \in \phi_{w_1}(J_S) \subset \bigcup_{i\in I}\phi_i(J_S).$$

On the other hand, let $x \in \cup_{i\in I}\phi_i(J_S)$. Note that there exists $i_0 \in I$ such that $x \in \phi_{i_0}(J_S)$. By lemma 2.10, there exists $i_0 \in I$ and $w \in I^\infty$ such that

$$x = \phi_{i_0}(\pi(w)) = \pi(i_0w) \in J_S.$$

Therefore, we have proved our lemma. □

By induction with respect to $n \in \mathbb{N}$, we obtain that $J_S = \cup_{w\in I^n}\phi_w(J_S)$ for each $n \in \mathbb{N}$.

Definition 2.14. We say that IFS $S = \{\phi_i : X \to X \mid i \in I\}$ is pointwise finite if for each $x \in X$, we have $|\{i \in I \mid x \in \phi_i(X)\}| < \infty$.

Note that if I is finite, then IFS $S = \{\phi_i : X \to X | i \in I\}$ is pointwise finite.

Proposition 2.15. Let $S = \{\phi_i : X \to X \mid i \in I\}$ be a pointwise finite IFS. Then, for each $x \in X$ and $n \ge 1$, we have $|\{w \in I^n \mid x \in \phi_w(X)\}| < \infty$.

Proof. We prove Theorem 2.15 by induction with respect to $n \in \mathbb{N}$. By the definition of the pointwise finite, the statement in Theorem 2.15 holds if $n = 1$. We assume that the statement holds if $n \in \mathbb{N}$. Let $x \in X$. We set $I_x^n := \{w \in I^n \mid x \in \phi_w(X)\}$. Note that $|I_x^n| < \infty$. We set $\tilde{I} := \{\tau \in I_x^n \mid \phi_\tau^{-1}(\{x\}) \neq \emptyset\}$. Note that for each $\tau \in \tilde{I}$, $x \in \phi_\tau(X)$ and ϕ_τ is injective. We set $I' := \cup_{\tau \in \tilde{I}} I^1_{\phi_\tau^{-1}(x)}$.

We show that $I_x^{n+1} \subset I_x^n \times I'$. Indeed, let $w = w_1 w_2 \cdots w_{n+1} \in I_x^{n+1}$. Note that since $\phi_w(X) \subset \phi_{w|_n}(X)$, we have $w|_n \in I_x^n$. In addition, since there exists $y \in X$ such that $\phi_w(y) = x$, we have $\phi_{w|_n}^{-1}(x) = \phi_{w_{n+1}}(y) \in \phi_{w_{n+1}}(X)$. Therefore, we have $w|_n \in I_x^n$, and $w_{n+1} \in I_{\phi_{w|_n}^{-1}(x)} \subset I'$ and we obtain that $I_x^{n+1} \subset I_x^n \times I'$. Moreover, since $|\tilde{I}| \leq |I_x^n| < \infty$ and for each $\tau \in \tilde{I}$, $|I^1_{\phi_\tau^{-1}(x)}| < \infty$, we have $|I'| < \infty$. It follows that $|I_x^{n+1}| \leq |I_x^n \times I'| < \infty$. Thus, we have proved our proposition. □

Proposition 2.16. Let $S = \{\phi_i : X \to X | i \in I\}$ be a pointwise finite IFS. Then, we have

$$J_S = \bigcap_{n \in \mathbb{N}} \bigcup_{w \in I^n} \phi_w(X).$$

In particular, J_S is a Borel subset in X.

Proof. Let $\pi(w) \in J_S$ $(w \in I^\infty)$. Then, for each $n \in \mathbb{N}$, we have $x \in \phi_{w|_n}(X) \subset \cup_{w \in I^n} \phi_w(X)$. It follows that $x \in \cap_{n \in \mathbb{N}} \cup_{w \in I^n} \phi_w(X)$. On the other hand, let $x \in \cap_{n \in \mathbb{N}} \cup_{w \in I^n} \phi_w(X)$. Then, for each $n \in \mathbb{N}$, there exists $w(n) \in I^n$ such that $x \in \phi_{w(n)}(X)$. By Proposition 2.15, there exists $\alpha_1 \in I$ such that $|\{n \in \mathbb{N} \mid w(n)_1 = \alpha_1\}| = \infty$. We denote $\{n \geq 2 \mid w(n)_1 = \alpha_1\}$ by $\{n(k) \mid k \in \mathbb{N}\}$ with $n(k) < n(k+1)$ for each $k \in \mathbb{N}$. Since for each $k \in \mathbb{N}$, $x \in \phi_{\alpha_1 w_2(n(k))}(X)$, there exists $\alpha_2 \in I$ such that $|\{k \in \mathbb{N} \mid w(n(k))_2 = \alpha_1\}| = \infty$. By induction, there exists $\alpha := \alpha_1 \alpha_2 \cdots \in I^\infty$ such that for each $n \in \mathbb{N}$, we have $x \in \phi_{\alpha|_n}(X)$ and we deduce that $x \in \cap_{n \in \mathbb{N}} \phi_{\alpha|_n}(X) = \{\pi(\alpha)\} \subset J_S$. Therefore, we have proved $J_S = \bigcap_{n \in \mathbb{N}} \bigcup_{w \in I^n} \phi_w(X)$. Note that for each $n \in \mathbb{N}$, I^n is countable set and for each $w \in I^n$, $\phi_w(X)$ is compact in metric space X. Thus, we have proved our proposition. □

Definition 2.17. Let I be a set with $|I| = \infty$ and $I' \subset I$ with $|I'| = \infty$. We say that $x_i \in X$ $(i \in I')$ converges to x if for each $\epsilon > 0$, there exists $F' \subset I'$ with $|F'| < \infty$ such that for each $i \in I' \setminus F'$, $\rho(x_i, x) < \epsilon$.

We set $X_S(\infty) := \{\lim_{i \in I'} z_i \in X | \ I' \subset I \text{ with } |I'| = \infty \text{ and } z_i \in \phi_i(X) \ (i \in I')\}$.

Lemma 2.18. If $\lim_{i \in I} \operatorname{diam}(\phi_i(X)) = 0$, then we have $\overline{J_S} = J_S \cup \bigcup_{w \in I^*} \phi_w(X(\infty)) \cup X(\infty)$.

Proof. We first show $X_S(\infty) \subset \overline{J_S}$. Let $x = \lim_{i \in I'} x_i \in X_S(\infty)$ with $(i \in I', x_i \in \phi_i(X))$. Let $\epsilon > 0$. Then, there exists finite subset $F_1 \subset I'$ such that for each $i \in I' \setminus F_1$, $\rho(x_i, x) < \epsilon/2$. In addition, since $\lim_{i \in I} \operatorname{diam}(\phi_i(X)) = 0$, there exists finite subset $F_2 \subset I$ such that for each $i \in I \setminus F_2$, $\operatorname{diam}(\phi_i(X)) < \epsilon/2$. We set $F := F_1 \cup F_2$. Note that $|F| < \infty$. Let $i \in I' \setminus F$. Since there exists $y \in \phi_i(J_S) \subset J_S$ such that

$$\rho(x, y) \leq \rho(x, x_i) + \rho(x_i, y) \leq \frac{\epsilon}{2} + \operatorname{diam}(\phi_i(X)) < \epsilon.$$

We deduce that $x \in \overline{J_S}$ and $X(\infty) \subset \overline{J_S}$. Since ϕ_w is continuous for each $w \in I^*$ and $J_S = \cup_{w \in I^n} \phi_w(J_S)$ for each $n \in \mathbb{N}$,

$$\phi_w(X(\infty)) \subset \phi_w(\overline{J_S}) \subset \overline{\phi_w(J_S)} \subset \overline{J_S}.$$

On the other hand, let $x \in \overline{J_S}$. Note that there exist $\pi(w(n)) \in J_S = \pi(I^\infty)$ $(n \in \mathbb{N})$ such that $\lim_{n\to\infty} \pi(w(n)) = x$. We show that if $\{w(n)_1 \mid n \in \mathbb{N}\}$ is infinite, then $x \in X(\infty)$. Indeed, We set $I(1) := \{w(n)_1 \mid n \in \mathbb{N}\}$. Since for each $n \in \mathbb{N}$, $\pi(w(n)) \in \phi_{w(n)_1}(X)$, if $|I(1)| = \infty$, then there exists $y(n) \in X$ such that $\pi(w(n)) = \phi_{w(n)_1}(y(n))$. Therefore, we deduce that $\lim_{w(n)_1 \in I(1)} \phi_{w(n)_1}(y(n)) = x$, which is equivalent to $x \in X(\infty)$. We assume that $I(1) := \{w(n)_1 \mid n \in \mathbb{N}\}$ is finite, then there exists $u_1 \in I$ such that $|\{n \in \mathbb{N} \mid w(n)_1 = u_1\}| = \infty$. We set $N_1 := \{n \in \mathbb{N} \mid w(n)_1 = u_1\}$. If $\{w(n)_2 \mid n \in N_1\}$ is infinite, then $x \in \phi_{u_1}(X(\infty))$ by the above same procedure. If $\{w(n)_2 \mid n \in N_1\}$ is finite, then there exists $u_2 \in I$ such that $|\{n \in N_1 \mid w(n)_2 = u_2\}| = \infty$. We set $N_2 := \{n \in N_1 \mid w(n)_2 = u_2\}$. By induction with respect to $m \in \mathbb{N}$, we deduce that if there exists $m \in \mathbb{N}$, N_{m+1} is infinite, we have $x \in \phi_{u_1 \cdots u_m}(X(\infty))$.

We assume that for all $m \in \mathbb{N}$, $N_{m+1} = \{n \in N_m \mid w(n)_{m+1} = u_{m+1}\}$ is finite. We set $u := u_1 u_2 \cdots \in I^\infty$. We show that for each $m \in \mathbb{N}$, $x \in \phi_{u|_m}(X)$. Indeed, if there exists $m_0 \in \mathbb{N}$, $x \notin \phi_{u|_{m_0}}(X)$. By induction, there exists a subsequence $\{j_n\}$ in $\mathbb{N}$ such that $j_n \in N_n$ and $j_{n+1} > j_n$ for each $n \in \mathbb{N}$. Note that $\lim_{n\to\infty} \pi(w(j_n)) = x$ since $\lim_{n\to\infty} \pi(w(n)) = x$. Since for each $m \in \mathbb{N}$ with $j_m \geq m_0$, $\pi(w(j_m)) \in \phi_{u|_{m_0}}(X)$ and $\phi_{u|_{m_0}}(X)$ is compact, we have $x \in \phi_{u|_{m_0}}(X)$. This contradicts there exists $m_0 \in \mathbb{N}$, $x \notin \phi_{u|_{m_0}}(X)$. Therefore, we have $x \in \cap_{m\in\mathbb{N}} \phi_{u|_m}(X) \subset J_S$ and we have proved our lemma. □

2.2 Conformal iterated function systems, pressure functions and conformal measures

Definition 2.19 (Conformal iterated function system). Let $X \subset \mathbb{R}^d$ be a non-empty compact and connected set with the Euclidean norm $|\cdot|$ and let I be a finite set or bijective to $\mathbb{N}$. Suppose that I has at least two elements. We say that $S := \{\phi_i : X \to X \mid i \in I\}$ is a conformal iterated function system (for short, CIFS) if S satisfies the following conditions.

(i) Injectivity: For all $i \in I$, $\phi_i : X \to X$ is injective.

(ii) Uniform Contractivity: There exists $c \in (0, 1)$ such that, for all $i \in I$ and $x, y \in X$, the following inequality holds.

$$|\phi_i(x) - \phi_i(y)| \leq c|x - y|.$$

(iii) Conformality: There exists a positive number ϵ and an open and connected subset $V \subset \mathbb{R}^d$ with $X \subset V$ such that for all $i \in I$, ϕ_i extends to a $C^{1+\epsilon}$diffeomorphism on V and ϕ_i is conformal on V i.e. for each $x \in V$ and $i \in I$, there exists $C_i(x) > 0$ such that for each $u, v \in \mathbb{R}^d$,

$$|f'(x)u - f'(x)v| = C_i(x)|u - v|.$$

(iv) Open Set Condition(OSC): For all $i, j \in I$ $(i \neq j)$, $\phi_i(\mathrm{Int}(X)) \subset \mathrm{Int}(X)$ and $\phi_i(\mathrm{Int}(X)) \cap \phi_j(\mathrm{Int}(X)) = \emptyset$. Here, $\mathrm{Int}(X)$ denotes the set of interior points of X with respect to the topology in $\mathbb{R}^d$.

(v) Bounded Distortion Property(BDP): There exists $K \geq 1$ such that for all $x, y \in V$ and for all $w \in I^* := \bigcup_{n=1}^\infty I^n$, the following inequality holds.

$$|\phi'_w(x)| \leq K \cdot |\phi'_w(y)|.$$

Here, for each $n \in \mathbb{N}$ and $w = w_1 w_2 \cdots w_n \in I^n$, we set $\phi_w := \phi_{w_1} \circ \phi_{w_2} \circ \cdots \circ \phi_{w_n}$ and $|\phi'_w(x)|$ denotes the norm of the derivative of ϕ_w at $x \in X$ with respect to the Euclidean norm on $\mathbb{R}^d$.

(vi) Cone Condition: For all $x \in \partial X$, there exists an open cone $\mathrm{Cone}(x, u, \alpha)$ with a vertex x, a direction u, an altitude $|u|$ and an angle α such that $\mathrm{Cone}(x, u, \alpha)$ is a subset of $\mathrm{Int}(X)$.

Note that if S is a CIFS, then S is an IFS. For each $f: Y \to \mathbb{R}^d$ with C^1 class defined on a subset $Y \subset \mathbb{R}^d$, we set $||f||_Y := \sup\{|f'(y)| \mid y \in Y\}$.

Remark 2.20. Since for each $i \in I$, ϕ_i is conformal, $\phi_i'(x)$ is similitude for each $x \in V$. In addition, we have for each $u, v \in \mathbb{R}^d$ with $u \neq 0$ and $v \neq 0$, we have

$$|\phi_i'(x)| = C_i(x), \quad \frac{\langle \phi_i(u), \phi_i(v) \rangle}{|\phi_i(u)| \cdot |\phi_i(v)|} = \frac{\langle u, v \rangle}{|u| \cdot |v|} \quad \text{and} \quad |(\phi_i'(u))^{-1}| = |(\phi_i'(u))|^{-1},$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^d$.

We consider the basic properties of the Bounded Distortion Property. Since X is compact and ∂V is closed in $\mathbb{R}^d$, we have the following lemma.

Lemma 2.21. Let S be a CIFS on $X \subset \mathbb{R}^d$ and $V \subset \mathbb{R}^d$ be a open subset which satisfies the definition of CIFS S. Then, we have $\rho(X, \partial V) > 0$, where ρ is the Euclidean metric on $V \subset \mathbb{R}^d$.

Lemma 2.22 ((BDP.1))**.** Let S be a CIFS on $X \subset \mathbb{R}^d$ and $V \subset \mathbb{R}^d$ be a open subset which satisfies the definition of CIFS S. Let B be a convex set in V, $w \in I^*$ and $r \in (0, \rho(X, \partial V))$. Let V' be a open set with $B \subset V' \subset V$ and $B(x, r) \subset V'$ for each $x \in X$. Then, for each $x \in X$, we have

$$\mathrm{diam}(\phi_w(B)) \leq ||\phi_w'||_{V'} \mathrm{diam}(B) \quad \text{and} \quad \phi_w(B(x, r)) \subset B(\phi_w(x), ||\phi_w'||_{V'}).$$

Lemma 2.23. Let S be a CIFS on $X \subset \mathbb{R}^d$ and $V \subset \mathbb{R}^d$ be a open subset which satisfies the definition of CIFS S. Then, there exists $q \in \mathbb{N}$, $x_1, x_2, \ldots, x_q \in X$ and $r_i \in (0, \rho(X, \partial V))$ $i = 1, 2, \ldots, q$ such that for all $j = 1, 2, \ldots, q-1$,

$$X \subset \bigcup_{i=1}^{q} B(x_i, r_i) \quad \text{and} \quad B(x_i, r_i) \cap B(x_{i+1}, r_{i+1}) \neq \emptyset.$$

In particular, we retake a open subset $V_0 := \cup_{i=1}^q B(x_i, r_i) \subset \mathbb{R}^d$ which satisfies the definition of CIFS S. In addition, we have $X \subset V_0 \subset V$ and V_0 is connected.

Proposition 2.24 ((BDP.2))**.** Let S be a CIFS on $X \subset \mathbb{R}^d$ and $V_0 \subset \mathbb{R}^d$ be a open connected subset which we defined in Lemma 2.23. Then, for each $D \geq \max\{q, \mathrm{diam}(V_0)\}$ and $w \in I^*$, we have $\mathrm{diam}(\phi_w(V_0)) \leq D||\phi_w'||_{V_0}$.

Proposition 2.25. Let S be a CIFS on $X \subset \mathbb{R}^d$ and $V_0 \subset \mathbb{R}^d$ be the open connected subset which we define in Lemma 2.23. Let $x \in X$, $r \in (0, \rho(X, \partial V))$ and $w \in I^*$. Then, $R := \max\{t \geq 0 \mid B(\phi_w(x), t) \subset \phi_w(B(x, r))\}$ exists and $\partial B(\phi_w(x), R) \cap \partial \phi_w(B(x, r)) \neq \emptyset$. In addition, we have

$$\phi_w^{-1}(B(\phi_w(x), R)) \subset B(x, R||(\phi_w^{-1})'||_{\phi_w(V_0)}) \subset B(xd, KR||(\phi_w)'||_{V_0}^{-1}).$$

Moreover, we have

$$KR||(\phi_w)'||_{V_0}^{-1} \geq r \quad \text{and} \quad \phi_w(B(x, r)) \supset B(\phi_w(x), K^{-1}r||(\phi_w)'||_{V_0}).$$

We set $\eta := \rho(X, \partial V)$, where ρ is the Euclidean metric on $V \subset \mathbb{R}^d$.

Proposition 2.26 ((BDP.4))**.** Let S be a CIFS on $X \subset \mathbb{R}^d$ and $V_0 \subset \mathbb{R}^d$ be the open connected subset which we define in Lemma 2.23. Then, there exists $D > 0$ such that for each $w \in I^*$ and $x \in \mathrm{Int}(X)$, we have

$$\mathrm{diam}(\phi_w(V_0)) \geq D^{-1}||(\phi_w)'||_{V_0} \quad \text{and} \quad \phi_w(V_0) \supset B(\phi_w(x), K^{-1}\eta||(\phi_w)'||_{V_0}).$$

In addition, for each $x \in \mathrm{Int}(X)$, there exists $D'_x > 0$ such that

$$\mathrm{diam}(\phi_w(X)) \geq (D'_x)^{-1}||(\phi_w)'||_{V_0} \quad \text{and} \quad \phi_w(X) \supset B(\phi_w(x), (D'_x)^{-1}||(\phi_w)'||_{V_0}).$$

We consider some properties deduced by the Cone Condition.

Lemma 2.27. Let S be a CIFS on $X \subset \mathbb{R}^d$ and $V_0 \subset \mathbb{R}^d$ be the open connected subset which we define in Lemma 2.23. Then, there exists $D \geq 1$ and $\beta \in (0, \alpha)$ such that for each $x \in X$ and $w \in I^*$, there exists $u \in S^{d-1}$ such that

$$\phi_w(\mathrm{Int}(X)) \supset \mathrm{Cone}(\phi_w(x), D^{-1}||(\phi_w)'||_{V_0}u, \beta) \supset \mathrm{Cone}(\phi_w(x), D^{-2}\mathrm{diam}(X)u, \beta).$$

Lemma 2.28. Let S be a CIFS on $X \subset \mathbb{R}^d$ and $V_0 \subset \mathbb{R}^d$ be the open connected subset which we define in Lemma 2.23. Then, we have $\sum_{i \in I} ||\phi'_i||^d_{V_0} \leq K^d$.

Note that by Lemma 2.28, we deduce that $\lim_{i \in I} \mathrm{diam}(\phi_i(X)) = 0$.

Lemma 2.29. Let S be a CIFS on $X \subset \mathbb{R}^d$. Then, we have $X = \overline{\mathrm{Int}(X)}$.

Definition 2.30. Let $w, \tau \in I^*$. We say that w is a extension of τ if there exists $\alpha \in I^*$ such that $w = \tau\alpha$. We say that w is comparable with τ if w is a extension of τ or τ is a extension of w, which is denoted by $w \approx \tau$. We say that w and τ are mutually incomparable if w is not comparable, which is denoted by $w \not\approx \tau$.

The following lemma shows that for all CIFS S, S is pointwise finite.

Lemma 2.31. Let S be a CIFS on $X \subset \mathbb{R}^d$. Suppose that $\pi_n^{-1}(x) \subset I^n$ satisfies the following conditions: for each $w, \tau \in \pi_n^{-1}(x)$,

(i) $x \in \phi_w(X)$.

(ii) if $w \approx \tau$, then $w = \tau$.

Then, we have $|\pi_n^{-1}(x)| \leq 1/\beta$, where β is introduced by Proposition 2.27. In particular, for each $x \in X$, we have

$$|\{w \in I \mid x \in \phi_i(X)\}| \leq \frac{1}{\beta}.$$

Lemma 2.32. Let S be a CIFS on $X \subset \mathbb{R}^d$. Suppose that $F(x, r) \subset I^*$ satisfies the following conditions: for each $w, \tau \in F(x, r)$,

(i) $B(x, r) \cap \phi_w(X) \neq \emptyset$ and $\mathrm{diam}(\phi_w(X)) \geq r$.

(ii) if $w \approx \tau$, then $w = \tau$.

Then, we have $|F(x, r)| \leq D^{2d}\beta^{-1}(1 + D^{-2})^d$, where β and D is introduced in Proposition 2.27.

For each IFS S, we set $h_S := \dim_{\mathcal{H}} J_S$, where $\dim_{\mathcal{H}}$ denote the Hausdorff dimension.

2.3 The pressure function for CIFSs and the Hausdorff dimension of the limit sets of CIFSs

For any CIFS S, we define the pressure function of S as follows.

Definition 2.33 (Pressure function)**.** For each $n \in \mathbb{N}$, $[0, \infty]$-valued function ψ_S^n is defined by

$$\psi_S^n(t) := \sum_{w \in I^n} ||\phi'_w(z)||_X^t \quad (t \geq 0).$$

We set $\theta_S := \inf\{t \geq 0 | \ \psi_S^1(t) < \infty\}$ and $F(S) := \{t \geq 0 \mid \psi_S^1(t) < \infty\}$. Note that by the following lemma, we deduce that $F(S) = (\theta_S, \infty)$ or $F(S) = [\theta_S, \infty)$.

Lemma 2.34. Let S be a CIFS. Then, $\psi_S^1(t)$ is non-incerasing on $[0, \infty)$ and decreasing and convex on $F(S)$. In addition, we have $\psi_S^1(d) \leq K^d$. In particular, $\theta_S \leq d$.

Proposition 2.35. Let S be a CIFS. For all $m, n \in \mathbb{N}$ and $t \geq 0$, we have

$$K^{-2t}\psi_S^k(t)\psi_S^n(t) \leq \psi_S^{m+n}(t) \leq \psi_S^m(t)\psi_S^n(t).$$

In particular, $\psi_S^n(t) < \infty$ for each $n \in \mathbb{N}$ is equivalent to $\psi_S^n(t) < \infty$ for some $n \in \mathbb{N}$ (or $n = 1$), and for all $t \geq 0$, $\log \psi_S^n(t)$ is subadditive with respect to $n \in \mathbb{N}$.

By subadditivity of $\log \psi_S^n(t)$, we define the pressure function of S as follows.

Definition 2.36. The function $P_S \colon [0, \infty) \to (-\infty, \infty]$ is called the pressure function of S, which is defined by

$$P_S(t) := \lim_{n \to \infty} \frac{1}{n} \log \psi_S^n(t) \in (-\infty, \infty] \ (t \geq 0).$$

Proposition 2.37. Let S be a CIFS and P_S be the pressure function of S. Then, for each $t \geq 0$, $P_S(t) < \infty$ is equivalent to $\psi_S^1(t) < \infty$. In particular, we have $\theta_S = \inf\{t \geq 0 \mid P_S(t) < \infty\}$.

