Nonintegrability of Dynamical Systems near Equilibria and Heteroclinic Orbits

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Chapter 1

Introduction

Integrability of dynamical systems is one of classical and important topics in the theory of differential equations. The most commonly used notion of integrability is the *Liouville integrability* for finite-dimensional Hamiltonian systems. An m-degree-of-freedom Hamiltonian system is called Liouville integrable if it has m Poisson commutative first integrals, where m is a positive integer. Until the nineteen century, some integrable systems, such as the harmonic oscillator, the two-body problem, and the integrable heavy tops studied by Euler, Lagrange and Kovalevskaya, were discovered. The general and restricted three-body problems, whose nonintegrability was proved by Bruns [14] and Poincaré [73], respectively, in the late nineteen century, are early examples of nonintegrable Hamiltonian systems. Although these results discouraged mathematicians from discovering them, integrable systems have been an central topic in the theory of differential equations again since the discovery of solitons in the KdV equations [61].

The techniques developed by Bruns and Poincaré are actually applicable to few dynamical systems. In the late twenty century, Ziglin [94] developed a technique to prove nonintegrability of Hamiltonian systems. The method now called the *Ziglin analysis* says that if non-resonant monodromy matrices of the variational equation along a particular solution are not commutative, then the Hamiltonian system is meromorphically nonintegrable. Using this method, he succeeded in determining when the heavy top is integrable. Ito [37] also proved nonintegrability of the Hénon-Heiles system using it except in their known integrable cases and one particular case which seems nonintegrable. Subsequently, Morales-Ruiz and Ramis [65] developed a new technique to prove nonintegrability of Hamiltonian systems using the differential Galois group of the variational equations. Their approach, which is now called the *Morales-Ramis theory* as the most strong theory for proving nonintegrability, has almost completely uncovered the nonintegrability of homogeneous Hamiltonian systems [56]. Moreover, Morales-Ruiz, Ramis and Simo [66] improved the theory by using the differential Galois groups of higher variational equations, which was applied to prove nonintegrability of the Hénon-Heiles system in the remaining case.

Mishchenko and Fomenko [60] generalized the Liouville integrability when first integrals may not be Poisson commutative and its number is not less than the number of degrees of freedom. Subsequently, Bogoyavlenskij [9] introduced a more general notion of integrability which admits any number of first integrals but needs a sufficient number of commutative vector fields instead. He also gave a two-degree-of-freedom Hamiltonian system which has only one first integral and three commuting vector fields: it is not Liouville integrable but integrable in his sense. It is especially important that his notion of integrability is applicable to non-Hamiltonian systems. Ayoul and Zung [6] extended the Morales-Ramis theory to the nonintegrability of non-Hamiltonian systems in the Bogoyavlenskij sense. Much work on nonintegrability of non-Hamiltonian systems has been done since their work [6].

In spite of the powerfulness of the Morales-Ramis theory, there still exist many dynamical systems whose nonintegrability is not determined and the implication of nonintegrability on the dynamics of dynamical systems is largely unknown. The double pendulum is a typical example which is believed to exhibit chaos but whose nonintegrability has not been proved. The reason for the former problem is that, to prove nonintegrability by the Morales-Ramis theory, we need a particular solution around which the variational equation is so simple that its differential Galois group is computable. Even finding a particular solution is a high hurdle when applying the Morales-Ramis theory. If such a solution is not found, then the theory is not applicable.

For the latter problem, what motions dynamical systems exhibit is unclear even if they are nonintegrable. Poincaré [73] concluded that the restricted three-body problem is nonintegrable due to the complexity of intersections between the stable and unstable manifolds. In fact, integrable systems are transformed to simple systems by the action-angle coordinates as stated in the Liouville-Arnold theorem [59]. This implies that chaotic dynamical systems are not integrable. On the other hand, the Morales-Ramis theory itself does not give information about dynamics of dynamical systems. Even after proving nonintegrability of dynamical systems, mathematicians often have to rely on numerical simulations to know their chaotic dynamics. So it is a remaining important problem to uncover a relationship between nonintegrability and chaotic dynamics.