Proposition 2.38. Let S be a CIFS and P_S be the pressure function of S. Then, for each $x \in V$ and $t \geq 0$,

$$P_S(t) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{w \in I^n} |\phi'_w(z)(x)|^t.$$

In addition, if $t \in F(S)$, then $(1/n) \log \sum_{w \in I^n} |\phi'_w(z)(x)|^t$ converges $P_S(t)$ as $n \to \infty$ uniformly with respect to $x \in V$.

Proposition 2.39. Let S be a CIFS and P_S be the pressure function of S. Then, P_S is non-increasing on $[0, \infty)$ and decreasing and convex on $F(S)$.

Note that $P_S(0) = \infty$ is equivalent to I is infinite. By using the pressure function, we define some properties of CIFSs.

Definition 2.40 (Regular, Strongly regular, Hereditarily regular)**.** Let S be a CIFS. We say that S is regular if there exists $t \geq 0$ such that $P_S(t) = 0$. We say that S is strongly regular if there exists $t \geq 0$ such that $P_S(t) \in (0, \infty)$. We say that S is hereditarily regular if, for all $I' \subset I$ with $|I \setminus I'| < \infty$, $S' := \{\phi_i \colon X \to X \mid i \in I'\}$ is regular. Here, for any set A, we denote by $|A|$ the cardinality of A.

Note that if a CIFS S is hereditarily regular, then S is strong regular and if S is strong regular, then S is regular. We set $F(I) := \{F \subset I | \ 2 \leq |F| < \infty\}$. For each $F \in F(I)$, we set $S_F := \{\phi_i \colon X \to X | \ i \in F\}$. Mauldin and Urbański showed the following results.

Theorem 2.41 ([22] Theorem 3.15)**.** Let S be a CIFS. Then we have

$$h_S = \inf\{t \geq 0 \mid P_S(t) < 0\} = \sup\{h_{S_F} | \ F \in F(I)\} \geq \theta_S.$$

Moreover, if there exists $t \geq 0$ such that $P_S(t) = 0$, then t is the unique zero of the pressure function P_S and we have $t = h_S$.

Theorem 2.42 ([22] Theorem 3.20)**.** Let I be infinite and let S be a CIFS. Then, the following conditions are equivalent:

(i) S is hereditarily regular.

(ii) $\psi_S^1(\theta_S) = \infty$.

Especially, if S is hereditarily regular, we have $\theta_S < h_S$.

If CIFS S is regular, then we obtain a following a "nice" probability measure on J_S, which is called h_S-conformal measure of S. In fact, by the existence of the conformal measure of CIFS, we deduce Theorem 2.41.

Proposition 2.43 ([22] Lemma 3.13)**.** Let S be a CIFS. If S is regular, then there exists the unique Borel probability measure m_S on X such that the following properties hold.

(i) $m_S(J_S) = 1$.

(ii) For all Borel subset A on X and $i \in I$, $m_S(\phi_i(A)) = \int_A |\phi_i'|^{h_S} \mathrm{d}m_S$.

(iii) For all $i, j \in I$ with $i \neq j$, $m_S(\phi_i(X) \cap \phi_j(X)) = 0$.

We call m_S the h_S-conformal measure of S. By the existence of the conformal measure of CIFSs, we obtain the following four theorems.

Theorem 2.44 ([22] Theorem 4.5)**.** Let S be a regular CIFS and λ_d be the d-dimensional Lebesgue measure. If $\lambda_d(\mathrm{Int}(X) \setminus X_1) > 0$, then $h_S < d$. Here, $X_1 := \cup_{i \in I} \phi_i(X)$.

Theorem 2.45 ([22] Theorem 4.9)**.** Let S be a regular CIFS and m_S be the h_S-conformal measure of S. We set $r_0 := \mathrm{dist}(X, \partial V)$. If there exist a sequence of $\{z_j\}_{j=1}^{\infty}$ in $X_S(\infty)$ and a sequence $\{r_j\}_{j=1}^{\infty}$ in $(0, r_0)$ such that

$$\limsup_{j \to \infty} \frac{m_S(B(z_j, r_j))}{r_j^{h_S}} = \infty,$$

then we have $\mathcal{H}^{h_S}(J_S) = 0$.

Theorem 2.46 ([22] Lemma 4.3)**.** Let S be a regular CIFS. If $J_S \cap \mathrm{Int}(X) \neq \emptyset$, then we have $\mathcal{P}^{h_S}(J_S) > 0$.

Theorem 2.47 ([22] Lemma 4.10)**.** Let S be a regular CIFS and m_S be the h_S-conformal measure of S. Soppose that there exist $L > 0$, $\xi > 0$ and $\gamma \geq 1$ such that for all $b \in I$ and $r > 0$ with $\gamma \mathrm{diam} \phi_b(X) \leq r \leq \xi$, there exists $y \in \phi_b(V)$ such that $m_S(B(y, r)) \geq L r^{h_S}$. Then, we have $\mathcal{P}^{h_S}(J_S) < \infty$.

2.4 Families of conformal iterated function systems

We now consider families of CIFSs. Let CIFS(X, I) be the family of all CIFSs with $X \subset \mathbb{C}$ and an infinite alphabet I. For each $S \in$ CIFS(X,I), let $\pi_S \colon I^\infty \to X$ be the coding map of S. In this paper, for any sequence $\{S^n\}_{n\in\mathbb{N}}$ in CIFS(X,I) and $S \in$ CIFS(X,I), we write $\lambda(\{S^n\}_{n\in\mathbb{N}}) = S$ if the following conditions are satisfied.

(L1) For every $i \in I$, $\lim_{n\to\infty}(||\phi_i^n - \phi_i|| + ||(\phi_i^n)' - (\phi_i)'||) = 0$.

(L2) There exist $C > 0$, $M \in \mathbb{N}$ and a finite set $F \subset I$ such that for all $i \in I \setminus F$ and $n \geq M$, $|\ \log||(\phi_i^n)'|| - \log||\phi_i'||\ | \leq C$.

Here, we write S^n as $\{\phi_i^n\}_{i\in I}$ and S as $\{\phi_i\}_{i\in I}$, and we set $||\phi_i'|| := \sup_{z\in X} |\phi_i'(z)|$, $||\phi_i^n - \phi_i|| := \sup_{z\in X} |\phi_i^n(z) - \phi_i(z)|$ and $||(\phi_i^n)' - (\phi_i)'|| := \sup_{z\in X} |(\phi_i^n)'(z) - (\phi_i)'(z)|$. If a sequence $\{S^n\}_{n\in\mathbb{N}}$ in CIFS(X,I) does not admit any $S \in$ CIFS(X,I) for which the above conditions are fulfilled, we declare that $\lambda(\{S^n\}_{n\in\mathbb{N}}) = \emptyset$. A sequence $\{S^n\}_{n\in\mathbb{N}} \in$ CIFS$(X, I)^{\mathbb{N}}$ is called λ-converging if $\lambda(\{S^n\}_{n\in\mathbb{N}}) \in$ CIFS(X, I). We endow CIFS(X,I) with the λ-topology ([29]).

Definition 2.48. Let Λ be an open and connected subset of $\mathbb{C}$. Let $\{S^\mu\}_{\mu\in\Lambda}$ be a family of elements of CIFS(X, I). We write S^μ as $\{\phi_i^\mu\}_{i\in I}$. We say that $\{S^\mu\}_{\mu\in\Lambda}$ is plane-analytic if for all $x \in X$ and $i \in I$, $\mu \mapsto \phi_i^\mu(x)$ is holomorphic in Λ.

Moreover, we say that plane-analytic $\{S^\mu\}_{\mu\in\Lambda}$ is regularly plane-analytic if there exists $\mu_0 \in \Lambda$ such that the following conditions are satisfied.

(i) S^{μ_0} is strongly regular.

(ii) There exists $\eta \in (0, 1)$ such that for all $w \in I^\infty$ and $\mu \in \Lambda$, $|\kappa_w^{\mu_0}(\mu) - 1| \leq \eta$. Here, for each $\mu_0 \in \Lambda$ and $w = w_1 w_2 \cdots \in I^\infty$, we set $\pi_\mu := \pi_{S_\mu}$ and

$$\kappa_w^{\mu_0}(\mu) := \frac{(\phi_{w_1}^\mu)'(\pi_\mu(\sigma w))}{(\phi_{w_1}^{\mu_0})'(\pi_{\mu_0}(\sigma w))} \quad (\mu \in \Lambda).$$

Roy and Urbański showed the following results [29].

Theorem 2.49 ([29] Theorem 5.10)**.** The Hausdorff dimension function $h\colon$ CIFS(X, I) $\to [0, \infty)$, $S \mapsto h_S$, is continuous when CIFS(X, I) is endowed with the λ-topology.

Theorem 2.50 ([29] Theorem 6.1)**.** Let Λ be an open and connected subset of $\mathbb{C}$. Let $\{S^\mu\}_{\mu\in\Lambda}$ be a family of elements of CIFS(X, I). If $\{S^\mu\}_{\mu\in\Lambda}$ is regularly plane-analytic, then $\mu \mapsto h_{S^\mu}$ is real-analytic in Λ.

Theorem 2.51 ([29] Theorem 6.3)**.** Let Λ be an open and connected subset of $\mathbb{C}$. Let $\{S^\mu\}_{\mu\in\Lambda}$ be a family of elements of CIFS(X, I). If $\{S^\mu\}_{\mu\in\Lambda}$ is plane-analytic, then $\mu \mapsto 1/h_{S^\mu}$ is superharmonic in Λ.

2.5 Conformal iterated function systems of generalized complex continued fractions

In this subsection, we prove some properties of the CIFSs of generalized complex continued fractions [33]. Note that they are important and interesting examples of infinite CIFSs. We introduce some additional notations. For each $\tau \in A_0$, we set $\pi_\tau := \pi_{S_\tau}$, $\theta_\tau := \theta_{S_\tau}$, $\psi_\tau^n(t) := \psi_{S_\tau}^n(t) \quad (t \geq 0, n \in \mathbb{N})$ and $P_\tau(t) := P_{S_\tau}(t) \quad (t \geq 0)$.

Proposition 2.52. For all $\tau \in A_0$, S_τ is a CIFS.

Proof. Let $\tau \in A_0$. Firstly, we show that for all $b \in I_\tau$, $\phi_b(X) \subset X$. Let $Y := \{z \in \mathbb{C} |\ \Re z \geq 1\}$ and let $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the Möbius transformation defined by $f(z) := 1/z$. Since $f(0) = \infty$, $f(1) = 1$, $f(1/2 + i/2) = 2/(1 + i) = (1 - i)$, we have $f(\partial X) = \partial Y \cup \{\infty\}$. Moreover, since f(1/2) = 2, we have $f(X) = Y \cup \{\infty\}$. Thus, $f\colon X \to Y \cup \{\infty\}$ is a homeomorphism. Let $g_b\colon X \to Y$ be the map defined by $g_b(z) := z + b$. We deduce that $\phi_b = f^{-1} \circ g_b$ and $\phi_b(X) \subset f^{-1}(Y) \subset X$. Therefore, we have proved $\phi_b(X) \subset X$.

We next show that for each $\tau \in A_0$, S_τ satisfies the conditions of Definition 2.19.

1. Injectivity.

Since each ϕ_b is a Möbius transformation, each ϕ_b is injective.

2. Uniform Contractivity.

Let $b = m + n\tau (= m + nu + inv)$ be an element of I_τ and let $z = x + iy$ and $z' = x' + iy'$ be elements of X. We have

$$\begin{aligned}|z + b|^2 &= |x + m + nu + i(y + nv)|^2 \\ &= (x + m + nu)^2 + (y + nv)^2 \geq (0 + 1 + 0)^2 + (-1/2 + 1)^2 = \frac{5}{4}.\end{aligned}$$

Therefore, we deduce that $|z + b| \geq \sqrt{5/4}$. We also deduce that $|z' + b| \geq \sqrt{5/4}$. Finally, we obtain that

$$\begin{aligned}|\phi_b(z) - \phi_b(z')| &= \left|\frac{1}{z + b} - \frac{1}{z' + b}\right| \\ &= \frac{|z - z'|}{|z + b||z' + b|} \leq \left(\sqrt{\frac{4}{5}}\right)^2 |z - z'| = \frac{4}{5}|z - z'|.\end{aligned}$$

Therefore, S_τ is uniformly contractive on X.

4. Open Set Condition.

Note that $\mathrm{Int}(X) = \{z \in \mathbb{C} |\ |z - 1/2| < 1/2\}$. Let $\tau \in A_0$ and let $b \in I_\tau$. Since $f(\partial X) = \partial Y \cup \{\infty\}$, we deduce that for all $b \in I_\tau$,

$$g_b(\mathrm{Int}(X)) \subset \{z = x + iy \in \mathbb{C} |\ x > 1\} = f(\mathrm{Int}(X)).$$

Moreover, if b and b' are distinct elements, then $g_b(\mathrm{Int}(X))$ and $g_{b'}(\mathrm{Int}(X))$ are disjoint. Therefore, we have that for all $b \in I_\tau$,

$$\phi_b(\mathrm{Int}(X)) = f^{-1} \circ g_b(\mathrm{Int}(X)) \subset f^{-1} \circ f(\mathrm{Int}(X)) = \mathrm{Int}(X).$$

And if b and b' are distinct elements,

$$\phi_b(\mathrm{Int}(X)) \cap \phi_{b'}(\mathrm{Int}(X)) = f^{-1}(g_b(\mathrm{Int}(X)) \cap g_{b'}(\mathrm{Int}(X))) = \emptyset.$$

Therefore, S_τ satisfies the Open Set Condition of S_τ.

5. Bounded distortion Property.

Let ϵ be a positive real number which is less than $1/12$ and let $V' := B(1/2, 1/2 + \epsilon)$ be the open ball with center $1/2$ and radius $1/2 + \epsilon$. We set $\tau := u + iv$. Then, for all $(m, n) \in \mathbb{N}^2$

and $z := x + iy \in V'$, we have that

$$
\begin{aligned}
|\phi'_{m+n\tau}(z)| &= \frac{1}{|z+m+n\tau|^2} = \frac{1}{(x+m+nu)^2+(y+nv)^2} \\
&\leq \frac{1}{(-\epsilon+1+0)^2+(-1/2-\epsilon+1)^2} \\
&= \frac{1}{2\epsilon^2-3\epsilon+5/4} = \frac{1}{2(\epsilon-3/4)^2+1/8} \\
&\leq \frac{1}{2(1/12-3/4)^2+1/8} = \frac{72}{73} < 1
\end{aligned}
$$

For each $z \in V'$, we set

$$
z' := \begin{cases} (|z-1/2|-\epsilon)\dfrac{(z-1/2)}{|z-1/2|} + 1/2 & (z \notin X) \\ z & (z \in X). \end{cases}
$$

Then, we have that $|z - z'| \leq \epsilon$ and $|z' - 1/2| < 1/2$. It implies that $z' \in X$. Thus, we obtain that $|\phi_b(z) - \phi_b(z')| \leq (72/73)|z - z'| < \epsilon$ and

$$
\left|\phi_b(z) - \frac{1}{2}\right| \leq |\phi_b(z) - \phi_b(z')| + \left|\phi_b(z') - \frac{1}{2}\right| < \frac{1}{2} + \epsilon.
$$

It follows that for all $b \in I_\tau$, $\phi_b(V') \subset V'$. In addition, ϕ_b is injective on V' and ϕ_b is holomorphic on $V' := B(1/2, 1/2 + \epsilon)$ since ϕ_b is holomorphic on $\mathbb{C} \setminus \{-b\}$.

Let b be an element of I_τ and $r_0 := 1/2 + \epsilon$. Let f_b be the function defined by

$$
f_b(z) := \frac{(\phi_b(r_0 z + 1/2) - \phi_b(1/2))}{r_0 \phi'_b(1/2)} \quad (z \in D := \{z \in \mathbb{C} | |z| < 1\}).
$$

Note that f_b is holomorphic on D and $f_b(0) = 0$ and $f'_b(0) = 1$. By using the Koebe distortion theorem, we deduce that for all $z \in D$,

$$
\frac{1-|z|}{(1+|z|)^3} \leq |f_b(z)| \leq \frac{1+|z|}{(1-|z|)^3}.
$$

Let $r_1 := (r_0 + 1/2)/2$. we deduce that there exist $C_1 \geq 1$ and $C_2 \leq 1$ such that for all $z \in B(0, r_1/r_0) (\subset D)$,

$$
C_2 \leq \frac{1-|z|}{(1+|z|)^3} \quad \text{and} \quad \frac{1+|z|}{(1-|z|)^3} \leq C_1.
$$

Let $C := C_1/C_2$. Then, we have that for all $z, z' \in B(0, r_1/r_0)$,

$$
\begin{aligned}
\frac{|\phi'_b(r_0 z + 1/2)|}{|\phi'_b(1/2)|} &= |f'_b(z)| \leq \frac{1+|z|}{(1-|z|)^3} \\
&\leq C_1 = CC_2 \leq C\frac{1-|z'|}{(1+|z'|)^3} \\
&\leq C|f'_b(z')| \leq C\frac{|\phi'_b(r_0 z' + 1/2)|}{|\phi'_b(1/2)|}.
\end{aligned}
$$

It follows that for all $z, z' \in B(0, r_1/r_0)$, $|\phi'_b(r_0 z + 1/2)| \leq C|\phi'_b(r_0 z' + 1/2)|$. Finally, let $V := B(1/2, r_1)$ be the open ball with center $1/2$ and radius r_1. Then, V is an open and connected subset of $\mathbb{C}$ with $X \subset V$ and for all $z, z' \in V$,

$$
|\phi'_b(z)| \leq C|\phi'_b(z')|.
$$

Therefore, S_τ satisfies the Bounded Distortion Property.

3. Conformality.

Let $\tau \in A_0$ and let $b \in I_\tau$. Since ϕ_b is holomorphic on $\mathbb{C} \setminus \{-b\}$, ϕ_b is C^2 and conformal on V. By the above argument, we have $\phi_b(V) \subset V$.

6. Cone Condition.

Since X is a closed disk, the Cone Condition is satisfied. □

For the rest of the paper, let $V := B(1/2, r_1)$, where r_1 is the number in the proof of Lemma 2.52.

Lemma 2.53. Let $\tau \in A_0$. Then, there exists $C \geq 1$ such that for all $z \in V$ and $b \in I_\tau$, we have $C^{-1}|b|^{-2} \leq |\phi_b'(z)| \leq C|b|^{-2}$.

Proof. Note that $|\phi_b'(0)| = |b|^{-2}$. By using the BDP, there exists $C \geq 1$ such that for all $z \in B(1/2, r_1)$, we have $C^{-1}|\phi_b'(0)| \leq |\phi_b'(z)| \leq C|\phi_b'(0)|$. We deduce that $C^{-1}|b|^{-2} \leq |\phi_b'(z)| \leq C|b|^{-2}$. □

Lemma 2.54. For all $\tau \in A_0$, S_τ is a hereditarily regular CIFS with $\theta_\tau = 1$.

Proof. Let $\tau \in A_0$. For each non-negative integer p, we set $K'(p) := \{b = m + n\tau \in I_\tau |\ (m, n) \in \mathbb{N}^2, m < 2^p, n < 2^p\}$ and $K(p) := K'(p) \setminus K'(p-1)$. Note that for each non-negative integer p, $|K'(p)| = (2^p - 1)^2$. We deduce that for each $p \in \mathbb{N}$, $|K(p)| = |K'(p)| - |K'(p-1)| = (2^p - 1)^2 - (2^{p-1} - 1)^2 = 3 \cdot 4^{p-1} - 2 \cdot 2^{p-1} = 2^{p-1}(3 \cdot 2^{p-1} - 2)$ and $4^{p-1} \leq |K(p)| \leq 3 \cdot 4^{p-1}$.

Let $b = m + n\tau = m + n(u + iv) \in K(p)$. We consider the following two cases.

(i) If $m \geq 2^{p-1}$ then we have

$$\begin{aligned} |b|^2 &= |m + nu + inv|^2 \\ &= (m + nu)^2 + (nv)^2 \\ &\geq (2^{p-1} + u)^2 + v^2 \\ &\geq (2^{p-1})^2 + |\tau|^2 = 4^{p-1}\left(1 + \frac{|\tau|^2}{4^{p-1}}\right). \end{aligned}$$

(ii) If $n \geq 2^{p-1}$ then we have

$$\begin{aligned} |b|^2 &= |m + nu + inv|^2 \\ &= (m + nu)^2 + (nv)^2 \\ &\geq n^2(u^2 + v^2) \geq 4^{p-1}|\tau|^2. \end{aligned}$$

Then for any $t \geq 0$, we have

$$\begin{aligned} \sum_{b \in I_\tau} |b|^{-2t} &= \sum_{p \in \mathbb{N}} \sum_{b \in K(p)} \left\{|b|^2\right\}^{-t} \\ &\leq \sum_{p \in \mathbb{N}} |K(p)| 4^{-t(p-1)} \left\{\min\{1 + \frac{|\tau|^2}{4^{p-1}}, |\tau|^2\}\right\}^{-t} \\ &\leq \sum_{p \in \mathbb{N}} 3 \cdot 4^{(p-1)(1-t)} \left\{\min\{1 + \frac{|\tau|^2}{4^{p-1}}, |\tau|^2\}\right\}^{-t}. \end{aligned}$$

Hence, we deduce that

$$\sum_{b\in I_\tau} |b|^{-2t} \le 3\sum_{p\in\mathbb{N}} 4^{(p-1)(1-t)} \left\{\min\{1+\frac{|\tau|^2}{4^{p-1}}, |\tau|^2\}\right\}^{-t}. \tag{2.2}$$

Moreover, by the inequality $|\tau|^2 \ge 1$ and the inequality $1+\dfrac{|\tau|^2}{4^{p-1}} \ge 1$, we deduce that for all $p \in \mathbb{N}$,

$$3\cdot 4^{(p-1)(1-t)} \left\{\min\{1+\frac{|\tau|^2}{4^{p-1}}, |\tau|^2\}\right\}^{-t} \le 3\cdot 4^{(p-1)(1-t)}. \tag{2.3}$$

Also, by the inequality $|b| \le |m|+|n||\tau| \le 2^p(1+|\tau|)$ $(p\in\mathbb{N}, b\in K(p))$, we have

$$\begin{aligned}\sum_{b\in I_\tau} |b|^{-2t} &= \sum_{p\in\mathbb{N}}\sum_{b\in K(p)} \{|b|^{-2}\}^t \\ &\ge \sum_{p\in\mathbb{N}} |K(p)| 4^{-pt}(1+|\tau|)^{-2t}.\end{aligned}$$

Thus, we deduce that

$$\sum_{b\in I_\tau} |b|^{-2t} \ge 4^{-1}\sum_{p\in\mathbb{N}} 4^{p(1-t)}(1+|\tau|)^{-2t}. \tag{2.4}$$

Finally, from Lemma 2.53, the inequality (2.2) and the inequality (2.4), it follows that $\psi^1_\tau(1)=\infty$ and if $t>1$, then $\psi^1_\tau(t)<\infty$. Therefore, we deduce that $\theta_\tau = 1$ and by Theorem 2.42, we obtain that for all $\tau\in A_0$, S_τ is hereditarily regular. Hence, we have proved our lemma. □

Lemma 2.55. We have $\lim_{\tau\to\infty,\tau\in A_0} h_\tau = 1$, i.e., for each $\epsilon>0$, there exists $N>0$ such that, for all $\tau\in A_0$ with $|\tau|\ge N$, we have $|h_\tau - 1|<\epsilon$.