An approach to solve the former problem is to use normal forms dating back to Poincaré to prove nonintegrability near equilibrium points. The normal forms for general differential equations and Hamiltonian systems are called *Poincaré-Dulac normal forms* and *Birkhoff normal forms*, respectively. There are many applications of these normal forms. For example, Arnold [3] used Birkhoff normal forms to prove the stability of the Lagrange points in the restricted three-body problems. Poincaré-Dulac normal forms were used to prove the existence of the Lorenz attractor based on numerical verification methods by Tucker [80] and to analyze Painlevé equations by Chiba [16, 17].

One of the most important problems related to the normal form is on the existence of analytic transformations. It is known that all differential equations with equilibrium points can be transformed to the normal forms by formal transformations called *normalizations*, which may be divergent and may not be unique. Poincaré [73] gave conditions for convergent normalizations of Poincaré-Dulac forms. We should also mention Bruno's work [10, 11], which gives the most general condition guaranteeing convergent normalizations. Some relationships between integrability and existence of convergent normalizations have been extensively studied [13, 38, 39, 78, 83, 97]. Vey [83] showed that n volume-preserving commuting vector fields on the $n+1$-dimensional plane are simultaneously analytically transformed to Poincaré-Dulac normal forms. Bruno and Walcher [13] proved that two-dimensional systems have covergent normalizations if and only if they have commuting vector fields. Ito [38] showed that non-resonant Hamiltonian systems have convergent normalizations. He also extended his result when resonance degrees are less than two [39]. Subsequently, Zung [97] proved that analytically integrable systems in the Bogoyavlenskij sense have analytic normalizations to Poincaré-Dulac normal forms. Moreover, he proved that Liouville integrable Hamiltonian systems also have convergent symplectic normalizations to Birkhoff normal forms. The importance of Zung's results is that no assumption is needed on the dimension and resonance degrees of dynamical systems. Stolovitch [78] also studied convergent normalizations when a sufficient number of commuting vector fields exist.

Zung's result [97] for Poincaré-Dulac normal forms is useful to study analytic integrability of dynamical systems near equilibria. If there exist convergent normalizations, then integrability of a given system is reduced to its normal form. If not, the system is analytically nonintegrable. However, it seems difficult to determine integrability of normal forms and divergence of all normalizations. Integrability of a special class of Birkhoff normal forms called $1:2:\omega$ resonance normal forms has been studied by Duistermaat [28], van der Aa and Verhulst [1]. Christov [18] proved nonintegrability of this class by using the Morales-Ramis theory. Ito [39, 40] dealt with convergent normalization and gave some important information on superintegrable Birkhoff normal forms. To the author's knowledge, integrability of Poincaré-Dulac normal forms in the Bogoyavlenskij sense has not been studied. Moreover, no dynamical system which does not have a convergent normalization to Poincaré-Dulac normal forms has not been reported in the literature except for the classical example dating back to Euler [30]. Some Hamiltonian systems whose normalizations to Birkhoff normal forms are all divergent were also discovered by Gong [32].

We now discuss relationships between dynamics and integrability. The most typical mechanism of chaos comes from infinitely many intersections between stable and unstable manifolds of some invariant sets. Though a one-degree-of-Hamiltonian system is always integrable, if it has a homoclinic orbit to an equilibrium and is perturbed periodically, then near the equilibrium there exists a periodic orbits whose stable and unstable manifolds may intersect transversely. If they intersect transversely once, then infinitely many intersections between them arise and a horseshoe map is embedded so that chaotic dynamics occurs [68]. Such transverse intersections can be detected analytically by the Melnikov method [58]. This method has been extended to several situations. Especially, Yagasaki [89] extended the method to Hamiltonian systems with saddle-centers which may not be nearly integrable. Here saddle-centers are equilibria at which the Jacobian matrices of the Hamiltonian vector fields have some pairs of eigenvalues with positive and negative real parts and some pairs of purely imaginary eigenvalues. Moreover, the method was extended to heteroclinic orbits [75].