Proof. Let $\epsilon>0$ and $t := 1+\epsilon > 1$. Let $\{\tau_n\}_{n\in\mathbb{N}}$ be any sequence in A_0 such that $|\tau_n|\to\infty$ as $n\to\infty$. Note that for all $p\in\mathbb{N}$, we have $\left\{\min\{1+\dfrac{|\tau_n|^2}{4^{p-1}}, |\tau_n|^2\}\right\}^{-t} \to 0$ as $n\to\infty$. By the inequality (2.2) and the inequality (2.3), we deduce that the function

$$f_n(p) := 3\cdot 4^{(p-1)(1-t)}\left\{\min\{1+\frac{|\tau_n|^2}{4^{p-1}}, |\tau_n|^2\}\right\}^{-t} \quad (p\in\mathbb{N})$$

is dominated by the integrable function $g(p) := 3\cdot 4^{(p-1)(1-t)}$ $(p\in\mathbb{N})$ with respect to the counting measure on $\mathbb{N}$. Then, by Lebesgue's dominated convergence theorem, we deduce that $\lim_{n\to\infty}\sum_{b\in I_{\tau_n}} |b|^{-2t} = 0$. By Lemma 2.53, we obtain $\lim_{n\to\infty}\psi^1_{\tau_n}(t)=0$. It follows that for any $\epsilon>0$, there exists $N\in\mathbb{N}$ such that for all $\tau\in A_0$ with $|\tau|\ge N$, we have $\psi^1_\tau(1+\epsilon)=\psi^1_\tau(t)<1$.

By Proposition 2.35, we obtain that $\psi^n_\tau(1+\epsilon) \le (\psi^1_\tau(1+\epsilon))^n < 1$. Therefore, we deduce that $P_\tau(1+\epsilon)\le 0$. Thus, for all $\epsilon>0$, there exists $N\in\mathbb{N}$ such that for all $\tau\in A_0$ with $|\tau|\ge N$, $h_\tau \le 1+\epsilon$.

Moreover, by Theorem 2.41 and Lemma 2.54, for all $\tau\in A_0$, we have $1-\epsilon<h_\tau$. Hence, we have proved our lemma. □

Theorem 2.56. Let $\tau\in A_0$. Then we have $1<h_\tau<2$.

Proof. Let $\tau \in A_0$. By Theorem 2.42, we have $1 = \theta_\tau < h_\tau$. We now show that $h_\tau < 2$. We use the notations in the proof of Proposition 2.52. Note that

$$\bigcup_{b \in I_\tau} g_b(X) \subset \{z \in \mathbb{C} | \ \Re z \geq 1 \text{ and } \Im z \geq 0\}.$$

Let U_0 be an open ball such that $U_0 \subset \{z \in \mathbb{C} | \ \Re z \geq 1 \text{ and } \Im z < 0\}$. Since $U_0 \subset Y$, we deduce that $f^{-1}(U_0) \subset f^{-1}(Y) = \text{Int}(X)$. We set $X_1 := \cup_{b \in I_\tau} \phi_b(X)$. Since $U_0 \cap \bigcup_{b \in I_\tau} g_b(X) = \emptyset$, we deduce that $f^{-1}(U_0) \cap X_1 = f^{-1}(U_0 \cap \bigcup_{b \in I_\tau} g_b(X)) = \emptyset$. It follows that $\text{Int}(X) \setminus X_1 \supset f^{-1}(U_0)$.

Therefore, we deduce that $\lambda_2(\text{Int}(X) \setminus X_1) > 0$ where, λ_2 is the 2-dimensional Lebesgue measure. By Theorem 2.44, we obtain that $h_\tau < 2$. Hence, we have proved $1 < h_\tau < 2$. □

2.6 Appendix: the proof of the fact $\overline{J_\tau} \setminus J_\tau$ is at most countable

Rest of this section, we give the proof of the fact for each $\tau \in A_0$, $\overline{J_\tau} \setminus J_\tau$ is at most countable and $h_\tau = \dim_{\mathcal{H}}(\overline{J_\tau})$ ([32, Theorem 6.11]).

Let $\{S_\tau\}_{\tau \in A_0}$ be the family of CIFSs of generalized complex continued fractions. We set $X_\tau(\infty) := X_{S_\tau}(\infty)$. H. Sugita showed the following result ([32]).

Theorem 2.57. Let $\{S_\tau\}_{\tau \in A_0}$ be the family of CIFSs of generalized complex continued fractions. Then, we have that for all $\tau \in A_0$, $X_\tau(\infty) = \{0\}$. In particular, for each $\tau \in A_0$,

$$\overline{J_\tau} = J_\tau \cup \bigcup_{w \in I^*} \phi_w(\{0\}) \cup \{0\} \quad \text{and} \quad \overline{J_\tau} \setminus J_\tau \subset \bigcup_{w \in I^*} \phi_w(\{0\}) \cup \{0\}.$$

Proof. We first show that for all $\tau \in A_0$, $0 \in X_\tau(\infty)$. We set $I'_\tau := \{m + \tau \in I_\tau | \ m \in \mathbb{N}\} \subset I_\tau$ and $b_m := m + \tau \in I'_\tau$. Then, we have that $|I'_\tau| = \infty$ and since $0 \in X$, $\phi_{b_m}(0) \in \phi_{b_m}(X)$. Let $\epsilon > 0$. Then, there exists $M \in \mathbb{N}$ such that $M > 1/\epsilon$. Let $F_\tau := \{m + \tau \in I_\tau | \ m \in \mathbb{N}, m \leq M\} \subset I'_\tau$. We obtain that $|F_\tau| < \infty$ and if $b_m \in I'_\tau \setminus F_\tau$, then $\phi_{b_m}(0) \in \phi_{b_m}(X)$ and

$$|\phi_{b_m}(0)| = \left| \frac{1}{m + \tau} \right| < \frac{1}{m} < \frac{1}{M} < \epsilon.$$

Thus, for all $\tau \in A_0$, $0 \in X_\tau(\infty)$.

We next show that for each $\tau \in A_0$, $a \in X_\tau(\infty)$ implies $a = 0$. Suppose that there exists $a \in X_\tau(\infty)$ such that $a \neq 0$. Then, there exist $I'_\tau \subset I_\tau$ and $\{z'_b\}_{b \in I'_\tau}$ such that $|I'_\tau| = \infty$, $z'_b \in \phi_b(X)$ $(b \in I'_\tau)$ and $\lim_{b \in I'_\tau} z'_b = a$. Let $\delta := |a|/2 > 0$. Then, there exists $F'_\tau \subset I'_\tau$ such that $|F'_\tau| < \infty$ and for all $b \in I'_\tau \setminus F'_\tau$, $|z'_b - a| < \delta$. In particular, for all $b \in I'_\tau \setminus F'_\tau$,

$$|z'_b| \geq |a| - |z'_b - a| > \delta. \tag{*}$$

Moreover, for each $z \in X$, $\tau \in A_0$ and $b \in I_\tau$, we write $z := x + yi$, $\tau := u + iv$ and $b := m + n\tau$. Note that

$$\begin{aligned} |z + b|^2 &= |x + m + nu + i(y + nv)|^2 \\ &= (x + m + nu)^2 + (y + nv)^2 \\ &\geq (0 + m + nu)^2 + (-1/2 + nv)^2 \geq m^2 + (n - 1/2)^2. \end{aligned}$$

Let $M := 1/\delta$. By using the above inequality, there exists $N_\delta \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $n \in \mathbb{N}$ and $x \in X$, if $m \geq N_\delta$ or $n \geq N_\delta$, then $|z + b| > M = 1/\delta$. In particular,

$b \in I_\tau \setminus F_\tau(N_\delta)$ implies that for all $z \in X$, $|\phi_b(z)| < \delta$. Here, $F_\tau(N_\delta) := \{b := m + n\tau \in I_\tau | \ n \leq N_\delta, m \leq N_\delta\}$.

By the inequality (*) and $|F_\tau(N_\delta)| < \infty$, this contradicts that there exist $b \in I'_\tau \setminus (F'_\tau \cup F_\tau(N_\delta))$ and $z'_b \in \phi_b(X)$ such that $|z'_b| > \delta$. Therefore, we have proved that for all $\tau \in A_0$, $X_\tau(\infty) = \{0\}$. □

Corollary 2.58. Let $\{S_\tau\}_{\tau \in A_0}$ be the family of CIFSs of generalized complex continued fractions. Then, we have that for all $\tau \in A_0$, $\dim_{\mathcal{H}}(\overline{J_\tau}) = h_\tau$.

Proof. By Theorem 2.57, we obtain that $\overline{J_\tau} \setminus J_\tau$ is at most countable. Note that if A is at most countable, then $\dim_{\mathcal{H}} A = 0$. Thus,

$$\dim_{\mathcal{H}}(\overline{J_\tau}) = \max\{\dim_{\mathcal{H}} J_\tau, \dim_{\mathcal{H}}(\overline{J_\tau} \setminus J_\tau)\} = \max\{\dim_{\mathcal{H}} J_\tau, 0\} = h_\tau$$

Therefore, we have proved Corollary 2.58. □

3 The Hausdorff dimension function of the family of conformal iterated function systems of generalized complex continued fractions

3.1 Proof of Main Theorem 1

We first show the following lemma.

Lemma 3.1. Let $\tau \in A_0$ and suppose that a sequence $\{\tau_n\}_{n \in \mathbb{N}}$ in A_0 satisfies $\lim_{n\to\infty} \tau_n = \tau$. Then, there exist $K \in \mathbb{N}$, $C_1 > 0$ and $C_2 > 0$ such that for all $k \geq K$, $(m, n) \in \mathbb{N}^2$ and $z, z' \in X$,

$$C_1 \leq \frac{|z' + m + n\tau_k|^2}{|z + m + n\tau|^2} \leq C_2. \tag{3.1}$$

Proof. We set $\tau = u + iv$ and we set for each $n \in \mathbb{N}$, $\tau_n = u_n + iv_n$. Since $\lim_{n\to\infty} \tau_n = \tau$, there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $|u - u_k| \leq 1$ and $|v - v_k| \leq v/3$. Then, for all $(m, n) \in \mathbb{N}^2$ and $z, z' \in X$,

$$\begin{aligned}
&\frac{|z' + m + n\tau_k|^2}{|z + m + n\tau|^2} \\
&\leq \frac{(1 + m + nu_k)^2 + (1/2 + nv_k)^2}{(m + nu)^2 + (-1/2 + nv)^2} \\
&\leq \frac{(1 + m + n(1 + u))^2 + (1/2 + n(4/3)v)^2}{(m + nu)^2 + (-1/2 + nv)^2} \\
&= \frac{(1 + m + n(1 + u))^2}{(m + nu)^2 + (-1/2 + nv)^2} + \frac{(1/2 + n(4/3)v)^2}{(m + nu)^2 + (-1/2 + nv)^2} \\
&\leq \max\left\{\frac{(1 + (1 + u) + 1)^2}{1^2}, \frac{(1 + (1 + u) + 1)^2}{u^2 + (v - 1/2)^2}\right\} + \frac{(1/2n + (4/3)v)^2}{(v - 1/2n)^2} \\
&\leq \max\left\{\frac{(1 + (1 + u) + 1)^2}{1^2}, \frac{(1 + (1 + u) + 1)^2}{u^2 + (v - 1/2)^2}\right\} + \frac{(1/2 + (4/3)v)^2}{(v - 1/2)^2} < \infty
\end{aligned}$$

and

$$\begin{aligned}\frac{|z' + m + n\tau_k|^2}{|z + m + n\tau|^2} &\geq \frac{(m + nu_k)^2 + (-1/2 + nv_k)^2}{(1 + m + nu)^2 + (1/2 + nv)^2} \\ &\geq \frac{m^2 + (-1/2 + n(2/3)v)^2}{2(1 + m + n\max\{u, v\})^2} \\ &\geq \frac{1}{2}\left(\min\left\{\frac{1}{1 + 1 + \max\{u, v\}}, \frac{(2/3)v - 1/2}{1 + 1 + \max\{u, v\}}\right\}\right)^2 > 0.\end{aligned}$$

Therefore, we have proved our lemma. □

We now prove Theorem 1.3.

Proof. By Lemma 2.54, for each $\tau \in A_0$, the value h_τ is equal to the unique zero of the pressure function of S_τ. Moreover, by Lemma 2.55 and Theorem 2.56, we have that $1 < h_\tau < 2$ for each $\tau \in A_0$ and $h_\tau \to 1$ as $\tau \to \infty$ in A_0.

We next show that if a sequence $\{\tau_n\}_{n\in\mathbb{N}}$ in A_0 satisfies $\lim_{n\to\infty} \tau_n = \tau$, then we have $\lambda(\{S_{\tau_n}\}_{n\in\mathbb{N}}) = S_\tau$. Since for all $(m, n) \in \mathbb{N}^2$, $\phi_{z+m+n\tau}(z) = 1/(z + m + n\tau)$ and $(\phi_{m+n\tau})'(z) = (-1)/(z + m + n\tau)^2$, condition (L1) is satisfied. Since X is compact, there exist $z_0, z_k \in X$ such that

$$\begin{aligned}\log\left(||\phi'_{m+n\tau}||_X/||\phi'_{m+n\tau_k}||_X\right) &= \log(|\phi'_{m+n\tau}(z_0)|/|\phi'_{m+n\tau_k}(z_k)|) \\ &= \log(|z_k + m + n\tau_k|^2/|z_0 + m + n\tau|^2).\end{aligned}$$

By Lemma 3.1, there exist $C > 0$ and $K \in \mathbb{N}$ such that for each $k \geq K$ and $(m, n) \in \mathbb{N}^2$,

$$\left|\log\left(||\phi'_{m+n\tau}||_X\right) - \log\left(||\phi'_{m+n\tau_k}||_X\right)\right| = \left|\log\left(||\phi'_{m+n\tau}||_X/||\phi'_{m+n\tau_k}||_X\right)\right| \leq C.$$

Therefore, we have proved that if a sequence $\{\tau_n\}_{n\in\mathbb{N}}$ in A_0 satisfies $\lim_{n\to\infty} \tau_n = \tau$, then $\lambda(\{S_{\tau_n}\}_{n\in\mathbb{N}}) = S_\tau$.

We next show that $\tau \mapsto h_\tau$ is continuous in A_0. By Theorem 2.49, $S_\tau \mapsto h_\tau$ is continuous with respect to the λ-topology. By Lemma 3.3 of [29], if $\lambda(\{S_{\tau_n}\}_{n\in\mathbb{N}}) = S_\tau$, then $\lim_{n\to\infty} h_{\tau_n} = h_\tau$. Thus, if $\lim_{n\to\infty} \tau_n = \tau$, then $\lim_{n\to\infty} h_{\tau_n} = h_\tau$. Therefore, we have proved that $\tau \mapsto h_\tau$ is continuous in A_0. □

3.2 Proof of Main Theorem 2

We now prove Theorem 1.4.

Proof. We first show that $\tau \mapsto h_\tau$ is subharmonic in $\mathrm{Int}(A_0)$. Let $z \in X$ and Let $(m, n) \in \mathbb{N}^2$. Note that since the real part of $-(m+z)/n$ is negative, $-(m+z)/n$ is not an element of $\mathrm{Int}(A_0)$. Therefore, we deduce that the function $\tau \mapsto \phi_{m+n\tau}(z) = 1/(z + m + n\tau)$ is holomorphic in $\mathrm{Int}(A_0)$. Hence, $\{S_\tau\}_{\tau\in\mathrm{Int}(A_0)}$ is plane-analytic. Therefore, by using Theorem 2.51, we obtain that $\tau \mapsto h_\tau$ is subharmonic in $\mathrm{Int}(A_0)$.

We next show that $\tau \mapsto h_\tau$ is real-analytic in $\mathrm{Int}(A_0)$. Since for each $\tau \in A_0$, S_τ is a hereditarily regular CIFS, we have that for each $\tau \in \mathrm{Int}(A_0)$, S_τ is a strongly regular CIFS. We now show that for any $\tau_0 \in \mathrm{Int}(A_0)$, there exists an open ball $U \subset \mathrm{Int}(A_0)$ with center τ_0 and $\eta \in (0, 1)$ such that for all $\tau \in U$ and $w := (m_i, n_i)_{i\in\mathbb{N}} \in (\mathbb{N}^2)^\infty$, $|\kappa_w^{\tau_0}(\tau) - 1| \leq \eta$, where we denote $(\phi'_{m_1+n_1\tau}(\pi_\tau\sigma w))/(\phi'_{m_1+n_1\tau_0}(\pi_{\tau_0}\sigma w))$ by $\kappa_w^{\tau_0}(\tau)$.

By Lemma 3.1, there exists an open ball $U'' \subset \mathrm{Int}(A_0)$ with center τ_0 such that $|\kappa_w^{\tau_0}|$ is bounded in U'' uniformly on $w \in (\mathbb{N}^2)^\infty$. Note that $\kappa_w^{\tau_0}$ is holomorphic in $\mathrm{Int}(A_0)$. By using the Cauchy formula

$$(\kappa_w^{\tau_0})'(\tau) = \frac{1}{2\pi i}\int_{\partial U''} \frac{\kappa_w^{\tau_0}(\xi)}{(\xi-\tau)^2}\mathrm{d}\xi \quad (\tau \in U''),$$

we deduce that there exists $M > 0$ such that for all $\tau \in U'$ and $w \in (\mathbb{N}^2)^\infty$, $|(\kappa_w^{\tau_0})'(\tau)| \leq M$. Here, U' is an open ball with center τ_0 such that $U' \subset U''$. Then, we have that

$$\begin{aligned} |\kappa_w^{\tau_0}(\tau) - 1| &= |\kappa_w^{\tau_0}(\tau) - \kappa_w^{\tau_0}(\tau_0)| \\ &= \left| \int_{\tau_0}^{\tau} (\kappa_w^{\tau_0})'(\xi)\mathrm{d}\xi \right| \\ &\leq \int_{\tau_0}^{\tau} |(\kappa_w^{\tau_0})'(\xi)||\mathrm{d}\xi| \leq M|\tau - \tau_0|. \end{aligned}$$

It follows that there exist an open ball $U(\subset U')$ with center τ_0 and an element $\eta \in (0,1)$ such that for all $\tau \in U$ and $w \in (\mathbb{N}^2)^\infty$, $|\kappa_w^{\tau_0}(\tau) - 1| \leq \eta$.

Thus, for any $\tau_0 \in \mathrm{Int}(A_0)$, there exists an open ball $U \subset \mathrm{Int}(A_0)$ with center τ_0 such that $\{S_\tau\}_{\tau \in U}$ is regularly plane-analytic. By Theorem 2.50, for any $\tau_0 \in \mathrm{Int}(A_0)$, there exists an open ball $U \subset \mathrm{Int}(A_0)$ with center τ_0 such that the map $\tau \mapsto h_\tau$ is real-analytic in U. Since τ_0 is arbitrary, we deduce that the map $\tau \mapsto h_\tau$ is real-analytic in $\mathrm{Int}(A_0)$. Combining this with Theorem 1.3 (Main Theorem 1), we obtain that the function $\tau \mapsto h_\tau$ is not constant on any non-empty open subset of A_0. □

3.3 Proof of Main Theorem 3

We now prove Corollary 1.5.

Proof. For each $n \in \mathbb{N}$, let $B_n := A_0 \cap \{z \in \mathbb{C} |\ |\Re z| \leq n \text{ and } |\Im z| \leq n\}$. Note that for all $n \in \mathbb{N}$, the map $\tau \mapsto h_\tau$ is subharmonic in $\mathrm{Int}(B_n)$ by Theorem 1.4. Let $\epsilon := (h_i - 1)/2 > 0$, where $i = \sqrt{-1}$. By Lemma 2.55, we deduce that there exists $N \in \mathbb{N}$ such that for all $\tau \in A_0 \setminus B_N$, $|h_\tau - 1| < \epsilon$. It follows that $(h_i - 1)/2 > h_\tau - 1$. Then, we obtain that for all $\tau \in A_0 \setminus B_N$,

$$h_i > 2h_\tau - 1 = h_\tau + (h_\tau - 1) > h_\tau.$$

Since the function $\tau \mapsto h_\tau$ is continuous in B_N, there exists a maximum point of the function $\tau \mapsto h_\tau$ in A_0 and

$$\max\{h_\tau |\ \tau \in A_0\} = \max\{h_\tau |\ \tau \in B_N\}.$$

Since the function $\tau \mapsto h_\tau$ is subharmonic in $\mathrm{Int}(A_0)$, there exists no maximum point of the function $\tau \mapsto h_\tau$ in $\mathrm{Int}(A_0)$. Thus, we have proved Corollary 1.5. □

4 The Hausdorff measures and the packing measures of the limit sets of CIFSs of generalized complex continued fractions

4.1 Proof of Main Theorem 4

In order to prove Theorem 1.7 (Main Theorem 4), we first show a basic estimate for the conformal measure. Note that for each $\tau \in A_0$, there exists the unique h_τ-conformal measure m_{S_τ} of S_τ by Proposition 2.43 since for each $\tau \in A_0$, S_τ is hereditarily regular. We set $m_\tau := m_{S_\tau}$.

Lemma 4.1. Let $\tau \in A_0$ and m_τ be the h_τ-conformal measure of S_τ. Then, there exists $K_0 \geq 1$ such that for each $b \in I_\tau$, we have $\phi_b(X) \subset B(0, K_0|b|^{-1})$ and

$$m_\tau(\phi_b(X)) \geq K_0^{-h_\tau}|b|^{-2h_\tau}.$$

Proof. By Lemma 2.53 with $K = K_0$, we deduce that for all $b \in I_\tau$ and $z \in V$, $\phi_b(V) \subset B(0, K_0|b|^{-1})$ and $K_0^{-1}|b|^{-2} \leq |\phi_b'(z)| \leq K_0|b|^{-2}$. Therefore, we have $\phi_b(X) \subset B(0, K|b|^{-1})$ and

$$m_\tau(\phi_b(X)) = \int_X |\phi_b'|^{h_\tau} \mathrm{d}m_\tau \geq (K_0^{-1}|b|^{-2})^{h_\tau} m_\tau(X) \geq K_0^{-h_\tau}|b|^{-2h_\tau}.$$

Thus, we have proved our lemma. $\square$

Recall that $X_\tau(\infty) = \{0\}$. By Lemma 4.1, for sufficiently small $r > 0$, $b \in I_\tau$ and $N > 0$ with $r/N < K_0|b|^{-1} < r$, we have $\phi_b(X) \subset B(0, K_0|b|^{-1}) \subset B(0, r)$ and

$$\frac{m_\tau(B(0,r))}{r^{h_\tau}} \geq \frac{m_\tau(\phi_b(X))}{r^{h_\tau}} \geq \frac{K_0^{-h_\tau}|b|^{-2h_\tau}}{r^{h_\tau}} \geq \frac{K_0^{-h_\tau}}{r^{h_\tau}} \left(\frac{r}{NK_0}\right)^{2h_\tau} \simeq r^{h_\tau}. \tag{4.1}$$

This inequality (4.1) does not satisfies the assumption of Theorem 2.45 unfortunately. However, since for all $b, b' \in I_\tau$ with $b \neq b'$, $m_\tau(\phi_b(X) \cap \phi_{b'}(X)) = 0$, we have a sharper estimate on the value of $m_\tau(B(0, r))$. To obtain this estimate, we set

$$I_\tau(r) := \{b \in I_\tau \mid r/N_\tau \leq K_0|b|^{-1} < r\},$$

where N_τ is the number we introduce later. Then, we have

$$\frac{m_\tau(B(0,r))}{r^{h_\tau}} \geq \sum_{b \in I_\tau(r)} \frac{m_\tau(\phi_b(X))}{r^{h_\tau}} \geq |I_\tau(r)| K_0^{-3h_\tau} N_\tau^{-2h_\tau} r^{h_\tau}. \tag{4.2}$$

Note that since $I_\tau(r) = \{b \in I_\tau \mid K_0 r^{-1} < |b| \leq N_\tau K_0 r^{-1}\}$, we have $|I_\tau(r)| \gtrsim r^{-2}$ intuitively since we have a intuition that the number of the points $b \in I_\tau(r)$ in the slant lattice I_τ is almost the same as the area of $I_\tau(r)$.

To prove this intuitive estimate rigorously, we introduce the following notations and prove lemma 4.2, Proposition 4.3 and Lemma 4.4. We identify $\mathbb{C}$ with $\mathbb{R}^2$, I_τ with $\{{}^t(a, b) \in \mathbb{R}^2 \mid a + ib \in I_\tau\}$ and $\mathbb{N}^2$ with $\{{}^t(m, n) \in \mathbb{R}^2 \mid m, n \in \mathbb{N}\}$, where for any matrix A, we denote by ${}^tA$ the transpose of A. For each $\tau = u + iv \in A_0$, we set

$$E_\tau := \begin{pmatrix} 1 & u \\ 0 & v \end{pmatrix} \quad \text{and} \quad F_\tau := {}^tE_\tau E_\tau = \begin{pmatrix} 1 & u \\ u & |\tau|^2 \end{pmatrix}.$$

Note that $E_\tau \mathbb{N}^2 = I_\tau$, since $\det(E_\tau) = v \neq 0$, E_τ is invertible and by direct calculations, there exist the eigenvalues $\lambda_1 > 0$ and $\lambda_2 > 0$ of F_τ with $\lambda_1 < \lambda_2$. Let $v_1 \in \mathbb{R}^2$ be a eigenvector with respect to λ_1 and $v_2 \in \mathbb{R}^2$ be a eigenvector with respect to λ_2. Note that since F_τ is a symmetric matrix, there exist eigenvectors $v_1 \in \mathbb{R}^2$ and $v_2 \in \mathbb{R}^2$ such that $V_\tau := (v_1, v_2)$ is an orthogonal matrix.

For each $R_1 > 0$ and $R_2 > 0$ with $R_1/\sqrt{\lambda_1} < R_2/\sqrt{\lambda_2}$, we set

$$D_1'(\tau, R_1, R_2) := \{{}^t(x, y) \in \mathbb{R}^2 \mid R_1^2/\lambda_1 < x^2 + y^2 \leq R_2^2/\lambda_2\} \quad \text{and}$$
$$D_2'(R_1, R_2) := \{{}^t(x, y) \in \mathbb{R}^2 \mid R_1^2 < x^2 + y^2 \leq R_2^2\}.$$

Lemma 4.2. Let $\tau \in A_0$ and let $R_1 > 0$ and $R_2 > 0$ with $R_1/\sqrt{\lambda_1} < R_2/\sqrt{\lambda_2}$. Then, we have that $E_\tau(D_1'(\tau, R_1, R_2)) \subset D_2'(R_1, R_2)$. In particular, we have that

$$E_\tau(\mathbb{N}^2 \cap D_1'(\tau, R_1, R_2)) \subset I_\tau \cap D_2'(R_1, R_2) \quad \text{and} \quad |\mathbb{N}^2 \cap D_1'(\tau, R_1, R_2)| \leq |I_\tau \cap D_2'(R_1, R_2)|.$$

Proof. By the above observation of F_τ, we deduce that

$$F_\tau = V_\tau \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} {}^t V_\tau.$$

Let ${}^t(x, y) \in D_1'(\tau, R_1, R_2)$. We set $(x', y') := (x, y)\ V_\tau$ and $(v, w) := (x, y)\ {}^t E_\tau$. Note that since V_τ is an orthogonal matrix, we deduce that $(x')^2 + (y')^2 = x^2 + y^2$. Since $\lambda_1 < \lambda_2$, we have

$$\begin{aligned} R_1^2 &< \lambda_1(x^2 + y^2) = \lambda_1((x')^2 + (y')^2) < \lambda_1(x')^2 + \lambda_2(y')^2 \\ &= (x', y') \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = (x, y)\ V_\tau \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} {}^t V_\tau \begin{pmatrix} x \\ y \end{pmatrix} \\ &= (x, y)\ F_\tau \begin{pmatrix} x \\ y \end{pmatrix} = (x, y)\ {}^t E_\tau E_\tau \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

By the above inequality, we deduce that $R_1^2 < v^2 + w^2$. Also,

$$\begin{aligned} R_2^2 &\geq \lambda_2(x^2 + y^2) = \lambda_2((x')^2 + (y')^2) \geq \lambda_1(x')^2 + \lambda_2(y')^2 \\ &= (x', y') \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = (x, y)\ V_\tau \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} {}^t V_\tau \begin{pmatrix} x \\ y \end{pmatrix} \\ &= (x, y)\ F_\tau \begin{pmatrix} x \\ y \end{pmatrix} = (x, y)\ {}^t E_\tau E_\tau \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

By the above inequality, we deduce that $v^2 + w^2 \leq R_2^2$. Therefore, we have proved our lemma. ☐

For each $R > 0$, we set $I(R) := \{{}^t(m, n) \in \mathbb{N}^2 \mid m^2 + n^2 \leq R^2\}$.

Proposition 4.3. Let $R > 0$. Then, for each $R \geq 6$,

$$0 < \frac{R^2 - 7R + 7}{2} \leq |I(R)| \leq R^2.$$

Proof. For each $a \in \mathbb{R}$, we denote by $\lfloor a \rfloor$ the maximum integer of the set $\{n \in \mathbb{Z} \mid n \leq a\}$. Let $R \geq 6$. We set $M := \lfloor \sqrt{R^2 - 1} \rfloor (\geq 1)$. For each $m_0 = 1, \ldots, M$, we set $N(m_0) := \lfloor \sqrt{R^2 - m_0^2} \rfloor (\geq 1)$. Note that since $M \leq \sqrt{R^2 - 1} < M + 1$, we deduce that

$$\sqrt{R^2 - 1} - 1 < M \leq \sqrt{R^2 - 1}. \tag{4.3}$$

Also, since $N(m_0) \leq \sqrt{R^2 - m_0^2} < N(m_0) + 1$, we deduce that

$$\sqrt{R^2 - m_0^2} - 1 < N(m_0) \leq \sqrt{R^2 - m_0^2}. \tag{4.4}$$

By using a geometric observation, we deduce that $|I(R)| = \sum_{m_0=1}^{M} N(m_0)$.

By the inequalities (4.3) and (4.4), we deduce that

$$|I(R)| \leq \sum_{m_0=1}^{M} \sqrt{R^2 - m_0^2} \leq RM \leq R\sqrt{R^2 - 1} \leq R^2.$$

We now show that $|I(R)| \geq (R^2 - 7R + 7)/2$. Since $\sqrt{R^2 - m_0^2} \geq R - m_0$ for each $m_0 = 1, \ldots, M$, by the inequalities (4.3) and (4.4) again, we deduce that

$$\begin{aligned}|I(R)| &\geq \sum_{m_0=1}^{M} \left(\sqrt{R^2 - m_0^2} - 1 \right) \geq \sum_{m_0=1}^{M} (R - m_0 - 1) = M(R-1) - \frac{M(M+1)}{2} \\ &= \frac{M(2R-3) - M^2}{2} \geq \frac{\left(\sqrt{R^2-1} - 1\right)(2R-3) - (R^2 - 1)}{2} \\ &\geq \frac{(R-2)(2R-3) - R^2 + 1}{2} = \frac{R^2 - 7R + 7}{2}.\end{aligned}$$

Therefore, we have proved our lemma. □

For each $\tau \in A_0$, we set $N_\tau := \sqrt{2\lambda_2}/\sqrt{\lambda_1} + 1$ (> 2). For each $R > 0$, we set $D_1(\tau, R) := D_1'(\tau, R, N_\tau R)$ and $D_2(\tau, R) := D_2'(R, N_\tau R)$. Note that since $\sqrt{\lambda_2}/\sqrt{\lambda_1} < N_\tau$, we have that $R/\sqrt{\lambda_1} < (N_\tau R)/\sqrt{\lambda_2}$.

Lemma 4.4. Let $\tau \in A_0$. Then, there exist $R_\tau > 0$ and $L_\tau > 0$ such that for all $R > R_\tau$,

$$|\mathbb{N}^2 \cap D_1(\tau, R)| \geq L_\tau R^2 - \frac{7N_\tau}{2\sqrt{\lambda_2}} R.$$

Proof. Let $\tau \in A_0$. We set $L_\tau := N_\tau^2/(2\lambda_2) - 1/\lambda_1$. Note that since $N_\tau > \sqrt{2\lambda_2}/\sqrt{\lambda_1}$, we deduce that $L_\tau > 0$. We set

$$R_\tau := \max\{(6\sqrt{\lambda_2})/N_\tau, 6\sqrt{\lambda_1}\}(> 0).$$

Let $R \geq R_\tau$. Note that $N_\tau R/\sqrt{\lambda_2} \geq 6$, $R/\sqrt{\lambda_1} \geq 6$ and

$$\mathbb{N}^2 \cap D_1(\tau, R) = I\left(\frac{N_\tau R}{\sqrt{\lambda_2}}\right) \setminus I\left(\frac{R}{\sqrt{\lambda_1}}\right). \tag{4.5}$$

Also, we have $I\left((N_\tau R)/\sqrt{\lambda_2}\right) \supset I\left(R/\sqrt{\lambda_1}\right)$. By (4.5) and Proposition 4.3, we deduce that

$$\begin{aligned}|\mathbb{N}^2 \cap D_1(\tau, R)| &= \left|I\left(\frac{N_\tau R}{\sqrt{\lambda_2}}\right)\right| - \left|I\left(\frac{R}{\sqrt{\lambda_1}}\right)\right| \\ &\geq \frac{1}{2}\left(\frac{(N_\tau R)^2}{\lambda_2} - 7\frac{N_\tau R}{\sqrt{\lambda_2}} + 7\right) - \frac{R^2}{\lambda_1} > L_\tau R^2 - \frac{7N_\tau}{2\sqrt{\lambda_2}} R.\end{aligned}$$

Therefore, we have proved our lemma. □

Note that by Lemma 4.4, there exists $Q_\tau > 0$ and $R_\tau' > 0$ such that for all $R > R_\tau'$, we have

$$|\mathbb{N}^2 \cap D_1(\tau, R)| > Q_\tau R^2. \tag{4.6}$$

We now give the proof of the main result Theorem 1.7.

Proof of Theorem 1.7. Let $\tau \in A_0$. Recall that there exists the unique h_τ-conformal measure m_τ of S_τ. We set $r_\tau := K_0 R_\tau^{-1}(> 0)$ and $M_\tau := (7N_\tau)/(2\sqrt{\lambda_2})$.

We first show that for all $r \in (0, r_\tau]$,

$$|I_\tau(r)| = |\{b \in I_\tau \mid r/N_\tau \leq K_0|b|^{-1} < r\}| \geq L_\tau K_0^2 r^{-2} - M_\tau K_0 r^{-1}. \tag{4.7}$$

Let $r \in (0, r_\tau]$. We set $R := K_0 r^{-1}$. Note that $r \le r_\tau$ if and only if $R \ge R_\tau$. Recall that $I_\tau(r) := \{b \in I_\tau \mid r/N_\tau \le K_0|b|^{-1} < r\}$ and

$$I_\tau(r) = \{b \in I_\tau \mid K_0 r^{-1} < |b| \le N_\tau K_0 r^{-1}\} = I_\tau \cap D_2'(K_0 r^{-1}, N_\tau K_0 r^{-1}).$$

Recall that $R := K_0 r^{-1}$, $D_1(\tau, R) := D_1'(\tau, R, N_\tau R)$ and $M_\tau := (7N_\tau)/(2\sqrt{\lambda_2})$. By Lemmas 4.2 and 4.4, it follows that

$$\begin{aligned}|I_\tau(r)| &= |I_\tau \cap D_2'(K_0 r^{-1}, N_\tau K_0 r^{-1})| = |I_\tau \cap D_2'(R, N_\tau R)| \ge |\mathbb{N}^2 \cap D_1'(\tau, R, N_\tau R)| \\ &= |\mathbb{N}^2 \cap D_1(\tau, R)| \ge L_\tau R^2 - M_\tau R = L_\tau K_0^2 r^{-2} - M_\tau K_0 r^{-1}.\end{aligned}$$

Thus, we have proved the inequality (4.15).

We next show that for all $r \in (0, r_\tau]$,

$$m_\tau(B(0,r)) \ge L_\tau K_0^{2-3h_\tau} N_\tau^{-2h_\tau} r^{2h_\tau - 2} - M_\tau K_0^{1-3h_\tau} N_\tau^{-2h_\tau} r^{2h_\tau - 1}. \tag{4.8}$$

By Lemma 4.1 and the definition of $I_\tau(r)$, we have that for all $b \in I_\tau(r)$, $\phi_b(X) \subset B(0, K_0|b|^{-1}) \subset B(0,r)$. It follows that

$$\bigcup_{b \in I_\tau(r)} \phi_b(X) \subset B(0,r). \tag{4.9}$$

In addtion, if $b, b' \in I_\tau$ with $b \neq b'$, then $m_\tau(\phi_b(X) \cap \phi_{b'}(X)) = 0$ by the definition of the conformal measure (Proposition 2.43). Thus, by inequality (4.9) and Lemma 4.1, it follows that

$$\begin{aligned}m_\tau(B(0,r)) &\ge m_\tau\left(\bigcup_{b \in I_\tau(r)} \phi_b(X)\right) = \sum_{b \in I_\tau(r)} m_\tau(\phi_b(X)) \ge \sum_{b \in I_\tau(r)} K_0^{-h_\tau}|b|^{-2h_\tau} \\ &\ge \sum_{b \in I_\tau(r)} K_0^{-h_\tau}\left(\frac{r}{N_\tau K_0}\right)^{2h_\tau} = |I_\tau(r)| K_0^{-3h_\tau} N_\tau^{-2h_\tau} r^{2h_\tau}.\end{aligned}$$

By the inequality (4.15), we obtain that

$$m_\tau(B(0,r)) \ge L_\tau K_0^{2-3h_\tau} N_\tau^{-2h_\tau} r^{2h_\tau - 2} - M_\tau K_0^{1-3h_\tau} N_\tau^{-2h_\tau} r^{2h_\tau - 1}.$$

Thus, we have proved inequality (4.8).

We now show that $\mathcal{H}^{h_\tau}(J_\tau) = 0$. For each $j \in \mathbb{N}$, we set $z_j := 0$ and $r_j := r_\tau / j$ ($\in (0, r_\tau]$). Note that $\{r_j\}_{j \in \mathbb{N}}$ is a sequence in the set of positive real numbers and by Lemma 2.57, $\{z_j\}_{j \in \mathbb{N}}$ is a sequence in $X_\tau(\infty)$. Thus, by the inequality (4.8), we deduce that for each $j \in \mathbb{N}$,

$$\begin{aligned}&\frac{m_\tau(B(z_j, r_j))}{r_j^{h_\tau}} = \frac{m_\tau(B(0, r_j))}{r_j^{h_\tau}} \\ &\ge L_\tau K_0^{2-3h_\tau} N_\tau^{-2h_\tau} r_j^{h_\tau - 2} - M_\tau K_0^{1-3h_\tau} N_\tau^{-2h_\tau} r_j^{h_\tau - 1} \\ &= L_\tau K_0^{2-3h_\tau} N_\tau^{-2h_\tau} r_\tau^{h_\tau - 2} j^{2-h_\tau} - M_\tau K_0^{1-3h_\tau} N_\tau^{-2h_\tau} r_\tau^{h_\tau - 1}\left(\frac{1}{j}\right)^{h_\tau - 1}.\end{aligned}$$

By Lemma 2.56, we have that $2 - h_\tau > 0$ and $h_\tau - 1 > 0$. It follows that

$$\limsup_{j \to \infty} \frac{m_\tau(B(z_j, r_j))}{r_j^{h_\tau}} = \infty.$$

By Theorem 2.45, we obtain that $\mathcal{H}^{h_\tau}(J_\tau) = 0$. □

4.2 Proof of Main Theorem 5

In this section, We show that $0 < \mathcal{P}^{h_\tau}(J_\tau) < \infty$ for each $\tau \in A_0$.

4.2.1 Proof of positiveness of the packing measures

Proof. Let $\tau = u + iv \in A_0$. We set $b_2 := 2 + \tau \in I_\tau$. We use some notations in Lemma 2.52. For any $z = x + iy \in X$,

$$\begin{aligned} g_{b_2}(z) &= z + (2+\tau) \\ &= (x + 2 + u) + i(y + v) \in \{z \in \mathbb{C} \mid \Re z > 1\} = \mathrm{Int}(Y). \end{aligned}$$

Since $f(\partial X) = \partial Y \cup \{\infty\}$ and $f\colon X \to Y \cup \{\infty\}$ is bijective, we have

$$\phi_{b_2}(X) = (f^{-1} \circ g_{b_2})(X) \subset \mathrm{Int}(X).$$

Therefore, we obtain that $J_\tau \cap \mathrm{Int}(X) \neq \emptyset$. Since S_τ is hereditarily regular and $J_\tau \cap \mathrm{Int}(X) \neq \emptyset$, we deduce that $\mathcal{P}^{h_\tau}(J_\tau) > 0$ by Theorem 2.46. □

4.2.2 Proof of finiteness of the packing measures

To prove the finiteness of the packing measure, we prove the following lemmas.

Lemma 4.5. Let $\tau \in A_0$, $\tilde{x} \in \mathbb{R}^2$ and $\tilde{R} > 0$. Then, we have

$$E_\tau(B(E_\tau^{-1}\tilde{x}, \tilde{R}/\sqrt{\lambda_2})) \subset B(\tilde{x}, \tilde{R}).$$

In particular, we have

$$E_\tau(\mathbb{N} \cap B(E_\tau^{-1}\tilde{x}, \tilde{R}/\sqrt{\lambda_2})) \subset I_\tau \cap B(\tilde{x}, \tilde{R}).$$

Proof. Let ${}^t(x,y) \in B(E_\tau^{-1}\tilde{x}, \tilde{R}/\sqrt{\lambda_2})$. We set $\tilde{y} = {}^t(\tilde{y}_1, \tilde{y}_2) := {}^t(x,y) - E_\tau^{-1}\tilde{x}$. Since V_τ is orthogonal and $|\tilde{y}|^2 = |{}^t(x,y) - E_\tau^{-1}\tilde{x}|^2 \leq \tilde{R}^2/\lambda_2$, we have

$$\begin{aligned} &|E_\tau {}^t(x,y) - \tilde{x}|^2 = |E_\tau \tilde{y}|^2 = {}^t\tilde{y}{}^t E_\tau E_\tau \tilde{y} = {}^t\tilde{y} F_\tau \tilde{y} \\ &= {}^t\tilde{y} V_\tau \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} {}^t V_\tau \tilde{y} = \lambda_1 z_1^2 + \lambda_2 z_2^2 \leq \lambda_2 |V_\tau \tilde{y}|^2 = \lambda_2 |\tilde{y}|^2 \leq \tilde{R}^2, \end{aligned}$$

where ${}^t(z_1, z_2) := {}^t V_\tau \tilde{y}$. Therefore, since $E_\tau {}^t(x,y) \in B(\tilde{x}, \tilde{R})$, we have proved our lemma. □

Lemma 4.6. Let $\tau \in A_0$, let $w \in \mathbb{R}^2$ and $\bar{R} > 0$ with $|w| > \bar{R}$. Then, for each $M \geq 2$, we have

$$B\left(w - \frac{\bar{R}}{M}\frac{w}{|w|}, \frac{\bar{R}}{M}\right) \subset B(0, |w|) \cap B(w, 2\bar{R}/M) \subset B(0,|w|) \cap B(w, \bar{R}).$$

In particular, by Lemma 4.5 with $\tilde{x} := w - \bar{R}w/M|w|$ and $\tilde{R} := \bar{R}/M$, we have

$$E_\tau B\left(E_\tau^{-1}\left(w - \frac{\bar{R}}{M}\frac{w}{|w|}\right), \frac{\bar{R}}{\sqrt{\lambda_2}M}\right) \subset B(0,|w|) \cap B(w, \bar{R}).$$

and since $I_\tau = E_\tau(\mathbb{N}^2)$ and E_τ is injective, we have

$$E_\tau\left(\mathbb{N}^2 \cap B\left(E_\tau^{-1}\left(w - \frac{\bar{R}}{M}\frac{w}{|w|}\right), \frac{\bar{R}}{\sqrt{\lambda_2}M}\right)\right) \subset I_\tau \cap B(0,|w|) \cap B(w, \bar{R}).$$

Proof. Let $\tau \in A_0$, $w \in \mathbb{R}^2$, $\bar{R} > 0$ with $|w| > \bar{R}$, $M \geq 2$ and $z \in B\left(w - \frac{\bar{R}}{M}\frac{w}{|w|}, \frac{\bar{R}}{M}\right)$. Then,

$$\begin{aligned}|z| &\leq \left|z - \left(w - \frac{\bar{R}}{M}\frac{w}{|w|}\right)\right| + \left|w - \frac{\bar{R}}{M}\frac{w}{|w|}\right| \\ &< \frac{\bar{R}}{M} + \left|1 - \frac{\bar{R}}{M|w|}\right||w| = \frac{\bar{R}}{M} + \left(1 - \frac{\bar{R}}{M|w|}\right)|w| = |w|,\end{aligned}$$

since $\bar{R} < |w|$ and $M \geq 2$, which deduce that $1 > 1/M > \bar{R}/M|w|$. In addition,

$$|z - w| \leq \left|z - \left(w - \frac{\bar{R}}{M}\frac{w}{|w|}\right)\right| + \left|\frac{\bar{R}}{M}\frac{w}{|w|}\right| < \frac{\bar{R}}{M} + \frac{\bar{R}}{M} = \frac{2\bar{R}}{M}$$

and since $M \geq 2$, we have $2\bar{R}/M \leq \bar{R}$. Therefore, we have proved our lemma. □

Note that by similar argument of Proposition 4.3, Lemma 4.4 and the equation (4.6), we obtain the following lemma.

Lemma 4.7. Let $\tau \in A_0$, $w \in \mathbb{R}^2$, $\bar{R} > 0$ with $|w| > \bar{R}$. Then, there exist $C_\tau > 0$ and $Q'_\tau > 0$ such that for all $\bar{R} \geq C_\tau$, we have

$$|I_\tau \cap B(0, |w|) \cap B(w, \bar{R})| \geq Q'_\tau \bar{R}^2$$

We now prove the finiteness of the packing measure of J_τ.

Proof of the finiteness of packing measures in Theorem 1.8. To prove Main Theorem 5, it suffice to show the assumption of Lemma 2.47 for each $\tau \in A_0$. Let $\tau \in A_0$. Let $r_0 < \min\{1/8, K_0 (R'_\tau)^{-1}\}$, where R'_τ is defined in the equation (4.6). Note that there exists $\tilde{R} > \max\{C_\tau, 1\}$ such that for each $R > \tilde{R}$, $(R-1)/R \geq 1/2$, where $C_\tau > 0$ is defined in Lemma 4.7. We set $L'_\tau := \min\{Q'_\tau/4, (\tilde{R}+1)^{-2}\}$ (> 0), where Q'_τ is defined in Lemma 4.7. We also set $\xi := r_0^2 (> 0)$, $\gamma := K (\geq 1)$ and

$$L_\tau := \min\left\{L'(4K)^{-h_\tau}, Q_\tau K_0^{1-2h_\tau} N_\tau^{-2h_\tau} 2^{2-2h_\tau}\right\} (> 0).$$

Let $b := m + n\tau \in I_\tau$ and $r > 0$ with $\gamma \text{diam}(\phi_b(X)) \leq r \leq \xi$. We set $x := 1/b = \phi_b(0) \in \phi_b(V)$. We consider the following three cases.

1. $r \leq |x|/2$.