There are some studies about relationships between nonintegrability and chaotic dynamics. Lerman [50] considered two-degree-of-freedom Hamiltonian systems with saddle-centers and homoclinic orbits. He gave a sufficient condition for the existence of horseshoe dynamics although it seems difficult to check the condition directly. Grotta Ragazzo [34] linked Lerman's result to a scattering problem of a Shrödinger equation. He proved that the sufficient condition equals a condition that the reflection coefficient is not zero and rephrased it by using monodromy matrices of the variational equations. These results are related to only sufficient conditions of chaos. Morales-Ruiz and Peris [64] showed that the sufficient condition of nonintegrability by the Morales-Ramis theory is equivalent to the sufficient condition of chaos given by Grotta-Ragazzo. This result is remarkable because it reveals that Hamiltonian systems proved to be nonintegrable by the the Morales-Ramis theory also exhibit chaotic motions, though a small error was contained as pointed out in Chapter 3. Subsequently, Yagasaki [91] showed that the condition for nonintegrability leads to existence of transverse homoclinic orbits for more general Hamiltonian systems by using the extended Melnikov method [89].

In this thesis, we first study integrability and nonintegrability of Poincaré-Dulac normal forms around equilibria. We next consider general systems with heteroclinic orbits and use the Morales-Ramis theory [6] to obtain sufficient conditions for their nonintegrability by the monodromy groups of variational equations along the heteroclinic orbits. Finally, we consider two-degree-of-freedom Hamiltonian systems with heteroclinic orbits and use the extended Melnikov method [75] along with the result of Chapter 3 to obtain some relationships between nonintegrability and chaos.

In Chapter 2, we consider dynamical systems in Poincaré-Dulac normal form having an equilibrium at the origin. We give a sufficient condition for their integrability, and prove that it is also necessary for existence of the maximal number of first integrals under some condition. Moreover, we show that they are integrable if their resonance degrees are zero or one and that they may be nonintegrable if their resonance degrees are greater than one, as in Birkhoff normal forms for Hamiltonian systems. We demonstrate the theoretical results for a normal form appearing in the codimension-two fold-Hopf bifurcation.

In Chapter 3, we consider general n-dimensional systems of differential equations having $(n-2)$-dimensional, locally invariant manifolds on which there exist equilibria connected by heteroclinic orbits for $n \geq 3$. The system may be non-Hamiltonian and have no saddle-centers, and the equilibria are allowed to be the same and connected by a homoclinic orbit. Under additional assumptions, we prove that the monodromy group for the normal variational equation, which is represented by components of the variational equation normal to the locally invariant manifold and defined on a Riemann surface, is diagonalizable or infinitely cyclic if the system is real-meromorphically integrable in the meaning of Bogoyavlenskij. We apply the theory to a three-dimensional volume-preserving system describing the streamline of a steady incompressible flow with two parameters, and show that it is real-meromorphically nonintegrable for almost all values of the two parameters.

In Chapter 4, we consider a class of two-degree-of-freedom Hamiltonian systems with saddle-centers connected by heteroclinic orbits and discuss some relationships between the existence of transverse heteroclinic orbits and nonintegrability. By the Lyapunov center theorem there is a family of periodic orbits near each of the saddle-centers, and the Hessian matrices of the Hamiltonian at the two saddle-centers are assumed to have the same number of positive eigenvalues. We show that if the associated Jacobian matrices have the same pair of purely imaginary eigenvalues, then the stable and unstable manifolds of the periodic orbits intersect transversely on the same Hamiltonian energy surface when the sufficient conditions obtained in Chapter 3 for real-meromorphic nonintegrability of the Hamiltonian systems hold; if not, then these manifolds intersect transversely on the same energy surface, have quadratic tangencies or do not intersect whether the sufficient conditions hold or not. Our theory is illustrated for a system with quartic single-well potential and some numerical results are given to support the theoretical results.

Finally we provide concluding remarks and give some comments on future work in Chapter 5.

Chapter 2

Local integrability of Poincaré-Dulac normal forms

2.1 Introduction

Consider n-dimensional dynamical systems of the form

$$\dot{x} = f(x), \quad x \in \mathbb{C}^n, \tag{2.1.1}$$

where $n \geq 1$, $f \colon \mathbb{C}^n \to \mathbb{C}^n$ is analytic and $f(0) = 0$, i.e., the origin is an equilibrium. Throughout this chapter, we assume that $f \not\equiv 0$. The vector field f is assumed to have the power series expansion

$$f(x) = \sum_{i=1}^{\infty} f^i(x),$$

where f^i represents a homogeneous vector field of degree i for $i = 1, 2, \ldots$. Let $A = Df(0)$ and write its SN decomposition as $A = S+N$, where S and N are semisimple and nilpotent matrices, respectively. So we have the SN decomposition $f^1(x) = f^{\mathrm{s}}(x) + f^{\mathrm{n}}(x)$, where $f^{\mathrm{s}}(x) = Sx$ and $f^{\mathrm{n}}(x) = Nx$. Throughout this chapter, we assume that A is in Jordan normal form without loss of generality, so that

$$f^{\mathrm{s}}(x) = \sum_{i=1}^{n} \lambda_i x_i e_i, \tag{2.1.2}$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are the eigenvalues of A and $e_i \in \mathbb{C}^n$ is the unit vector of which the ith component is one and the others are zero.