Note that by the assumption, we have $0 < r$ $(\leq |x|/2) < |x|$ and

$$|x|^2 = K \cdot K^{-1}|b|^{-2} \leq \gamma \cdot \text{diam}\phi_b(X) \leq r \tag{4.10}$$

We set $f(z) := 1/z$ $(z \in \mathbb{C} \setminus \{0\})$. We show that for each $B(x, r)$ with $r < |x|$, we have

$$f(B(x, r)) = B\left(\frac{|x|^2}{|x|^2 - r^2} \cdot \frac{1}{x}, \ \frac{r}{|x|^2 - r^2}\right).$$

Indeed, for each $a \in \mathbb{C}$, $|1/a - \overline{x}/(|x|^2 - r^2)| = r/(|x|^2 - r^2)$ is equiavalent to $|r^2 - \overline{x}(x-a)| = r|a|$ which is also equivalent to

$$r^4 - r^2(\overline{x}(x-a) + x(\overline{x} - \overline{a}) + a\overline{a}) + x\overline{x}(x-a)(\overline{x} - \overline{a}) = 0. \tag{4.11}$$

Since $(\overline{x}(x-a)+x(\overline{x}-\overline{a})+a\overline{a}) = x\overline{x}+(x-a)(\overline{x}-\overline{a})$, the equation (4.11) is equivalent to $(r^2-|x-a|^2)(r^2-|x|^2)=0$. Since $r<|x|$, we deduce that $|1/a-\overline{x}/(|x|^2-r^2)| = r/(|x|^2-r^2)$ is equivalent to $r=|x-a|$. In addition, since $f(x)=1/x$ satisfies

$$\left|f(x)-\frac{|x|^2}{|x|^2-r^2}\cdot\frac{1}{x}\right| = \frac{r}{|x|^2-r^2}\frac{r}{|x|} < \frac{r}{|x|^2-r^2},$$

we obtain that

$$f(B(x,r)) = B\left(\frac{|x|^2}{|x|^2-r^2}\cdot\frac{1}{x},\ \frac{r}{|x|^2-r^2}\right). \tag{4.12}$$

We set

$$w := \frac{|x|^2}{|x|^2-r^2}\cdot\frac{1}{x} \quad\text{and}\quad R := \frac{r}{|x|^2-r^2}.$$

Note that $|w| := |x|/(|x|^2-r^2)$. We set

$$I_\tau(x,r) := \{b\in I_\tau \mid \phi_b(X)\subset B(x,r)\} = \{b\in I_\tau \mid B(b+1/2,1/2)\subset f(B(x,r))\} \tag{4.13}$$

and $I_{\tau,1} := I_\tau(x,r)\cap B(0,|w|)$.

We now show that $b\in I_{\tau,1}$. Note that since $|b| = 1/|x| \le |x|/(|x|^2-r^2) = |w|$, we have $b\in B(0,|w|)$. Therefore, it is sufficient to show that $|w-(b+1/2)|\le R-1/2$. Indeed, we set $A := 1-|b|^2r^2$ for simplicity. Since $r\le |x|$, we have $A>0$. Then, since $|b|^2r-1\ge 0$, which is equivalent to $r\ge |x|^2$, we have

$$\begin{aligned}
&\left|2|b|^2r-A\right|^2 - \left(2r^2|b|^2b-A\right)\left(2r^2|b|^2\overline{b}-A\right)\\
&= 4|b|^4r^2 - 2A\cdot 2|b|^2r + A^2 - (2|b|^2r^2)^2\cdot|b|^2 + 2|b|^2r^2A\cdot(b+\overline{b}) - A^2\\
&= 4|b|^4r^2 - 4|b|^6r^4 - 4A|b|^2r + 4|b|^2r^2A\cdot\Re(b) = 4|b|^4r^2(1-|b|^2r^2) + 4A|b|^2r(r\Re(b)-1)\\
&= 4A|b|^4r^2 + 4A|b|^2r(r\Re(b)-1) = 4A|b|^2r(|b|^2r + r\Re(b)-1)\ge 0,
\end{aligned}$$

where $\Re(b)\ge 0$ is the real part of b. In addition, since $|b|^2r-1\ge 0$, we have

$$2|b|^2r - A = |b|^2r^2 + |b|^2r + |b|^2r - 1 \ge |b|^2r-1\ge 0.$$

Therefore, we have $|2r^2|b|^2b-A|\le 2|b|^2r-A$. It follows that

$$\begin{aligned}
\left|w-\left(b+\frac{1}{2}\right)\right| &= \left|\frac{|x|^2}{|x|^2-r^2}\frac{1}{x} - b - \frac{1}{2}\right| = \left|\frac{b}{1-|b|^2r^2} - b - \frac{1}{2}\right|\\
&= \frac{|2b-2b(1-|b|^2r^2)-(1-|b|^2r^2)|}{2(1-|b|^2r^2)} = \frac{|2b|b|^2r^2-A|}{2(1-|b|^2r^2)}\\
&\le \frac{2|b|^2r-A}{2(1-|b|^2r^2)} = \frac{|b|^2r}{1-|b|^2r^2} - \frac{1-|b|^2r^2}{2(1-|b|^2r^2)} = \frac{r}{|x|^2-r^2} - \frac{1}{2} = R-\frac{1}{2}.
\end{aligned}$$

Thus, we have proved $B(b+1/2,1/2)\subset B(w,R)$.

Note that by the inclusion (4.13) and (4.12) we have

$$\begin{aligned}
I_{\tau,1} &= I_\tau(x,r)\cap B(0,|w|) = \{b\in I_\tau \mid B(b+1/2,1/2)\subset f(B(x,r))\}\cap B(0,|w|)\\
&= \{b\in I_\tau \mid B(b+1/2,1/2)\subset B(w,R)\}\cap B(0,|w|)\\
&\supset \{b\in I_\tau \mid b\in B(w,R-1)\}\cap B(0,|w|) = I_\tau\cap B(w,R-1)\cap B(0,|w|) \qquad (4.14)
\end{aligned}$$

Recall that there exists $\tilde{R} > \max\{C_\tau, 1\}$ such that for each $R>\tilde{R}$, $(R-1)/R\ge 1/2$, where $C_\tau>0$ is defined in Lemma 4.7.

We next show that $|I_{\tau,1}| \geq L'_\tau R^2$ for each $R > 0$. To prove this, we consider the following two cases.

1-1. $R \geq \tilde{R} + 1$.

Recall that since $|x| > r$, we have $|w| > R (> \tilde{R} > 0)$. By the inclusion (4.14) and Lemma 4.7 with $\bar{R} = R - 1$ $(= \tilde{R} > 0)$, we have

$$|I_{\tau,1}| \geq |I_\tau \cap B(w, R-1) \cap B(0, |w|)| \geq Q'_\tau (R-1)^2 \geq \frac{Q'_\tau}{4} R^2 \geq L'_\tau R^2.$$

1-2. $\tilde{R} + 1 \geq R$.

Since $I_{\tau,1}$ is not empty set and $\tilde{R} + 1 \geq R$, we have $|I_{\tau,1}| \geq 1 \geq (R/\tilde{R}+1)^2 \geq L'_\tau R^2$. Hence, we have proved $|I_{\tau,1}| \geq L'_\tau R^2$ for each $R > 0$.

We finally show Main Theorem 5 if $r \leq |x|/2$. Since $|x|^2 - r^2 \geq 3|x|^2/4$ and therefore $|w| = |x|/(|x|^2 - r^2) \leq 4/3|x|$, and $h_\tau < 2$, we obtain that

$$\begin{aligned}
m_\tau(B(x,r)) &\geq m_\tau\left(\bigcup_{b \in I_{\tau,1}} \phi_b(X)\right) = \sum_{b \in I_{\tau,1}} \int_X |\phi'_b|^{h_\tau} \mathrm{d}m_\tau \geq \sum_{b \in I_{\tau,1}} K^{-h_\tau} |b|^{-2h_\tau} m_\tau(X) \\
&= \sum_{b \in I_{\tau,1}} K^{-h_\tau} |x|^{2h_\tau} \geq |I_{\tau,1}| K^{-h_\tau} |x|^{2h_\tau} \geq L'_\tau \left(\frac{r}{|x|^2 - r^2}\right)^2 K^{-h_\tau} |x|^{2h_\tau} \\
&\geq L'_\tau \frac{r^2}{|x|^4} K^{-h_\tau} |x|^{2h_\tau} = L'_\tau K^{-h_\tau} r^2 |x|^{2h_\tau - 4} \geq L'_\tau K^{-h_\tau} r^2 r^{h_\tau - 2} \geq L_\tau r^{h_\tau}.
\end{aligned}$$

By Threorem 2.47, we have proved Main Theorem 5 if $r \leq |x|/2$.

2. $|x|/2 \leq r$ and $r < 2|x|$.

We set $\tilde{r} := r/4$. Then, by the assumption, we have $\tilde{r} < |x|/2$. In addition, since $|x|^2 = K \cdot K^{-1}|b|^{-2} \leq \gamma \cdot \mathrm{diam}\phi_b(X) \leq r \leq \xi = r_0^2$, we have

$$|x|^2 \leq r_0|x| \leq \frac{1}{8} \cdot 2r = \tilde{r}.$$

Therefore, instead of $r > 0$, $\tilde{r} > 0$ satisfies the assumption $r \geq |x|/2$ and the inequality (4.10) in the case 1. $r \leq |x|/2$. Thus, by the similar argument discussed in the case 1. $r \leq |x|/2$, we have

$$\begin{aligned}
m_\tau(B(x,r)) &\geq m_\tau(B(x,\tilde{r})) \geq L'_\tau K^{-h_\tau} \tilde{r}^{h_\tau} \\
&\geq L'_\tau K^{-h_\tau} 4^{-h_\tau} r^{h_\tau} \geq L'_\tau (4K)^{-h_\tau} r^{h_\tau} \geq L_\tau r^{h_\tau}.
\end{aligned}$$

By Threorem 2.47, we have proved Main Theorem 5 if $|x|/2 \leq r$ and $r < 2|x|$.

3. $2|x| \leq r$.

We set $\tilde{r} := r/2$. Note that $B(0,\tilde{r}) \subset B(x,r)$ since, if $y \in B(0,\tilde{r})$, then $|y - x| \leq |x| + |y| \leq r/2 + \tilde{r} = r$. We set $r_\tau := K_0 R_\tau^{-1} (> 0)$, where R_τ is defined in the inequality (4.6). We show that

$$|I_\tau(\tilde{r})| = |\{b \in I_\tau \mid \tilde{r}/N_\tau \leq K_0|b|^{-1} < \tilde{r}\}| \geq Q_\tau K_0^2 \tilde{r}^{-2}. \tag{4.15}$$

Indeed, note that $\tilde{r} = r/2 < r_0 < r_\tau$. We set $R := K_0 \tilde{r}^{-1}$. Note that $\tilde{r} \leq r_\tau$ if and only if $R \geq R_\tau$. Recall that $I_\tau(\tilde{r}) := \{b \in I_\tau \mid \tilde{r}/N_\tau \leq K_0|b|^{-1} < \tilde{r}\}$ and

$$I_\tau(\tilde{r}) = \{b \in I_\tau \mid K_0\tilde{r}^{-1} < |b| \leq N_\tau K_0 \tilde{r}^{-1}\} = I_\tau \cap D'_2(K_0\tilde{r}^{-1}, N_\tau K_0 \tilde{r}^{-1}).$$

By Lemmas 4.2 and 4.4, we obtain that

$$\begin{aligned}
|I_\tau(\tilde{r})| &= |I_\tau \cap D'_2(K_0\tilde{r}^{-1}, N_\tau K_0\tilde{r}^{-1})| = |I_\tau \cap D'_2(R, N_\tau R)| \geq |\mathbb{N}^2 \cap D'_1(\tau, R, N_\tau R)| \\
&= |\mathbb{N}^2 \cap D_1(\tau, R)| \geq Q_\tau R^2 = Q_\tau K_0^2 \tilde{r}^{-2}.
\end{aligned}$$

Thus, we have proved the inequality (4.15). Therefore, we obtain that

$$\begin{aligned} m_\tau(B(x,r)) &\geq m_\tau(B(0,\tilde{r})) \geq m_\tau\left(\bigcup_{b\in I_\tau(\tilde{r})} \phi_b(X)\right) = \sum_{b\in I_\tau(\tilde{r})} m_\tau\left(\phi_b(X)\right) \\ &\geq \sum_{b\in I_\tau(\tilde{r})} \int_X |\phi_b'|^{h_\tau} \mathrm{d}m_\tau \geq \sum_{b\in I_\tau(\tilde{r})} K_0^{-1}|b|^{-2h_\tau} \geq \sum_{b\in I_\tau(\tilde{r})} K_0^{-1}\left(\frac{\tilde{r}}{KN_\tau}\right)^{2h_\tau} \\ &= |I_\tau(\tilde{r})| K_0^{-(1+2h_\tau)} N_\tau^{-2h_\tau} \tilde{r}^{2h_\tau} \geq Q_\tau K_0^{1-2h_\tau} N_\tau^{-2h_\tau} \tilde{r}^{2h_\tau - 2} \\ &= Q_\tau K_0^{1-2h_\tau} N_\tau^{-2h_\tau} 2^{2-2h_\tau} r^{2h_\tau-2} \geq Q_\tau K_0^{1-2h_\tau} N_\tau^{-2h_\tau} 2^{2-2h_\tau} r^{h_\tau} \geq L_\tau r^\tau. \end{aligned}$$

By Threorem 2.47, we have proved Main Theorem 5 if $2|x| \leq r$. Hence, by the three cases, we have proved Main Theorem 5. □

5 Non-autonomous iterated function systems and the limit sets

5.1 Basic properties

In this subsection, we consider non-autonomous iterated fucntion systems (for short, NAIFS). Before we study the NAIFSs, we first consider some properties of sequences generated by contractive mappings on complete metric spaces.

Definition 5.1. Let (X,ρ) be a complete metric space. We say that $f_j\colon X \to X$ ($j \in \mathbb{N}$) is a sequence of contractive mappings on X with an uniform contraction constant $c \in (0,1)$ if there exists $c \in (0,1)$ such that for all $j \in \mathbb{N}$ and $x, y \in X$,

$$\rho(f_j(x), f_j(y)) \leq c\ \rho(x,y).$$

For each $m, n \in \mathbb{N}$ with $m < n$, we set $g(m,n)\colon X \to X$ defined by

$$g(m,n) := f_m \circ \cdots \circ f_{n-1} \tag{5.1}$$

and $g(m,m) := \mathrm{id}_X$ (We call $\{g(m,n)\}_{n=m}^\infty$ the non-autonomous right iterated sequence of the contractive mappings on X with $\{f_j\}_{j\in\mathbb{N}}$).

Note that since X is complete, for each $j \in \mathbb{N}$, there exists the unique fixed point z_j of the f_j. We set $Z := \{z_j \in X \mid j \in \mathbb{N}\}$. For each $r > 0$ and $A \subset X$, we set $A_r := \{y \in X \mid \exists a \in A, \rho(y,a) \leq r\}$.

Lemma 5.2. Let $f_j\colon X \to X$ ($j \in \mathbb{N}$) be a sequence of contractive mappings on a complete metric space (X,d) with an uniform contraction constant $c \in (0,1)$. Suppose that Z is bounded. Then, there exists $r_Z > 0$ such that for each $r \geq r_Z$ and $j \in \mathbb{N}$, $f_j(Z_r) \subset Z_r$.

Proof. We set $M := \operatorname{diam} Z < \infty$. We set $r_Z := cM/(1-c)$. Let $r \geq r_Z$, $j \in \mathbb{N}$ and $x \in Z_r$. Then, there exists $j_0 \in \mathbb{N}$ such that $d(x, z_{j_0}) \leq r$. In addition, we have

$$\begin{aligned} \rho(f_j(x), z_j) = \rho(f_j(x), f_j(z_j)) &\leq c\rho(x, z_j) \\ &\leq c(\rho(x, z_{j_0}) + \rho(z_{j_0}, z_j)) \leq c(r+M) \leq r \end{aligned}$$

since $cM/(1-c) \leq r$. Therefore, we have proved our lemma. □

The following lemma shows that if we consider a sequence of contractive mappings defined on a bounded set in some sense, then there exists the sequence of limit points generated by contractive mappings and it is bounded.

Lemma 5.3. Let $\{f_j\}_{j\in\mathbb{N}}$ $(j \in \mathbb{N})$ be a sequence of contractive mappings on a complete metric space (X,ρ) with an uniform contraction constant $c \in (0,1)$. Suppose that Z is bounded. Then, for each $m \in \mathbb{N}$, there exists $x_m(\infty) \in X$ such that for all $x \in X$, the sequence $\{g(m,n)(x)\}_{n=m}^{\infty}$ converges to $x_m(\infty)$. In addition, $\{x_m(\infty) \in X \mid m \in \mathbb{N}\}$ is bounded and for each $m_1, m_2 \in \mathbb{N}$ with $m_1 \leq m_2$, we have

$$g(m_1,m_2)(x_{m_2}(\infty)) = x_{m_1}(\infty).$$

Proof. Let $m \in \mathbb{N}$ and $x \in X$. We set $x_m(n) := g(m,m+n)(x)$ $(n \in \mathbb{N})$. Note that by Lemma 5.2, there exists $r \geq r_Z$ such that $x \in Z_r$ and for each $k \in \mathbb{N}$, $f_{m+k}(x) \in f_{m+k}(Z_r) \subset Z_r$. Then, for all $n_1, n_2 \in \mathbb{N}$ with $n_1 < n_2$,

$$\begin{aligned}
&\rho(x_m(n_1), x_m(n_2)) \\
&\leq \sum_{k=n_1}^{n_2-1} \rho(x_m(k), x_m(k+1)) = \sum_{k=n_1}^{n_2-1} \rho(g(m,m+k)(x), g(m,m+k+1)(x)) \\
&= \sum_{k=n_1}^{n_2-1} c^k \rho(x, f_{m+k}(x)) = \sum_{k=n_1}^{n_2-1} c^k \text{diam } Z_r.
\end{aligned}$$

Therefore, we deduce that $\{x_m(n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence on the complete metric space X and there exists $x_m(\infty) \in X$ such that $x_m(n) \longrightarrow x_m(\infty)$ as $n \longrightarrow \infty$. In addition, Let $m \in \mathbb{N}$ and $y \in X$. We set $y_m(n) := g(m,m+n)(y)$ $(n \in \mathbb{N})$. By the same argument, there exists $y_m(\infty)$ such that $y_m(n) \longrightarrow y_m(\infty)$ as $n \longrightarrow \infty$. Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\rho(x_m(\infty), y_m(\infty)) &\leq \rho(x_m(\infty), x_m(n)) + \rho(x_m(n), y_m(n)) + \rho(y_m(n), y_m(\infty)) \\
&\leq \rho(x_m(\infty), x_m(n)) + c^n \rho(x,y) + \rho(y_m(n), y_m(\infty)) \overset{n\longrightarrow\infty}{\longrightarrow} 0.
\end{aligned}$$

Let $x \in X$. Since Z is bounded, so is Z_r. Since there exists $r \geq r_Z$ such that $x_m(n) \in Z_r$ for all $m, n \in \mathbb{N}$, $\{x_m(\infty) \in X \mid m \in \mathbb{N}\}$ is included by the closure of Z_r. Therefore, $\{x_m(\infty) \in X \mid m \in \mathbb{N}\}$ is bounded. Finally, let $m_1, m_2 \in \mathbb{N}$ with $m_1 \leq m_2$. Then, for all $n \in \mathbb{N}$ and $x \in X$,

$$g(m_1, m_2+n)(x) = g(m_1,m_2) \circ g(m_2, m_2+n)(x) = g(m_1,m_2)(x_{m_2}(n)).$$

Since $x_{m_1}((m_2 - m_1) + n) = g(m_1, m_2+n)(x) \overset{n\longrightarrow\infty}{\longrightarrow} x_{m_1}(\infty)$, $x_{m_2}(n) \overset{n\longrightarrow\infty}{\longrightarrow} x_{m_2}(\infty)$ and $g(m_1,m_2)$ is continuous, we have $g(m_1,m_2)(x_{m_2}(\infty)) = x_{m_1}(\infty)$. Therefore, we have proved our lemma. □

Corollary 5.4. Under the assumption of Lemma 5.3, for all $m \in \mathbb{N}$ and $x \in X$, the sequence $\{g(m,n)(x)\}_{n=m}^{\infty}$ convrges to $x_m(\infty)$ exponentially fast with the rate c.

Proof. Let $x \in X$. There exists $r > r_Z$ such that $x \in Z_r$ and $\{x_m(\infty) \in X \mid m \in \mathbb{N}\} \subset \overline{Z_r}^{\rho}$, where $\overline{Z_r}^{\rho} \subset X$ is the closure of $Z_r \subset X$. Then, for all $m, n \in \mathbb{N}$ with $m \leq n$, we have

$$\begin{aligned}
\rho(x_m(\infty), g(m,n)(x)) &\leq \rho(g(m,n)(x_n(\infty)), g(m,n)(x)) \\
&\leq c^{n-m} \rho(x_n(\infty), x) \leq c^{n-m} \text{diam} \overline{Z_r}^{\rho}.
\end{aligned}$$

Since $\overline{Z_r}^{\rho}$ is bounded, we have proved our corollary. □

We now present the general definition of NAIFSs. Let I be a set and (X,ρ) be a complete metric space.

Definition 5.5. We say that $(\{f_i\}_{i\in I}, \{J_n\}_{n\in\mathbb{N}})$ satisfy the setting (NAIFS) if

(i) $\{J_n\}_{n\in\mathbb{N}}$ is a sequence in $\{J \subset I \mid J \text{ is finite}\}$, and

(ii) $\{f_i \colon X \to X\}_{i\in I}$ is a family of contractive mappings on X with the uniform contraction constant $c \in (0,1)$, that is, there exists $c \in (0,1)$ such that for all $i \in I$ and $x, y \in X$,
$$\rho(f_i(x), f_i(y)) \le c\ \rho(x,y).$$

Note that for each $i \in I$, there exists $z_i \in X$ such that z_i is the unique fixed point of f_i since X is complete and f_i is contractive on X.

Let $\mathcal{K}(X)$ be the set of non-empty compact subsets in a complete metric space (X,ρ). For each $\epsilon > 0$ and set $A \subset X$, we set $A_\epsilon := \{x \in X \mid {}^{\exists}a \in A,\ \text{s.t. } \rho(a,x) \le \epsilon\}$. Let d_H be the Hausdorff distance on $\mathcal{K}(X)$ which is defined by

$$d_H(A,B) := \inf\{\epsilon > 0 \mid A \subset B_\epsilon, B \subset A_\epsilon\} \quad (A, B \in \mathcal{K}(X)).$$

Let $F_n \colon \mathcal{K}(X) \to \mathcal{K}(X)$ $(n \in \mathbb{N})$ be mappings defined by

$$F_n(A) := \bigcup_{i\in J_n} f_i(A),$$

which is well-defined since f_i is continuous and J_n is finite. Each mapping F_n is often called the Barnsley operator.

Note that since (X,ρ) is complete, $(\mathcal{K}(X), d_H)$ is also complete (For example, see [17]). In addition, if $\{L_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{K}(X), d_H)$, then $\{L_n\}_{n\in\mathbb{N}}$ tend to

$$L := \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{k\ge n} L_k}^{\rho} \in \mathcal{K}(X) \tag{5.2}$$

as n tends to infinity, where $\overline{A}^{\rho}$ is the closure of $A \subset X$ with respect to the metric ρ. Note that $\{F_n\}_{n\in\mathbb{N}}$ is the sequence of the contractive mappings on $(\mathcal{K}(X), d_H)$ with an uniform contraction constant $c \in (0,1)$ since for all $A, B \in \mathcal{K}(X)$ and $n \in \mathbb{N}$,

$$d_H\left(\bigcup_{i\in J_n} f_i(A), \bigcup_{i\in J_n} f_i(B)\right) \le \max_{i\in J_n} d_H(f_i(A), f_i(B)) \le \max_{i\in J_n} c\ d_H(A,B) = c\ d_H(A,B).$$

Note that since $\mathcal{K}(X)$ is complete and F_n is contractive on $\mathcal{K}(X)$ for each $n \in \mathbb{N}$, there exists $A_n \in X$ such that A_n is the unique fixed point of F_n for each $n \in \mathbb{N}$. We set

$$Z_H := \{A_n \in \mathcal{K}(X) \mid n \in \mathbb{N}, A_n \text{ is the unique fixed point of } F_n\}.$$

To apply Lemma 5.3, we consider a sufficient condition to show Z_H is bounded in $\mathcal{K}(X)$. Recall that for each $i \in I$, there exist the unique fixed points z_i of f_i.

Lemma 5.6. Let $(\{f_i\}_{i\in I}, \{J_n\}_{n\in\mathbb{N}})$ satisfy the setting (NAIFS). Suppose that $Z := \{z_i \in X \mid i \in I\}$ is bounded. Then, Z_H is bounded.

Proof. By Lemma 5.2, there exists $r_Z > 0$ such that for all $r \geq r_Z$ and $i \in I$, $f_i(Z_r) \subset Z_r$. We set $r \geq r_Z$. Let $A \in \mathcal{K}(X)$ with $A \subset Z_r$. (for example, we can take A as set of one point in Z). Since $f_i(A) \subset f_i(Z_r) \subset Z_r$ for each $i \in I$, we have for each $n \in \mathbb{N}$, $\bigcup_{i \in J_n} f_i(A) \subset Z_r$, i.e. $F_n(A) \subset Z_r$. We deduce that for any $m \in \mathbb{N}$,

$$F_n^{(m)}(A) \subset F_n^{(m-1)}(Z_r) \subset Z_r,$$

where $F_n^{(m)}$ is the m-th composition of F_n (i.e. $F_n^{(m)}(A) := (\overbrace{F_n \circ \cdots \circ F_n}^{m})(A)$ $A \in \mathcal{K}(X)$). In addition, for each $n \in \mathbb{N}$, $\{F_n^{(m)}(A)\}_{m \in \mathbb{N}}$ is a Cauchy sequence in $(\mathcal{K}(X), d_H)$ and $F_n^{(m)}(A)$ tend to the unique fixed point A_n in $\mathcal{K}(X)$ as $m \longrightarrow \infty$ by the Banach fixed point theorem. Therefore, we have for each $n \in \mathbb{N}$,

$$A_n = \bigcap_{m \in \mathbb{N}} \overline{\bigcup_{l \geq m} F_n^{(m)}(A)}^{\rho} \subset \overline{Z_r}^{\rho}.$$

We set $R := \mathrm{diam}\overline{Z_r}^{\rho} = \mathrm{diam}Z_r < \infty$. We show that for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$, $d_H(A_n, A_m) \leq R$. Let $x \in A_m$. For any $y \in A_n$, $d(x, y) \leq \mathrm{diam}\overline{Z_r}^{d} = R$ since $A_m \subset \overline{Z_r}^{d}$ and $A_n \subset \overline{Z_r}^{d}$. We deduce that $A_m \subset (A_n)_R$. By the same argument, we have $A_n \subset (A_m)_R$. Therefore, $d_H(A_n, A_m) \leq R$. Thus, we have proved our lemma. □

Theorem 5.7. Let $(\{f_i\}_{i \in I}, \{J_n\}_{n \in \mathbb{N}})$ satisfy the setting (NAIFS). Suppose that $Z := \{z_i \in X \mid i \in I\}$ is bounded. Then, there exists a sequence $\{K_m\}_{m \in \mathbb{N}} \subset \mathcal{K}(X)$ such that for each $m \in \mathbb{N}$ and $A \in \mathcal{K}(X)$, $G(m, n)(A)$ converges to K_m as n tends to infinity, where for each $m, n \in \mathbb{N}$ with $m < n$, $G(m, n) := F_m \circ \cdots \circ F_{n-1}$.

In addition, $\{K_m \in \mathcal{K}(X) \mid m \in \mathbb{N}\}$ is a bounded set in $(\mathcal{K}(X), d_H)$, for each $m_1, m_2 \in \mathbb{N}$ with $m_1 \leq m_2$, we have $G(m_1, m_2)(K_{m_2}) = K_{m_1}$ and for all $m \in \mathbb{N}$ and $A \in \mathcal{K}(X)$, $\{G(m, n)(A)\}_{n=m}^{\infty}$ convrges to K_m exponentially fast with the rate c.

Proof. By Lemma 5.6, we have that Z_H is bounded. Therefore, by applying the Lemma 5.3 and Corollary 5.4. Thus, we have proved our theorem. □

By Theorem 5.7, we define the limit set (or the sequence of the limit sets) as the limit of the sequence in $\mathcal{K}(X)$ which is constructed by the Barnsley operators of an NAIFS defiend on a bounded set. Note that if the set of the unique fixed point of the contractive mappings is bounded, then we deduce to the case that we consider NAIFSs defined on a bounded set. This is the same case of [29]. However, we consider NAIFSs with the weaker assumption in Main Theorem 6 and Main Theorem 7.

We next consider the sequence of the limit sets and the sequence of the limt measures generated by NAIFSs with weights $(\{f_i\}_{i \in I}, \{J_n\}_{n \in \mathbb{N}}, \{p_n\}_{n \in \mathbb{N}})$ (for short, wNAIFSs). Let I be a set and (X, ρ) be a complete separable metric space.

Definition 5.8. We say that $(\{f_i\}_{i \in I}, \{J_n\}_{n \in \mathbb{N}}, \{p_n\}_{n \in \mathbb{N}})$ satisfy the setting (wNAIFS) if

(i) $\{J_n\}_{n \in \mathbb{N}}$ is a sequence in $\{J \subset I \mid J \text{ is finite}\}$,

(ii) $\{f_i \colon X \to X\}_{i \in I}$ is a family of contractive mappings on X with an uniform contraction constant $c \in (0, 1)$, that is, there exists $c \in (0, 1)$ such that for all $i \in I$ and $x, y \in X$,
$$\rho(f_i(x), f_i(y)) \leq c\ \rho(x, y), \text{ and}$$

(iii) for each $n \in \mathbb{N}$, p_n is $[0,\infty)$-valued functions on I with $p_n(i) > 0$ if and only if $i \in J_n$, and p_n satisfies
$$\sum_{i \in J_n} p_n(i) = 1.$$

Let $\mathcal{P}_1(X)$ be the set of Borel probability measures defined on a complete separable metric space (X,ρ) for which there exists $a \in X$ such that the function $x \mapsto \rho(a,x)$ is integrable. Note that for each $b \in X$ and $P \in \mathcal{P}_1(X)$, we have $\int_X \rho(b,x)P(\mathrm{d}x) < \infty$ since

$$\int_X \rho(b,x)\ P(\mathrm{d}x) \leq \int_X \rho(b,a)\ P(\mathrm{d}x) + \int_X \rho(a,x)\ P(\mathrm{d}x) = \rho(b,a) + \int_X \rho(a,x)\ P(\mathrm{d}x) < \infty.$$

Let $\mathrm{Lip}_1(X)$ be a set of $\mathbb{R}$-valued functions f on X with $|f(x) - f(y))| \leq \rho(x,y)$ for all $x, y \in X$. Let d_{MK} be the Monge-Kantrovich distance on $\mathcal{P}_1(X)$ which is defined by

$$d_{MK}(\mu,\nu) := \sup\left\{\int_X f\mathrm{d}\mu - \int_X f\mathrm{d}\nu \mid f \in \mathrm{Lip}_1(X)\right\} \quad (\mu,\nu \in \mathcal{P}_1(X)).$$

Let $M_n \colon \mathcal{P}_1(X) \to \mathcal{P}_1(X)$ $(n \in \mathbb{N})$ be mappings defined by

$$M_n(\mu)(B) := \sum_{i \in J_n} p_n(i)\ \mu(f_i^{-1}(B)) \quad (B \in \mathcal{B}(X)),$$

where $\mathcal{B}(X)$ is the set of the Borel sets in X. Note that for each $n \in \mathbb{N}$, M_n is well-defined since

$$\begin{aligned}
&\int_X \rho(x,a)\ \mathrm{d}M_n(\mu) = \sum_{i \in J_n} p_n(i) \int_X \rho(x,a)\ \mathrm{d}(\mu \circ f_i^{-1}) \\
&= \sum_{i \in J_n} p_n(i) \int_X \rho(f_i(x),a)\ \mathrm{d}\mu \leq \sum_{i \in J_n} p_n(i) \int_X \rho(f_i(x), f_i(z_i)) + \rho(f_i(z_i),a)\ \mathrm{d}\mu \\
&= \sum_{i \in J_n} p_n(i) \left\{\int_X c\ \rho(z_i,x)\ \mathrm{d}\mu + \rho(z_i,a)\right\} < \infty.
\end{aligned}$$

Each mapping M_n is often called the Foias operator.

Note that $(\mathcal{P}_1(X), d_{MK})$ is a complete separable metric space since (X,ρ) is complete and separable (For example, [18], [34]). In addition, $\{M_n\}_{n\in\mathbb{N}}$ is the sequence of the contractive mappings on $(\mathcal{P}_1(X), d_{MK})$ with an uniform contraction constant $c \in (0,1)$. Indeed, since $(f \circ f_i)/c \in \mathrm{Lip}_1(X)$ for each $i \in I$, for all $\mu,\nu \in \mathcal{P}_1(X)$, $f \in \mathrm{Lip}_1(X)$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\int_X f\mathrm{d}M_n(\mu) - \int_X f\mathrm{d}M_n(\nu) &\leq \sum_{i \in J_n} p_n(i) \left(\int_X f_i \circ f\mathrm{d}\mu - \int_X f_i \circ f\mathrm{d}\nu\right) \\
&\leq \sum_{i \in J_n} p_n(i)\ c\ d_{MK}(\mu,\nu) = c\ d_{MK}(\mu,\nu).
\end{aligned}$$

Note that for each $n \in \mathbb{N}$, there exists $\mu_n \in \mathcal{P}_1(X)$ such that μ_n is the unique fixed point of M_n since $\mathcal{P}_1(X)$ is complete and for each $n \in \mathbb{N}$, M_n is contractive on $\mathcal{P}_1(X)$. Later, we consider the limit sets and the limit measures gnerated by the wNAIFSs under some assumption.

5.2 Proof of Main Theorem 6

To prove the existence of the limit (1.1) and the equation (1.2) in Main Theorem 6, we first prove the following lemma.

Lemma 5.9. Let $\{f_j\colon X \to X\}_{j\in\mathbb{N}}$ be a sequence of contractive mappings on a complete metric space (X, ρ) with an uniform contraction constant $c \in (0, 1)$. For each $j \in \mathbb{N}$, let $z_j \in X$ be the unique fixed point of f_j. If there exists $x' \in X$ such that $\sum_{j\in\mathbb{N}} c^j \rho(x', z_j) < \infty$, then for each $x \in X$, we have $\sum_{j\in\mathbb{N}} c^j \rho(x, z_j) < \infty$.

Proof. Let $x \in X$ and Let $x' \in X$ satisfy the inequality $\sum_{j\in\mathbb{N}} c^j \rho(x', z_j) < \infty$. Then,

$$\sum_{j\in\mathbb{N}} c^j \rho(x, z_j) \le \rho(x, x') \sum_{j\in\mathbb{N}} c^j + \sum_{j\in\mathbb{N}} c^j \rho(x', z_j) < \infty$$

since $\sum_{j\in\mathbb{N}} c^j$ is finite. Therefore, we have proved our lemma. □

Remark 5.10. By Lemma 5.9 if there exists $x' \in X$ such that $\sum_{j\in\mathbb{N}} c^j \rho(x', z_j) = \infty$, then for each $x \in X$, we have $\sum_{j\in\mathbb{N}} c^j \rho(x, z_i) = \infty$.

Lemma 5.11. Let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence of contractive mappings on a complete metric space (X, ρ) with an uniformly contractive constant $c \in (0, 1)$. For each $j \in \mathbb{N}$, let $z_j \in X$ be the unique fixed point of f_j. Suppose that there exists $x' \in X$ such that

$$\sum_{j\in\mathbb{N}} c^j \rho(x', z_j) < \infty. \tag{5.3}$$

Then, for all $m \in \mathbb{N}$, there exists $x_m(\infty) \in X$ such that for all $x \in X$, $g(m, m+n)(x)$ converges to $x_m(\infty)$ as n tends to infinity. In addition, for all $m_1, m_2 \in \mathbb{N}$ with $m_1 \le m_2$,

$$g(m_1, m_2)(x_{m_2}(\infty)) = x_{m_1}(\infty).$$

Proof. Let $m \in \mathbb{N}$ and $x \in X$. For each $n \in \mathbb{N}$, we set $x_m(n) := g(m, m+n)(x)$. Note that by Lemma 5.9,

$$\sum_{j\in\mathbb{N}} c^j \rho(x, z_j) < \infty.$$

Then, for all $n_1, n_2 \in \mathbb{N}$ with $n_1 < n_2$,

$$\begin{aligned}
\rho(x_m(n_1), x_m(n_2)) &\le \sum_{k=n_1}^{n_2-1} \rho(x_m(k), x_m(k+1)) \\
&= \sum_{k=n_1}^{n_2-1} \rho(g(m, m+k)(x), g(m, m+k+1)(x)) \\
&= \sum_{k=n_1}^{n_2-1} c^k \rho(x, f_{m+k}(x)) \le \sum_{k=n_1}^{n_2-1} c^k (\rho(x, z_{m+k}) + \rho(z_{m+k}, f_{m+k}(x))) \\
&= c^{-m} \sum_{k=n_1}^{n_2-1} c^{k+m} (\rho(x, z_{m+k}) + \rho(f_{m+k}(z_{m+k}), f_{m+k}(x))) \\
&\le c^{-m}(1+c) \sum_{k=n_1}^{n_2-1} c^{k+m} \rho(x, z_{m+k}).
\end{aligned} \tag{5.4}$$

Therefore, we deduce that $\{x_m(n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X and there exists $x_m(\infty) \in X$ such that $x_m(n)$ converges to $x_m(\infty)$ as n tends to infinity. In addition, let $m \in \mathbb{N}$ and $y \in X$. For each $n \in \mathbb{N}$, we set $y_m(n) := g(m, m+n)(y)$. By the same argument, there exists $y_m(\infty) \in X$ such that $y_m(n)$ converges to $y_m(\infty)$ as n tends to infinity. Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}\rho(x_m(\infty), y_m(\infty)) &\le \rho(x_m(\infty), x_m(n)) + \rho(x_m(n), y_m(n)) + \rho(y_m(n), y_m(\infty)) \\ &\le \rho(x_m(\infty), x_m(n)) + c^n\rho(x, y) + \rho(y_m(n), y_m(\infty)) \longrightarrow 0\end{aligned}$$

as n tends to infinity. Finally, let $m_1, m_2 \in \mathbb{N}$ with $m_1 \le m_2$. Then, for all $n \in \mathbb{N}$,

$$x_{m_1}((m_2 - m_1) + n) = g(m_1, m_2 + n)(x) = g(m_1, m_2)(x_{m_2}(n)).$$

Since $x_{m_1}((m_2 - m_1) + n)$ converges to $x_{m_1}(\infty)$ as n tends to infinity, $x_{m_2}(n)$ converges to $x_{m_2}(\infty)$ as n tends to infinity and $g(m_1, m_2)$ is continuous on X, we have $g(m_1, m_2)(x_{m_2}(\infty)) = x_{m_1}(\infty)$. Therefore, we have proved our lemma. □

To prove the existence of the limit (1.1) and the equation (1.2) in Main Theorem 6, we give the following lemma without proof.

Lemma 5.12 (Collage theorem, Inverse collage theorem [18])**.** Let $f\colon X \to X$ be a contractive mapping on a complete metric space (X, ρ) with a contraction constant $c \in (0, 1)$. Let $z \in X$ be the unique fixed point of f. Then, for each $a \in X$, we have

$$\rho(f(a), a) \le (1 + c)\ \rho(z, a) \quad \text{and} \quad \rho(z, a) \le \frac{\rho(f(a), a)}{1 - c}.$$

We now show the limit (1.1) and the equation (1.2) in Main Theorem 6.

Proof of the limit (1.1) and the equation (1.2). Let $x_0 \in X$ satisfy the assumption in Main Theorem 6. Recall that $A_m \in \mathcal{K}(X)$ be the unique fixed point of F_m for each $m \in \mathbb{N}$. Note that $\{x_0\} \in \mathcal{K}(X)$ and $F_n(\{x_0\}) = \cup_{i\in J_n}\{f_i(x_0)\}$ for each $n \in \mathbb{N}$. By Lemma 5.12, we have

$$d_H(\{x_0\}, A_n) \le \frac{d_H(F_n(\{x_0\}), \{x_0\})}{1 - c} = \frac{d_H(\cup_{i\in J_n}\{f_i(x_0)\}, \{x_0\})}{1 - c}.$$

In addition, by the properties of the Hausdorff distance and Lemma 5.12, we have

$$d_H(\cup_{i\in J_n}\{f_i(x_0)\}, \{x_0\}) = \max_{i\in J_n} \rho(f_i(x_0), x_0) \le \max_{i\in J_n}\ (1 + c)\rho(x_0, z_i).$$

Therefore, for each $n \in \mathbb{N}$, we have

$$d_H(\{x_0\}, A_n) \le \frac{1 + c}{1 - c}\ \max_{i\in J_n} \rho(x_0, z_i). \tag{5.5}$$

By the assumption of Main Theorem 6, we deduce that $\sum_{n\in\mathbb{N}} d_H(\{x_0\}, A_n) < \infty$. By Lemma 5.11, we have proved our theorem. □

To prove the rest of Main Theorem 6, we prove the following lemma and corollary.

Lemma 5.13. Let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence of contractive mappings on a complete metric space (X, ρ) with an uniform contraction constant $c \in (0, 1)$. For each $j \in \mathbb{N}$, let $z_j \in X$ be the unique fixed point of f_j. Suppose that there exist $x' \in X$, $r \in [c, 1)$ and $C' > 0$ such that for all $j \in \mathbb{N}$,

$$c^j \rho(x', z_j) \le C' r^j \tag{5.6}$$

Then, for all $m \in \mathbb{N}$, there exists $x_m(\infty) \in X$ such that for all $x \in X$, we have $g(m,n)(x)$ converges to $x_m(\infty)$ as n tends to infinity exponentially fast with the rate r. In addition, for all $m_1, m_2 \in \mathbb{N}$ with $m_1 \le m_2$,

$$g(m_1, m_2)(x_{m_2}(\infty)) = x_{m_1}(\infty).$$

Proof. Let $m \in \mathbb{N}$ and $x \in X$. For each $n \in \mathbb{N}$, we set $x_m(n) := g(m, m+n)(x)$. Note that by the assumption (5.6), for each $i \in \mathbb{N}$,

$$c^i \rho(x, z_i) \le \rho(x, x')c^i + c^i \rho(x', z_i) \le C(x)\ r^i,$$

where $C(x) := C' + \rho(x, x')$. Then, for all $n_1, n_2 \in \mathbb{N}$ with $n_1 < n_2$,

$$\begin{aligned}
\rho(x_m(n_1), x_m(n_2)) &\le \sum_{k=n_1}^{n_2-1} \rho(x_m(k), x_m(k+1)) \\
&= \sum_{k=n_1}^{n_2-1} \rho(g(m, m+k)(x), g(m, m+k+1)(x)) \\
&= \sum_{k=n_1}^{n_2-1} c^k \rho(x, f_{m+k}(x)) \le \sum_{k=n_1}^{n_2-1} c^k(\rho(x, z_{m+k}) + \rho(z_{m+k}, f_{m+k}(x))) \\
&= c^{-m} \sum_{k=n_1}^{n_2-1} c^{k+m}(\rho(x, z_{m+k}) + \rho(f_{m+k}(z_{m+k}), f_{m+k}(x))) \\
&\le c^{-m}(1+c) \sum_{k=n_1}^{n_2-1} c^{k+m} \rho(x, z_{m+k}) \le c^{-m} C(x)(1+c) \sum_{k=n_1}^{n_2-1} r^{k+m} \quad (5.7)
\end{aligned}$$

Therefore, we deduce that $\{x_m(n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X and there exists $x_m(\infty) \in X$ such that $x_m(n)$ converges to $x_m(\infty)$ as n tends to infinity. In addition, let $m \in \mathbb{N}$ and $y \in X$. For each $n \in \mathbb{N}$, we set $y_m(n) := g(m, m+n)(y)$. By the same argument, there exists $y_m(\infty) \in X$ such that $y_m(n)$ converges to $y_m(\infty)$ as n tends to infinity. Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\rho(x_m(\infty), y_m(\infty)) &\le \rho(x_m(\infty), x_m(n)) + \rho(x_m(n), y_m(n)) + \rho(y_m(n), y_m(\infty)) \\
&\le \rho(x_m(\infty), x_m(n)) + c^n \rho(x, y) + \rho(y_m(n), y_m(\infty)) \longrightarrow 0
\end{aligned}$$

as n tends to infinity. In addition, as n_2 tends to infinity in the inequality (5.7), we have

$$\rho(x_m(n_1), x_m(\infty)) \le c^{-m} C(x)(1+c) \sum_{k=n_1}^{\infty} r^{k+m} = \left(\frac{r}{c}\right)^m \frac{C(x)(1+c)}{1-r}\ r^{n_1}$$

for all $n_1 \in \mathbb{N}$. Therefore, we deduce that for each $m \in \mathbb{N}$, $g(m, m+n)(x)$ converges to $x_m(\infty)$ as n tends to infinity exponentially fast with the rate r. Finally, let $m_1, m_2 \in \mathbb{N}$ with $m_1 \le m_2$. Then, for all $n \in \mathbb{N}$,

$$x_{m_1}((m_2 - m_1) + n) = g(m_1, m_2 + n)(x) = g(m_1, m_2)(x_{m_2}(n)).$$

Since $x_{m_1}((m_2 - m_1) + n)$ converges to $x_{m_1}(\infty)$ as n tends to infinity, $x_{m_2}(n)$ converges to $x_{m_2}(\infty)$ as n tends to infinity and $g(m_1, m_2)$ is continuous on X, we have $g(m_1, m_2)(x_{m_2}(\infty)) = x_{m_1}(\infty)$. Thus, we have proved our lemma. □

Note that the constant $C(x) > 0$ depends on $x \in X$. However, if X is bounded, then instead of $C(x) > 0$, we can take the constant $C > 0$ which does not depend on $x \in X$. Indeed, we set $C := \mathrm{diam}_{x,y\in X}\rho(x,y) + C'(< \infty)$ and we have $C(x) = d(x,x') + C' \le C$.

Corollary 5.14. Let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence of contractive mappings on a complete metric space (X,ρ) with an uniformly contraction constant $c \in (0,1)$. For each $j \in \mathbb{N}$, let $z_j \in X$ be the unique fixed point of f_j. Suppose that there exists $x' \in X$,

$$a := \limsup_{j\longrightarrow\infty} \sqrt[j]{\rho(x',z_j)} < \frac{1}{c}.$$

Then, for all $m \in \mathbb{N}$ and $r \in \{r > 0 \mid c \le r < 1, ac < r\}$, there exists $x_m(\infty) \in X$ such that for all $x \in X$, $g(m,m+n)(x)$ converges to $x_m(\infty)$ as n tends to infinity exponentially fast with the rate r. In addition, for all $m_1, m_2 \in \mathbb{N}$ with $m_1 \le m_2$,

$$g(m_1,m_2)(x_{m_2}(\infty)) = x_{m_1}(\infty).$$

Proof. Let $r \in \{r > 0 \mid c \le r < 1, ac < r\}$. Then, by the assumption (5.14), there exists $M \in \mathbb{N}$ such that for all $j \ge M$, we have $c\ \sqrt[j]{\rho(x',z_j)} < r$, which is equivalent to

$$c^j\rho(x',z_j) < r^j.$$

We set $C'' := \max\{c^j\rho(x',z_j)/r^j \mid j < M\} \cup \{1\}(> 0)$. Then, we have for all $j \in \mathbb{N}$,

$$c^j\rho(x',z_j) \le C''r^j.$$

By Lemma 5.13, our statement of corollary holds. □

We now prove the rest of Main Theorem 6.