Definition 2.1.1. *Equation* (2.1.1) *and the vector field f are said to be in* Poincaré-Dulac normal form *if the Lie bracket of f^{s} and f, $[f^{\mathrm{s}}, f](x) = Df(x)f^{\mathrm{s}}(x) - Df^{\mathrm{s}}(x)f(x)$, vanishes for any $x \in \mathbb{C}^n$. Henceforth, we simply say "normal form" frequently instead of "Poincaré-Dulac normal form".*

When equation (2.1.1) is an m-degree-of-freedom Hamiltonian system with Hamiltonian $H \colon \mathbb{C}^{2m} \to \mathbb{C}$, i.e.,

$$f_j(x) = \frac{\partial H}{\partial x_{j+m}}(x), \quad f_{j+m}(x) = -\frac{\partial H}{\partial x_j}(x), \quad j = 1, \ldots, m, \tag{2.1.3}$$

where $f_j(x)$ is the jth-component of $f(x)$ and $n = 2m$, we say that it is in *Birkhoff normal form* if

$$\{H^{\mathrm{s}},\, H\} := \sum_{i=1}^{m} \left(\frac{\partial H^{\mathrm{s}}}{\partial x_i} \frac{\partial H}{\partial x_{i+m}} - \frac{\partial H^{\mathrm{s}}}{\partial x_{i+m}} \frac{\partial H}{\partial x_i} \right) = 0,$$

where H^{s} is the semisimple part of H, i.e., the vector field for H^{s} is the semisimple part of (2.1.3).

We first review previous results for analytic integrability of Poincaré-Dulac and Birkhoff normal forms. It is a well-known fact (e.g. [12]) that there exists a formal transformation $x = \phi(y) = y + O(|y|^2)$ under which equation (2.1.1) becomes

$$\dot{y} = g(y), \quad g(y) = D\phi(y)^{-1} f(\phi(y)),$$

in normal form. The transformation ϕ is not generally analytic since small denominators may appear in its power series expansion. So it is an important problem to determine when the formal transformation is analytic at least in a neighborhood of the origin, i.e., equation (2.1.1) has an *analytic normalization*. Poincaré and Sigel gave well known sufficient conditions for (2.1.1) to have an analytic normalization: Poincaré's condition is that the convex hull of the points $\lambda_1, \ldots, \lambda_n$ does not contain 0 on $\mathbb{C}$, and Sigel's is that there exist C_0, $\mu > 0$ such that

$$\left| \sum_{i=1}^{n} \lambda_i p_i - \lambda_j \right| \geq \frac{C_0}{|p|^{\mu}}, \quad j = 1, \ldots, n$$

for $p \in \mathbb{Z}^n_{\geq 0} = \{p \in \mathbb{Z}^n \mid p_i \geq 0, i = 1, \ldots, n\}$ and $|p| \geq 2$, where $|p| = \sum_{i=1}^{n} p_i$ for $p = (p_1, \ldots, p_n) \in \mathbb{Z}^n$. See, e.g., [20, 77]. The former holds only when there are at most finite number of $p \in \mathbb{Z}^n_{\geq 0}$ with $|p| \geq 2$ such that

$$\lambda_j = \sum_{i=1}^{n} p_i \lambda_i \quad \text{for some } j = 1, \ldots n, \tag{2.1.4}$$

while the latter does only when $\lambda_j \neq \sum_{i=1}^{n} p_i \lambda_i$ for any j and $p \in \mathbb{Z}^n_{\geq 0}$ with $|p| \geq 2$.