Proof of the rest of Main Theorem 6. Let $r \in \{r > 0 \mid c \le r < 1, ac < r\}$. Let $x_0 \in X$ satisfy the assumption in Main Theorem 6. Recall that $A_m \in \mathcal{K}(X)$ be the unique fixed point of F_m for each $m \in \mathbb{N}$. By the argument in the proof of Corollary 5.14, there exists $C''' > 0$ such that for all $n \in \mathbb{N}$,

$$\max_{i\in J_n} d(x_0,z_i)c^n \le C'''r^n.$$

By the similar argument to deduce the inequality (5.5), we have

$$d_H(\{x_0\},A_n)c^n \le \frac{1+c}{1-c}\max_{i\in J_n} d(x_0,z_i)c^n \le C'''\frac{1+c}{1-c}r^n \tag{5.8}$$

for each $n \in \mathbb{N}$. By Lemma 5.13, the statement of our corollary holds. □

5.3 Examples of the sequence of contractive mappings

Let $(\{f_i\}_{i\in I}, \{J_n\}_{n\in\mathbb{N}})$ satisfy the setting (NAIFS). If we do not assume that there exists $x' \in X$ such that $\sum_{j\in\mathbb{N}} c^j\rho(x',z_i) < \infty$, conculusion in Lemma 5.11 does not hold in general. To show this, we give the following counterexample.

Example 5.15. Let $f_j\colon \mathbb{R} \to \mathbb{R}$ $(j \in \mathbb{N})$ is defined by

$$f_j(x) := c(x - a_j) + a_j = cx + (1-c)a_j \ (x \in \mathbb{R}),$$

where $c \in (0, 1)$ and $a_j \geq 0$. Note that for each $j \in \mathbb{N}$, $a_j \geq 0$ is the unique fixed point of f_j i.e. $z_j = a_j$. Then, for all $m, k \in \mathbb{N}$ and $x \in \mathbb{R}$, we have

$$g(m, m+k)(x) = c^k x + (1-c) \sum_{l=0}^{k-1} c^l a_{m+l}. \tag{5.9}$$

Indeed, we show this by induction with respect to $k \in \mathbb{N}$, let $m \in \mathbb{N}$. Since

$$g(m, m+1)(x) = f_m(x) = cx + (1-c)a_m,$$

we deduce the equation (5.9) if $k = 1$.

Assume that $g(m, m+k)(x) = c^k x + (1-c) \sum_{l=0}^{k-1} c^l a_{m+l}$ holds with $k \in \mathbb{N}$. Then, we have

$$\begin{aligned} g(m, m+k+1)(x) &= g(m, m+k)(f_{m+k}(x)) = c^k f_{m+k}(x) + (1-c) \sum_{l=0}^{k-1} c^l a_{m+l} \\ &= c^k \left(cx + (1-c)a_{m+k}\right) + (1-c) \sum_{l=0}^{k-1} c^l a_{m+l} = c^{k+1} x + (1-c) \sum_{l=0}^{k} c^l a_{m+l}. \end{aligned}$$

Thus, we deduce that the equation (5.9) holds with $k + 1 \in \mathbb{N}$.

If $c = 1/2$, $a_i = 2^{i+1}$ and $x \in \mathbb{R}$, then for each $k \in \mathbb{N}$, the equation (5.9) is the following.

$$g(m, m+k)(x) = \frac{1}{2^k} x + \frac{1}{2} \sum_{l=0}^{k-1} \frac{1}{2^l} 2^{m+l+1} = \frac{1}{2^k} x + k 2^m,$$

which deduce that for each $m \in \mathbb{N}$ and $x \in \mathbb{R}$, $g(m, m+k)(x)$ does not converge as k tends to infinity.

In addition, there exists the example of the setting (NAIFS) in which the condition (5.3) holds but the set Z of the unique fixed point of f_j and $\{x_m(\infty) \in X \mid m \in \mathbb{N}\}$ are not bounded. To show this, we give Example 5.16.

Example 5.16. In the previous Exmaple 5.15, we set $c = 1/2$, $a_i = i$ and $x = 0$. Then, we first show that

$$\sum_{i=0}^{\infty} c^i |z_i| = \sum_{i=0}^{\infty} \frac{i}{2^i} < \infty. \tag{5.10}$$

To prove this, let $k \in \mathbb{N}$ and we set $S_k := \sum_{i=1}^{k} i/2^i$. Then, we have

$$\begin{aligned} S_k - \frac{1}{2} S_k &= \sum_{i=1}^{k} \frac{i}{2^i} - \sum_{i=1}^{k} \frac{i}{2^{i+1}} = \frac{1}{2} + \sum_{i=1}^{k-1} \frac{i+1}{2^{i+1}} - \sum_{i=1}^{k-1} \frac{i}{2^{i+1}} - \frac{k}{2^{k+1}} \\ &= \frac{1}{2} + \sum_{i=1}^{k-1} \frac{1}{2^{i+1}} - \frac{k}{2^{k+1}} = \frac{1}{2} + 2\left(\frac{1}{2^2} - \frac{1}{2^{k+1}}\right) - \frac{k}{2^{k+1}} \\ &= 1 - \frac{1}{2^k} - \frac{k}{2^{k+1}}. \end{aligned}$$

Therefore, we deduce that

$$\sum_{i=1}^{k} \frac{i}{2^i} = 2 - \frac{1}{2^{k-1}} - \frac{k}{2^k}.$$

Thus, we have proved the inequality (5.10). Note that $\{z_i \mid i \in \mathbb{N}\} = \mathbb{N}$ is not bounded. In addition, for all $m, k \in \mathbb{N}$ and $x \in \mathbb{R}$, we have

$$g(m, m+k)(x) = \frac{x}{2^k} + \sum_{l=0}^{k-1} \frac{m+l}{2^l}.$$

Therefore, we have $x_m(\infty) = 2m + 2$ for all $m \in \mathbb{N}$ since

$$\sum_{l=0}^{k} \frac{m+l}{2^l} = \sum_{l=0}^{k} \frac{1}{2^l}\ m + \sum_{l=0}^{k} \frac{l}{2^l} = 2(1 - 1/2^{k+1})\ m + \sum_{l=1}^{k} \frac{l}{2^l} \longrightarrow 2m+2$$

as k tends to infinity. Thus, we deduce that Z is also unbounded.

5.4 Proof of Main Theorem 7

Definition 5.17. Let $(\{f_i\}_{i \in I}, \{J_n\}_{n \in \mathbb{N}})$ satisfy the setting (NAIFS). For each $k, l \in \mathbb{N}$ with $1 \leq k < l < \infty$, we set

$$J^{(k,l)} := \Pi_{j=k}^{l-1} J_j, \quad J^{(k,\infty)} := \Pi_{j=k}^{\infty} J_j.$$

For each $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$ with $1 \leq m < n \leq \infty$ and $w \in J^{(m,n)}$, we write w as $w_m \cdots w_{n-1}$ ($w_k \in J_k, k = m, \ldots, n-1$). For each $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$ with $1 \leq m < n \leq \infty$ and $w \in J^{(m,n)}$, we set $|w| := n - m$. For each $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$ with $1 \leq m < n \leq \infty$, $w \in J^{(m,n)}$ and $s \in \mathbb{N}$ with $1 \leq s \leq |w|$, we set

$$w|_s := w_m \cdots w_{m+s-1} \in \Pi_{j=m}^{m+s-1} J_j.$$

For each $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$ with $1 \leq m < n - 1 \leq \infty$, the shift map $\sigma \colon J^{(m,n)} \to J^{(m+1,n)}$ is defined by

$$\sigma(w) := w_{m+1} w_{m+2} \cdots w_{n-1} \in J^{(m+1,n)} \quad (w = w_m w_{m+1} \cdots w_{n-1} \in J^{(m,n)}).$$

For each $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$ with $1 \leq m < n - 1 \leq \infty$ and $w_m \in J_m$, the map $\sigma_{w_m} \colon J^{(m+1,n)} \to J^{(m,n)}$ is defined by

$$\sigma_{w_m}(w) := w_m w_{m+1} \cdots w_{n-1} \in J^{(m,n)} \quad (w = w_{m+1} w_{m+2} \cdots w_{n-1} \in J^{(m+1,n)}).$$

Note that the definition of σ depends on $m \in \mathbb{N}$. However, we omit $m \in \mathbb{N}$ from the representation of the map. For each $m \in \mathbb{N}$, we introduce a metric $d_{(m,\infty)}$ on $J^{(m,\infty)}$ which is defined by

$$d_{(m,\infty)}(w, \tau) := \sum_{n=m}^{\infty} d_n(w_n, \tau_n)/2^n \quad (w = w_m w_{m+1} \cdots, \tau = \tau_m \tau_{m+1} \cdots \in J^{(m,\infty)}),$$

where for each $n \in \mathbb{N}$, d_n is the metric on J_n defined by

$$d_n(w_n, \tau_n) := \begin{cases} 0 & \text{if } w_n = \tau_n \\ 1 & \text{otherwise} \end{cases}.$$

For each $m, n \in \mathbb{N}$ with $m < n$ and $w = w_m \cdots w_{n-1} \in J^{(m,n)}$, we set $f_w := f_{w_m} \circ \cdots \circ f_{w_{n-1}}$. We show the following lemma holds even if Z is not bounded.

Lemma 5.18. Let $(\{f_i\}_{i\in I}, \{J_n\}_{n\in\mathbb{N}})$ satisfy the setting (NAIFS). For each $n \in \mathbb{N}$, let $A_n \in \mathcal{K}(X)$ be the unique fixed point of F_n. Suppose that there exists $A \in \mathcal{K}(X)$ such that

$$\sum_{n\in\mathbb{N}} d_H(A, A_n)c^n < \infty.$$

Then, for each $m' \in \mathbb{N}$ and $n_1 \in \mathbb{N} \cup \{0\}$, we have

$$d_H\left(A_{m'}(n_1), K_{m'}\right) \le c^{-m'}(1+c) \sum_{k=m'+n_1}^{\infty} c^k d_H(A, A_k),$$

where $A_{m'}(n_1) := F_{m'} \circ \cdots \circ F_{m'+n_1-1}(A)$ if $n_1 \ge 1$ and $A_{m'}(0) := A$.

Proof. Let $m' \in \mathbb{N}$ and $A \in \mathcal{K}(X)$. By the same argument of the inequality (5.4) in Lemma 5.11, we have

$$d_H(A_{m'}(n_1), A_{m'}(n_1+n_2)) \le c^{-m'}(1+c) \sum_{k=n_1}^{n_1+n_2-1} c^{m'+k} d_H(A, A_{m'+k}) \tag{5.11}$$

for each $n_1, n_2 \in \mathbb{N}$. Since $A_{m'}(n_1+n_2)$ converges to $K_{m'}$ as n_2 tends to infinity by Lemma 5.11, we have

$$d_H\left(A_{m'}(n_1), K_{m'}\right) \le c^{-m'}(1+c) \sum_{k=n_1}^{\infty} c^{m'+k} d_H(A, A_{m'+k})$$

as n_2 tends to infinity in the inequality (5.11). Therefore, we have proved our lemma. □

Lemma 5.19. Let $(\{f_i\}_{i\in I}, \{J_n\}_{n\in\mathbb{N}})$ satisfy the setting (NAIFS). For each $n \in \mathbb{N}$, let $A_n \in \mathcal{K}(X)$ be the unique fixed point of F_n. Suppose that there exists $A \in \mathcal{K}(X)$ such that

$$\sum_{n\in\mathbb{N}} d_H(A, A_n)c^n < \infty.$$

Then, for each $m \in \mathbb{N}$ and $w \in J^{(m,\infty)}$, we have $\operatorname{diam}(f_{w|_n}(K_{m+n}))$ tends to zero as n tends to infinity uniformly with respect to $w \in J^{(m,\infty)}$ and there exists the unique element $a(m,w) \in K_m$ such that

$$a(m,w) \in \bigcap_{n\in\mathbb{N}} f_{w|_n}(K_{m+n}),$$

where we wirte w as $w_m w_{m+1} \cdots$ and $f_{w|_n} := f_{w_m} \circ f_{w_{m+1}} \circ \cdots \circ f_{w_{m+n-1}}$.

Proof. Let $m \in \mathbb{N}$ and $w \in J^{(m,\infty)}$. We first show that $\bigcap_{n\in\mathbb{N}} f_{w|_n}(K_{m+n}) \neq \emptyset$. Note that by Lemma 5.11, for each $m_1, m_2 \in \mathbb{N}$,

$$G(m_1, m_1+m_2)(K_{m_1+m_2}) = K_{m_1},$$

where $G(m_1, m_1+m_2) := F_{m_1} \circ \cdots \circ F_{m_1+m_2-1}$. Especially, for each $n \ge m$ and $w_n \in J_n$, we have

$$f_{w_{m+n}}(K_{m+n+1}) \subset G(m+n, m+n+1)(K_{m+n+1}) = K_{m+n}.$$

Therefore, for each $n \in \mathbb{N}$, we have

$$f_{w_m} \circ f_{w_{m+1}} \circ \cdots \circ f_{w_{m+n}}(K_{m+n+1}) \subset f_{w_m} \circ f_{w_{m+1}} \circ \cdots \circ f_{w_{m+n-1}}(K_{m+n}).$$

Since $\{f_{w|_n}(K_{m+n})\}_{n\in\mathbb{N}}$ is a decreasing sequence of non-empty compact sets, we have

$$\bigcap_{n\in\mathbb{N}} f_{w|_n}(K_{m+n}) \neq \emptyset.$$

We next show that $\bigcap_{n\in\mathbb{N}} f_{w|_n}(K_{m+n})$ is a single set. It is sufficient to show

$$\lim_{n\to\infty} \operatorname{diam}(f_{w|_n}(K_{m+n})) = 0. \tag{5.12}$$

Indeed, admitting (5.12), if $z, w \in \bigcap_{n\in\mathbb{N}} f_{w|_n}(K_{m+n})$, then $d(z,w) \leq \operatorname{diam}(f_{w|_n}(K_{m+n}))$ for each $n \in \mathbb{N}$, which deduce that $z = w$ as n tends to infinity.

In order to prove (5.12), let $m \in \mathbb{N}$, $w \in J^{(m,\infty)}$ and $z \in X$. Note that for each $n \in \mathbb{N}$,

$$\operatorname{diam}(f_{w|_n}(K_{m+n})) \leq 2c^n d_H(\{z\}, K_{m+n}). \tag{5.13}$$

Indeed, for each $n \in \mathbb{N}$ and $x, y \in f_{w|_n}(K_{m+n})$,

$$\begin{aligned}\rho(x,y) &\leq \rho(x, f_{w|_n}(z)) + \rho(f_{w|_n}(z), y) \leq 2 \sup_{y\in f_{w|_n}(K_{m+n})} \rho(f_{w|_n}(z), y)\\ &\leq 2c^n \sup_{y\in K_{m+n}} \rho(z,y) \leq 2c^n d_H(\{z\}, K_{m+n}).\end{aligned}$$

Therefore, we have proved the inequality (5.13).

Note that by the assumption and Lemma 5.9, we have

$$\sum_{n\in\mathbb{N}} d_H(\{z\}, A_n)c^n < \infty.$$

Moreover, by Lemma 5.18 with $A := \{z\}$, $n_1 := 0$ and $m' := m+n$, we have

$$\begin{aligned}\operatorname{diam} f_{w|_n}(K_{m+n}) &\leq 2c^n d_H(\{z\}, K_{m+n}) \leq 2c^n c^{-(m+n)}(1+c) \sum_{k=m+n}^{\infty} c^k d_H(\{z\}, A_k)\\ &= 2c^{-m}(1+c) \sum_{k=m+n}^{\infty} c^k d_H(\{z\}, A_k).\end{aligned}$$

We have proved $\operatorname{diam}(f_{w|_n}(K_{m+n}))$ converges to zero as n tends to infinity uniformly with respect to $w \in J^{(m,\infty)}$. Hence, we have proved our lemma. □

Definition 5.20. Let $(\{f_i\}_{i\in I}, \{J_n\}_{n\in\mathbb{N}})$ satisfy the setting (NAIFS). For each $n \in \mathbb{N}$, let $A_n \in \mathcal{K}(X)$ be the unique fixed point of F_n. Suppose that there exists $A \in \mathcal{K}(X)$ such that

$$\sum_{n\in\mathbb{N}} d_H(A, A_n)c^n < \infty.$$

Then, for each $m \in \mathbb{N}$, the projection map $\pi\colon J^{(m,\infty)} \to K_m$ is defined by

$$\pi(w) := a(m,w) \ (w \in J^{(m,\infty)}).$$

Note that the definition of π depends on $m \in \mathbb{N}$. However, we omit $m \in \mathbb{N}$ from the representation of the map.

Lemma 5.21. Let $(\{f_i\}_{i\in I}, \{J_n\}_{n\in\mathbb{N}})$ satisfy the setting (NAIFS). For each $n \in \mathbb{N}$, let $A_n \in \mathcal{K}(X)$ be the unique fixed point of F_n. Suppose that there exists $A \in \mathcal{K}(X)$ such that

$$\sum_{n\in\mathbb{N}} d_H(A, A_n)c^n < \infty.$$

Then, for each $m \in \mathbb{N}$, we have $\pi\colon J^{(m,\infty)} \to K_m$ is uniformly continuous and surjective, and for each $w_m \in J_m$,

$$\pi = f_{w_m} \circ \pi \circ \sigma, \quad f_{w_m} \circ \pi = \pi \circ \sigma_{w_m}.$$

Proof. We first show that π is uniformly continuous. Let $\epsilon > 0$ and $m \in \mathbb{N}$. By Lemma 5.19, there exists $M \in \mathbb{N}$ such that $\mathrm{diam} f_{w|_M}(K_{m+M}) < \epsilon$ uniformly with respect to $w \in J^{(m,\infty)}$. We set $\delta := 2^{-(M+m)}$.

We deduce that for each $w, \tau \in J^{(m,\infty)}$ with $d_{(m,\infty)}(w, \tau) < \delta$, $w_n = \tau_n$ for each $n = m, \dots m + M$. Therefore, we have $\pi(w), \pi(\tau) \in \bigcap_{n=1}^{M+1} f_{w|_n}(K_{m+n}) = \bigcap_{n=1}^{M+1} f_{\tau|_n}(K_{m+n})$. Especially, we have $\pi(w), \pi(\tau) \in f_{w|_M}(K_{m+M})$. We deduce that

$$\rho(\pi(w), \pi(\tau)) \le \mathrm{diam} f_{w|_M}(K_{m+M}) < \epsilon.$$

Thus, we have proved that π is uniformly continuous.

We next show that π is surjective. Let $m \in \mathbb{N}$ and $z_m \in K_m = F_m(K_{m+1})$. By the definition of F_m, there exists $w_m \in J_m$ such that $z_m \in f_{w_m}(K_{m+1})$. We deduce that there exists $z_{m+1} \in K_{m+1} = F_{m+1}(K_{m+2})$ such that $z_m = f_{w_m}(z_{m+1})$. By the definition of F_{m+1}, there exists $w_{m+1} \in J_{m+1}$ such that $z_{m+1} \in f_{w_{m+1}}(K_{m+2})$. We deduce that there exists $z_{m+2} \in K_{m+2} = F_{m+2}(K_{m+3})$ such that $z_{m+1} = f_{w_{m+1}}(z_{m+2})$.

By induction with respect to $n \in \mathbb{N}$, there exists $w = w_m w_{m+1} \cdots \in J^{(m,\infty)}$ and $z_{m+n} \in K_{m+n}$ $(n \in \mathbb{N})$ such that for all $n \in \mathbb{N}$,

$$z_m = f_{w_m} \circ \cdots \circ f_{w_{m+n-1}}(z_{m+n}) \in f_{w|_n}(K_{m+n}).$$

Since z_m is the unique element of $\bigcap_{n\in\mathbb{N}} f_{w|_n}(K_{m+n})$, we deduce that $z_m = \pi(w)$. Therefore, we have proved that $\pi\colon J^{(m,\infty)} \to K_m$ is surjective.

We finally show for each $w_m \in J_m$, we have $\pi = f_{w_m} \circ \pi \circ \sigma$ and $f_{w_m} \circ \pi = \pi \circ \sigma_{w_m}$. Let $w \in J^{(m,\infty)}$. Note that $\pi(\sigma(w)) \in \bigcap_{n\in\mathbb{N}} f_{w_{m+1}} \circ \cdots \circ f_{w_{m+n}}(K_{m+1+n})$. Then, we have

$$\begin{aligned} f_{w_m}(\pi(\sigma(w))) &\in f_{w_m}\left(\bigcap_{n\in\mathbb{N}} f_{w_{m+1}} \circ \cdots \circ f_{w_{m+n}}(K_{m+1+n})\right) \\ &\subset \bigcap_{n\in\mathbb{N}} f_{w_m} \circ f_{w_{m+1}} \circ \cdots \circ f_{w_{m+n}}(K_{m+n+1}). \end{aligned}$$

Since $f_{w_m}(\pi(\sigma(w)))$ is the unique element of $\bigcap_{n\in\mathbb{N}} f_{w|_n}(K_{m+n})$, we have $f_{w_m}(\pi(\sigma(w))) = \pi(w)$.

Let $w \in J^{(m+1,\infty)}$ and $w_m \in J_m$. Note that $\pi(w) \in \bigcap_{n\in\mathbb{N}} f_{w_{m+1}} \circ \cdots \circ f_{w_{m+n}}(K_{m+1+n})$ and $\pi(\sigma_{w_m}(w)) \in \bigcap_{n\in\mathbb{N}} f_{w_m} \circ \cdots \circ f_{w_{m+n-1}}(K_{m+n})$. Then, we have

$$f_{w_m}(\pi(w)) \in f_{w_m}\left(\bigcap_{n\in\mathbb{N}} f_{w_{m+1}} \circ \cdots \circ f_{w_{m+n}}(K_{m+1+n})\right) \subset \bigcap_{n\in\mathbb{N}} f_{w_m} \circ \cdots \circ f_{w_{m+n-1}}(K_{m+n}).$$

Since $f_{w_m}(\pi(w))$ is the unique element of $\bigcap_{n\in\mathbb{N}} f_{(\sigma_{w_m}(w))|_n}(K_{m+n})$, we have $f_{w_m}(\pi(w)) = \pi(\sigma_{w_m}(w))$. Therefore, we have proved our lemma. ☐

We now prove the Main Theorem 7.

Proof of Main Theorem 7. Let $x_0 \in X$ satisfy the assumption in Main Theorem 7. By the assumption (1.3) and the inequality (5.5), we have

$$\sum_{n\in\mathbb{N}} d_H(A, A_n)c^n < \infty.$$

Therefore, by Lemma 5.19, Definition 5.20, Lemma 5.21 and the inequality (5.8), we have proved our Theorem. $\square$

5.5 Proof of Main Theorem 8

We now show the limit (1.4) and the equation (1.5) in Main Theorem 8.

Proof of the limit (1.4) and the equation (1.5). Let $x_0 \in X$ be a element of X in the assumption of Main Theorem 8. Note that $\delta_{x_0} \in \mathcal{P}_1(X)$. By Lemma 5.12, for each $n \in \mathbb{N}$, we have

$$d_{MK}(\delta_{x_0}, \mu_n) \leq \frac{d_{MK}(M_n(\delta_{x_0}), \delta_{x_0})}{1-c} = \frac{d_{MK}(\sum_{i\in J_n} p_n(i)\ \delta_{x_0}\circ f_i^{-1}, \sum_{i\in J_n} p_n(i)\ \delta_{x_0})}{1-c}.$$

In addition, by Lemma 5.12, for each $f \in \mathrm{Lip}_1(X)$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\int_X f \sum_{i\in J_n} p_n(i) \mathrm{d}\delta_{x_0}\circ f_i^{-1} - \int_X f \sum_{i\in J_n} p_n(i)\mathrm{d}\delta_{x_0} = \sum_{i\in J_n} p_n(i)\left(\int_X f\mathrm{d}\delta_{x_0}\circ f_i^{-1} - \int_X f\mathrm{d}\delta_{x_0}\right) \\
&= \sum_{i\in J_n} p_n(i)\left(f\circ f_i(x_0) - f(x_0)\right) \leq \sum_{i\in J_n} p_n(i)\rho(f_i(x_0), x_0) \\
&= \sum_{i\in J_n} p_n(i)(1+c)\rho(z_i, x_0) = (1+c)\max_{i\in J_n}\rho(z_i, x_0)
\end{aligned}$$

and we have

$$d_{MK}(\delta_{x_0}, \mu_n) \leq \frac{1+c}{1-c}\max_{i\in J_n}\rho(x_0, z_i). \tag{5.14}$$

By assumption, we deduce that $\sum_{n\in\mathbb{N}} d_{MK}(\delta_{x_0}, \mu_n)c^n < \infty$. By Lemma 5.11, we have proved our theorem. $\square$

We next prove the rest of Main Theorem 8.

Proof of the rest of Main Theorem 8. Let $r \in \{r > 0 \mid c \leq r < 1, ac < r\}$ and $\mu_m \in \mathcal{P}_1(X)$ be the unique fixed point of M_m for each $m \in \mathbb{N}$. Let $x_0 \in X$ satisfy the assumption in Main Theorem 8. By the same argument in the proof of Corollary 5.14, there exists $C''' > 0$ such that for all $n \in \mathbb{N}$,

$$\max_{i\in J_n}\rho(x_0, z_i)c^n \leq C''' r^n.$$

In addition, by the inequality (5.14), we have

$$d_{MK}(\delta_{x_0}, \mu_n)c^n \leq \frac{1+c}{1-c}\max_{i\in J_n}\rho(x_0, z_i)c^n \leq C'''\frac{1+c}{1-c}r^n$$

for each $n \in \mathbb{N}$. By lemma 5.13, the statement of our theorem holds. $\square$

5.6 Proof of Main Theorem 9

In this subsection, we show the support of the limit measures equals the corresponding limit sets when $(\{f_i\}_{i\in I}, \{J_n\}_{n\in\mathbb{N}}, \{p_n\}_{n\in\mathbb{N}})$ satisfies the setting (wNAIFS) (Main Theorem 9). To prove Main Theorem 9, we first show the following lemmas. Note that since (X,ρ) is complete and separable, for for each finite Borel measures μ on X, $\mathrm{supp}(\mu) \in \mathcal{B}(X)$ is the minimum closed set F in X such that $\mu(F) = \mu(X)$, which is equaivalent to the maximum open set O in X such that $\mu(O) = 0$. In addition, if $x \in X \setminus \mathrm{supp}(\mu)$, there exists $\epsilon > 0$ such that $\mu(B(x,\epsilon)) = 0$.

Lemma 5.22. Let (X,ρ) be a complete separable metric space and let μ be a finite Borel measure with a compact support. If $f\colon X \to X$ is continuous on X, then $\mathrm{supp}(\mu\circ f^{-1})$ is also compact in X and we have $\mathrm{supp}(\mu\circ f^{-1}) = f(\mathrm{supp}(\mu))$.

Proof. Let μ be a finite Borel measure with a compact support and $f\colon X\to X$ is continuous on X. Since $\mathrm{supp}(\mu) \subset f^{-1}\circ f(\mathrm{supp}(\mu))$, we have

$$(\mu\circ f^{-1})(f(\mathrm{supp}(\mu))) = \mu(f^{-1}\circ f(\mathrm{supp}(\mu))) \geq \mu(\mathrm{supp}(\mu)) = \mu(X).$$

Since $f(\mathrm{supp}(\mu))$ is compact, we have $\mathrm{supp}(\mu\circ f^{-1}) \subset f(\mathrm{supp}(\mu))$ and $\mathrm{supp}(\mu\circ f^{-1})$ is also compact.

We next show $\mathrm{supp}(\mu\circ f^{-1})^c \subset f(\mathrm{supp}(\mu))^c$. Let $x \in \mathrm{supp}(\mu\circ f^{-1})^c$. Note that there exists $\epsilon > 0$ such that $\mu\circ f^{-1}(B(x,\epsilon)) = 0$, which is equivalent to $\mu(f^{-1}(B(x,\epsilon))) = 0$.

We now assume that $x \in f(\mathrm{supp}(\mu))$. Then, there exists $z \in \mathrm{supp}(\mu)$ such that $x = f(z)$. Since f is continuous on X, we have there exists $\delta > 0$ such that $f(B(z,\delta)) \subset B(f(z),\epsilon)$, which is equivalent to $B(z,\delta) \subset f^{-1}(B(x,\epsilon))$.

Since if $z \in \mathrm{supp}(\mu)$, then for each $\delta > 0$, $\mu(B(z,\delta)) > 0$, we deduce that $\mu(B(z,\delta)) \leq \mu(f^{-1}(B(x,\epsilon))) = 0$, which contradicts $z \in \mathrm{supp}(\mu)$. Therefore, We have proved our lemma. □

Lemma 5.23. Let (X,ρ) be a complete separable metric space and μ,ν be a Borel measures on X. Then, for each $c > 0$, $\mathrm{supp}(c\mu) = \mathrm{supp}(\mu)$ and $\mathrm{supp}(\mu+\nu) = \mathrm{supp}(\mu)\cup\mathrm{supp}(\nu)$.

Proof. Since for each open subset $U \subset X$, $c\mu(U) = 0$ is equivalent to $\mu(U) = 0$. Therefore, we have $\mathrm{supp}(c\mu) = \mathrm{supp}(\mu)$. We now show $\mathrm{supp}(\mu+\nu)^c = \mathrm{supp}(\mu)^c \cap \mathrm{supp}(\nu)^c$. Let $x \in \mathrm{supp}(\mu+\nu)^c$. Then, there exists $\delta > 0$ such that $\mu(B(x,\delta)) + \nu(B(x,\delta)) = 0$. Therefore, we have $\mu(B(x,\delta)) = \nu(B(x,\delta)) = 0$. It follows that $x \in \mathrm{supp}(\mu)^c \cap \mathrm{supp}(\nu)^c$.

Conversely, let $x \in \mathrm{supp}(\mu)^c \cap \mathrm{supp}(\nu)^c$. Then, there exists $\delta_1, \delta_2 > 0$ such that $\mu(B(x,\delta_1)) = 0$ and $\nu(B(x,\delta_2)) = 0$. We set $\delta := \min\{\delta_1,\delta_2\} > 0$. Then, we have

$$(\mu+\nu)(B(x,\delta)) \leq \mu(B(x,\delta_1)) + \nu(B(x,\delta_2)) = 0.$$

We deduce that $x \in \mathrm{supp}(\mu+\nu)^c$. Therefore, we have proved our lemma. □

Lemma 5.24. Let $(\{f_i\}_{i\in I}, \{J_n\}_{n\in\mathbb{N}}, \{p_n\}_{n\in\mathbb{N}})$ satisfy the setting (wNAIFS). For each $n \in \mathbb{N}$, let $\nu_n \in \mathcal{P}_1(X)$ be a limit measure of the setting (wNAIFS) in Main Theorem 8. Let $n \in \mathbb{N}$. Then, for each $L \in \mathbb{N}$, $w \in J^{(n,\infty)}$ and $B \in \mathcal{B}(X)$, we have

$$\nu_n(f_{w|_L}(B)) \geq \prod_{j=0}^{L-1} p_{n+j}(w_{n+j})\nu_{n+L}(B).$$

Proof. Let $n \in \mathbb{N}$ and $w = w_n w_{n+1} \cdots \in J^{(n,\infty)}$. We show the statement of our lemma by induction with respect to $L \in \mathbb{N}$. Since $\nu_n = M_n \nu_{n+1}$ for each $B \in \mathcal{B}(X)$, we have

$$\begin{aligned}\nu_n(f_{w_n}(B)) = M_n\nu_{n+1}(f_{w_n}(B)) &= \sum_{i\in J_n} p_n(i)\nu_{n+1}(f_i^{-1}\circ f_{w_n}(B)) \\ &\geq p_n(w_n)\nu_{n+1}(f_{w_n}^{-1}\circ f_{w_n}(B)) \geq p_n(w_n)\nu_{n+1}(B).\end{aligned}$$

Therefore, we have proved the statement of our lemma if $L = 1$.

We assume that the statement of our lemma holds if $L \in \mathbb{N}$. Then, for each $B \in \mathcal{B}(X)$, we have

$$\nu_n(f_{w|_{L+1}}(B)) = \nu_n(f_{w|_L}(f_{w_{n+L}}(B))) \geq \prod_{j=0}^{L-1} p_{n+j}(w_{n+j})\nu_{n+L}(f_{w_{n+L}}(B)).$$

Since $\nu_{n+L} = M_{n+L}\nu_{n+L+1}$, we deduce that for each $B \in \mathcal{B}(X)$,

$$\begin{aligned}\nu_{n+L}(f_{w_{n+L}}(B)) = M_{n+L}\nu_{n+L+1}(f_{w_{n+L}}(B)) &= \sum_{i\in J_{n+L}} p_{n+L}(i)\nu_{n+L+1}(f_i^{-1}\circ f_{w_{n+L}}(B)) \\ &\geq p_{n+L}(w_{n+L})\nu_{n+L+1}(f_{w_{n+L}}^{-1}\circ f_{w_{n+L}}(B)) \geq p_{n+L}(w_{n+L})\nu_{n+L+1}(B).\end{aligned}$$

Therefore, for each $B \in \mathcal{B}(X)$, we have

$$\nu_n(f_{w|_{L+1}}(B)) \geq \prod_{j=0}^{L-1} p_{n+j}(w_{n+j})p_{n+L}(w_{n+L})\nu_{n+L+1}(B) = \prod_{j=0}^{L} p_{n+j}(w_{n+j})\nu_{n+L+1}(B)$$

and the statement of our Lemma holds when $L + 1 \in \mathbb{N}$. Thus, we have proved our lemma. ☐

Note that by Lemmas 5.22 and 5.23, if $\mu \in \mathcal{P}_1(X)$ has a compact support, then we have proved then $\mathrm{supp}(M_n(\mu)) = F_n(\mathrm{supp}(\mu))$. We now prove the equation (1.6) in Main Theorem 9.

Proof of the limit (1.6). Let $x_0 \in X$ satisfy the assumption in Main Theorem 9, $n \in \mathbb{N}$ and $\mu \in \mathcal{P}_1(X)$ with compact support. Note that $\mathrm{supp}(M_n(\mu)) = F_n(\mathrm{supp}(\mu))$ for each $n \in \mathbb{N}$. We set $\mu_l := M_n \circ \cdots \circ M_{n+l}(\mu)$ and

$$B_l := \mathrm{supp}(\mu_l) = \mathrm{supp}(M_n \circ \cdots \circ M_{n+l}(\mu)) = F_n \circ \cdots \circ F_{n+l}(\mathrm{supp}(\mu))$$

for each $n \in \mathbb{N}$ and $l \in \mathbb{N}$. By Theorem 1.11 and the equation (5.2), we have B_l converges to K_n as l tends to infinity in $(\mathcal{K}(X), d_H)$ and $K_n = \cap_{m\in\mathbb{N}}\overline{\cup_{l\geq m}B_l}^{\rho}$ since $\{B_l\}_{l\in\mathbb{N}}$ is a Cauchy esquence in $(\mathcal{K}(X), d_H)$.

We first show that $\nu_n(K_n) = 1$. Let $m, l \in \mathbb{N}$ with $l \geq m$. Then, we have

$$1 = \mu_l(\mathrm{supp}(\mu_l)) \leq \mu_l(\overline{\cup_{l\geq m}B_l}^{\rho}).$$

By the Portmanteau theorem (for example, see [16]), we have

$$1 \leq \limsup_{l\to\infty} \mu_l(\overline{\cup_{l\geq m}B_l}^{\rho}) \leq \nu_n(\overline{\cup_{l\geq m}B_l}^{\rho}).$$

Since $\overline{\cup_{l\geq m}B_l}^{\rho}$ is a decresing sequence with respect to $m \in \mathbb{N}$, we obtain that

$$1 \leq \lim_{m\to\infty} \nu_n(\overline{\cup_{l\geq m}B_l}^{\rho}) = \nu_n(\cap_{m\in\mathbb{N}}\overline{\cup_{l\geq m}B_l}^{\rho}) = \nu_n(K_n).$$

Therefore, we have $\nu_n(K_n) = 1$. Thus, we have $\mathrm{supp}(\nu_n) \subset K_n$ and $\mathrm{supp}(\nu_n)$ is compact.

We next show $K_n \subset \mathrm{supp}(\nu_n)$. Let $a \in K_n$ and $\epsilon > 0$. By Theorem 1.12, there exists $w \in J^{(n,\infty)}$ such that $\{a\} = \cap_{l\in\mathbb{N}} f_{w|_l}(K_{n+l})$. In addition, since $\mathrm{diam} f_{w|_l}(K_{n+l})$ converges to zero as l tends to infinity by Theorem 1.12, there exists $L \in \mathbb{N}$ such that $f_{w|_L}(K_{n+L}) \subset B(a, \epsilon)$. By lemma 5.24 with $B = K_{n+L}$ and the equation $\nu_{n+L}(K_{n+L}) = 1$, we have

$$\nu_n(B(a,\epsilon)) \geq \nu_n(f_{w|_L}(K_{n+L})) \geq \prod_{j=0}^{L-1} p_{n+j}(w_{n+j})\nu_{n+L}(K_{n+L}) = \prod_{j=0}^{L-1} p_{n+j}(w_{n+j}) > 0.$$

Therefore, we obtain that $a \in \mathrm{supp}(\nu_n)$ and $\mathrm{supp}(\nu_n) = K_n$ for each $n \in \mathbb{N}$. In addition, we have

$$\lim_{l\to\infty} \mathrm{supp}(M_n \circ \cdots \circ M_{n+l}(\mu)) = \lim_{l\to\infty} B_l = K_n = \mathrm{supp}(\nu_n) \quad \text{in } \mathcal{K}(X).$$

Thus, we have proved the limit (1.6). $\square$

We now prove the rest of Main Theorem 9.

Proof of the rest of Main Theorem 9. By the argument of the proof of the limit (1.6), for each $n \in \mathbb{N}$ and $l \in \mathbb{N}$, we have

$$B_l := \mathrm{supp}(M_n \circ \cdots \circ M_{n+l}(\mu)) = F_n \circ \cdots F_{n+l}(\mathrm{supp}(\mu)).$$

By Theorem 1.11 and the equation (5.2), for each $r \in \{r > 0 \mid c \leq r < 1, ac < r\}$, we have B_l converges to K_n as l tends to infinity in $(\mathcal{K}(X), d_H)$ exponentially fast with the rate r and $K_n = \cap_{m\in\mathbb{N}} \overline{\cup_{l\geq m} B_l}^{\rho}$ since $\{B_l\}_{l\in\mathbb{N}}$ is a Cauchy esquence in $(\mathcal{K}(X), d_H)$. By the similar argument in the proof of the limit (1.6), we have $\mathrm{supp}(\nu_n)$ is compact and $\mathrm{supp}(\nu_n) = K_n$ for each $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we have

$$F_n \circ \cdots F_{n+l}(\mathrm{supp}(\mu)) = B_l \longrightarrow K_n = \mathrm{supp}(\nu_n) \quad \text{in } \mathcal{K}(X)$$

as l tends to infinity exponentially fast with the rate r. Thus, we have proved the rest of Main Theorem 9. $\square$

6 The entopy of iterated function systems and the Hausdorff dimension of the limit sets

In this section, we introduce the notion of entropy of general (finite) iterated function systems and we estimate the Hausdorff dimension of the limit sets of general (finite) iterated function systems.

Let (X, ρ) be a complete metric space, let I be a finite set with $|I| = m$ and let $\{f_i\}_{i\in I}$ be a (finite) family of contractive mappings $f_i \colon X \to X$ with a contraction constant $c_i \in (0, 1)$. The family of the contractive mappings $\{f_i\}_{i\in I}$ is called a (finite) iterated function systems on X (for short, IFS on X). Note that by Hutchinson's idea (for example, see [17]), there exists the non-empty compact set K in X uniquely such that $K = \cup_{i\in I} f_i(K)$. The compact set K is called the limit set of an IFS $\{f_i\}_{i\in I}$.

For each $l \in \mathbb{N}$ and $w = w_1w_2 \cdots w_l \in I^l$, we set $f_w := f_{w_1} \circ f_{w_2} \circ \cdots \circ f_{w_l}$ and $|w| := l$. In addition, for each $k, l \in \mathbb{N}$, $w = w_1w_2 \cdots w_k \in I^k$ and $w' = w'_1w'_2 \cdots w'_l \in I^l$, we set

$$ww' := w_1w_2 \cdots w_kw'_1w'_2 \cdots w'_l \in I^{k+l}.$$

We now define the compact covering of the limit set of IFSs.

Definition 6.1. Let K be the limit set of an IFS $\{f_i\}_{i\in I}$ on complete metric space X and let $l \in \mathbb{N}$. The compact covering of K with the level l is defined by $\alpha_l := \{f_w(K) \mid w \in I^l\}$ and the minimun covering number of α_l is defined by

$$N(\alpha_l) := \min\{k \in \mathbb{N} \mid {}^{\exists}w_1, w_2, \dots, w_k \in I^l \text{ s.t. } K \subset \cup_{i=1}^{k} f_{w_i}(K)\}.$$

We set $H(\alpha_l) := \log N(\alpha_l)$.

Note that by the definition of the limit set, for each $l \in \mathbb{N}$, $N(\alpha_l) \le m^l$.

Lemma 6.2. Let K be the limit set of an IFS $\{f_i\}_{i\in I}$ on complete metric space X. Let $N(\alpha_l)$ is the minimum covering number of compact covering α_l of K. Then, $H(\alpha_l)/l$ converges to $\inf_{l\in\mathbb{N}} H(\alpha_l)/l$ as l tend to infinity.

Proof. By the general theory of subadditive sequence (for example, see [35]), we suffice to show that for each $k, l \in \mathbb{N}$, $N(\alpha_{k+l}) \le N(\alpha_k)N(\alpha_l)$.

Let $k, l \in \mathbb{N}$. By the definition of $N(\alpha_k)$, there exists $w_1, w_2, \dots, w_{N(\alpha_k)} \in I^k$ such that $K \subset \cup_{i=1}^{N(\alpha_k)} f_{w_i}(K)$. In addition, by the definition of $N(\alpha_l)$, for each $i = 1, \dots, N(\alpha_l)$, there exist $w'_1, w'_2, \dots, w'_{N(\alpha_l)} \in I^l$ such that $K \subset \cup_{j=1}^{N(\alpha_l)} f_{w'_j}(K)$. Therefore, we have

$$K \subset \bigcup_{i=1}^{N(\alpha_k)} f_{w_i}(K) \subset \bigcup_{i=1}^{N(\alpha_k)} \bigcup_{j=1}^{N(\alpha_l)} f_{w_i}(f_{w'_j}(K)) = \bigcup_{i=1}^{N(\alpha_k)} \bigcup_{j=1}^{N(\alpha_l)} f_{w_i w'_j}(K).$$

Since $f_{w_i w_j}(K)$ $(w_i w_j \in I^{k+l})$ is elements in α_{k+l}, K is covered by the elements of the set

$$\alpha' := \{f_{w_i w_j}(K) \mid i = 1, \dots, N(\alpha_k), j = 1, \dots, N(\alpha_l)\}.$$

In addition, we have $|\alpha'| = N(\alpha_k)N(\alpha_l)$, where $|\alpha'|$ is the cardinality of α'. Therefore, we deduce that $N(\alpha_{k+l}) \le N(\alpha_k)N(\alpha_l)$. Thus, we have proved our lemma. $\square$

We set $h(\{f_i\}_{i\in I}) := \lim_{l\to\infty} H(\alpha_l)/l = \inf_{l\in\mathbb{N}} H(\alpha_l)/l$. Note that the definition of $h(\{f_i\}_{i\in I})$ is derived from the definition of the entropies in the ergodic theory (for example, see [35]). In addition, we have $0 \le h(\{f_i\}_{i\in I}) \le \log m$ since for each $l \in \mathbb{N}$, $H(\alpha_l)/l \ge 0$ and $h(\{f_i\}_{i\in I}) \le H(\alpha_1)/1 \le \log m$.

6.1 Proof of Main Theorem 10

We now prove Theorem 1.17 (Main Theorem 10).

Proof of Main Theorem 10. We first show that if $h(\{f_i\}_{i\in I}) = \log m$, then f_w $(w \in I^* := \cup_{i\in\mathbb{N}} I^i)$ is distinct. We assume that there exist $\omega, \tau \in I^*$ such that $f_\omega = f_\tau$. Without loss of generality, we assume that $|\omega| \ge |\tau|$. We consider the following two cases.

1. $|\omega| = |\tau| = l$ for some $l \in \mathbb{N}$.

By the assumption, we have $K \subset \cup_{w\in I^l\setminus\{\tau\}} f_w(K)$. We deduce that $N(\alpha_l) \le m^l - 1$. Therefore, we obtain that

$$h(\{f_i\}_{i\in I}) \le H(\alpha_l)/l \le \frac{\log(m^l - 1)}{l} < \log m.$$

2. $l = |\omega| > |\tau| = l'$ for some $l, l' \in \mathbb{N}$ with $l > l'$.

We set $\tau I^{l-l'} := \{\tau w' \in I^l \mid w' \in I^{l-l'}\}$. Note that $|\tau I^{l-l'}| = m^{l-l'}$. Since $f_{\tau w'}(K) \subset$

$f_\tau(K) = f_\omega(K)$ for each $w' \in I^{l-l'}$, we deduce that $K \subset \cup_{w \in I^l \setminus \tau I^{l-l'}} f_w(K)$. Therefore, we obtain that

$$h(\{f_i\}_{i\in I}) \leq H(\alpha_l)/l \leq \frac{\log(m^l - m^{l-l'})}{l} < \log m.$$

Therefore, we have proved if $h(\{f_i\}_{i\in I}) = \log m$, then f_w ($w \in I^* := \cup_{i\in\mathbb{N}} I^i$) is distinct.

we next show that $\dim_{\mathcal{H}}(K) \leq h(\{f_i\}_{i\in I})/(-\log \max_{i\in I} c_i)$. We set $c := \max_{i\in I} c_i$. Note that K is bounded and for each $l \in \mathbb{N}$ and $w \in I^l$, $\mathrm{diam} f_w(K) \leq c^l \mathrm{diam} K$ since $d(f_w(x), f_w(y)) \leq c^l d(x,y)$ for each $x, y \in X$.

Let $\epsilon, \delta > 0$. We set $s_\epsilon := h(\{f_i\}_{i\in I})/(-\log c) + \epsilon$. Note that there exists $L \in \mathbb{N}$ such that for all $l \geq L$, $c^l \mathrm{diam} K \leq \delta$. Since $\mathrm{diam} f_{w_i}(K) \leq \delta$ for all $i = 1, \ldots, N(\alpha_l)$, we have

$$\begin{aligned}
\log \mathcal{H}^{s_\epsilon}_\delta(K) &\leq \log\left(\sum_{i=1}^{N(\alpha_l)} (\mathrm{diam} f_{w_i}(K))^{s_\epsilon}\right) \leq \log\left(\sum_{i=1}^{N(\alpha_l)} (c^l \mathrm{diam} K)^{s_\epsilon}\right) \\
&= \log\left(N(\alpha_l)(c^l \mathrm{diam} K)^{s_\epsilon}\right) \leq \log N(\alpha_l) + s_\epsilon(l \log c + \mathrm{diam} K) \\
&\leq -l\left\{-\frac{1}{l} H(\alpha_l) + (-s_\epsilon)\left(\log c + \frac{\mathrm{diam} K}{l}\right)\right\}.
\end{aligned}$$

Since $\lim_{l\to\infty} -\frac{1}{l} H(\alpha_l) = -h(\{f_i\}_{i\in I})$ and

$$\lim_{l\to\infty} (-s_\epsilon)\left(\log c + \frac{\mathrm{diam} K}{l}\right) = (-s_\epsilon)\log c = h(\{f_i\}_{i\in I}) - \epsilon \log c,$$

we have

$$\lim_{l\to\infty} -\frac{1}{l} H(\alpha_l) + (-s_\epsilon)\left(\log c + \frac{\mathrm{diam} K}{l}\right) = -\epsilon \log c > 0.$$

Therefore, we obtain that $\log \mathcal{H}^{s_\epsilon}_\delta(K) = -\infty$, which is equivalent to $\mathcal{H}^{s_\epsilon}_\delta(K) = 0$. Thus, we have $\dim_{\mathcal{H}}(K) \leq s_\epsilon = h(\{f_i\}_{i\in I})/(-\log c) + \epsilon$ for each $\epsilon > 0$. Hence, we have proved our theorem. □

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