Definition 2.1.2. *We say that equation* (2.1.1) *is in* case 1 *if the interior of the convex hull of the points* $\lambda_1, \ldots, \lambda_n$ *does not contain* 0 *on* $\mathbb{C}$, *and it is in* case 2 *otherwise.*

When there exist an infinite number of $p \in \mathbb{Z}^n_{\geq 0}$ with $|p| \geq 2$ such that condition (2.1.4) holds, Bruno [10, 11] obtained the following result (see also [12, 86]).

Theorem 2.1.1 (Bruno [10, 11])**.** *Equation* (2.1.1) *has an analytic normalization if it is in case 2 and the following two conditions hold:*

(A_2) *Equation* (2.1.1) *is formally transformed to the normal form*

$$\dot{y}_i = \alpha(y)\lambda_i y_i + \beta(y)\bar{\lambda}_i y_i, \quad i = 1, \ldots, n, \tag{2.1.5}$$

where $\alpha(y), \beta(y)$ *are scalar valued power series of* $y \in \mathbb{C}^n$ *with* $\alpha(0) = 1, \beta(0) = 0$*;*

(ω) $\sum_{k=1}^{\infty} 2^{-k} \log \omega_k^{-1} < \infty$, *where*

$$\omega_k := \min \left\{ \left| \sum_{i=1}^{n} p_i \lambda_i - \lambda_j \right| : 1 \le j \le n,\, p \in \mathbb{Z}_{\ge 0}^n, \right.$$
$$\left. 1 < \sum_{i=1}^{n} p_i < 2^k,\, \sum_{i=1}^{n} p_i \lambda_i \ne \lambda_j \right\}.$$

Bruno [10] also gave sufficient conditions for (2.1.1) in case 1 to have an analytic normalization.

On the other hand, for general dynamical systems which may not be Hamiltonian, Bogoyavlenskij [9] introduced the following definition of integrability.

Definition 2.1.3 (Bogoyavlenskij [9]). *Let m be an integer such that $1 \le m \le n$. Equation* (2.1.1) *is called* $(m, n-m)$-integrable *if there exist m vector fields $f_1 (= f), f_2, \ldots, f_m$ and $n-m$ functions $F_1, \ldots, F_{n-m}$ such that the following conditions hold:*

(i) $f_1, \ldots, f_m$ are linearly independent and $DF_1, \ldots, DF_{n-m}$ are linearly independent almost everywhere;

(ii) The vector fields $f_1, \ldots, f_m$ commute pairwise, i.e., $[f_i, f_j] = 0$ for $i, j = 1, \ldots, m$;

(iii) The functions $F_1, \ldots, F_{n-m}$ are first integrals of $f_1, \ldots, f_m$, i.e., $\mathscr{L}_{f_i} F_j = 0$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n-m$, where $\mathscr{L}_f F(x) := DF(x) f(x)$ represents the Lie derivative of F with respect to f.

We simply say that equation (2.1.1) *is* integrable *if it is $(m, n-m)$-integrable for some integer m. In particular, we say that equation* (2.1.1) *is* analytically integrable *and* meromorphically integrable *if the vector fields $f_1, \ldots, f_m$ and functions $F_1 \ldots, F_{n-m}$ are analytic and meromorphic, respectively.*

If a Hamiltonian system is Liouville integrable, then it is also integrable in the sense of Bogoyavlenskij. Zung [97] obtained the following result.

Theorem 2.1.2 (Zung [97]). *If equation* (2.1.1) *is analytically integrable, then it has an analytic normalization.*

This theorem means that an integrable system is transformed to an integrable normal form in the analytic framework. Zung [98] also proved that Liouville integrable Hamiltonian systems are transformed to Birkhoff normal forms in the same framework. However, normal forms may not be integrable, like Birkhoff normal forms of Hamiltonian systems [98]: There exists an analytically Liouville nonintegrable Birkhoff normal form with $\gamma_H \ge 2$, while Birkhoff normal forms with $\gamma_H \le 1$ are always analytically Liouville integrable [18, 28], where γ_{H} represents the resonance degree and is defined as

$$\gamma_H = \dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}} \left\{ p \in \mathbb{Z}^m \,\middle|\, \sum_{i=1}^{m} \lambda_i p_i = 0 \right\}$$

for the Hamiltonian vector field (2.1.3) when we take $\lambda_{m+j} = -\lambda_j$, $j = 1, \ldots, m$, for the eigenvalues of $Df(0)$. We recommend [78, 79, 83, 84] and [38, 39, 40, 43] for related results on the analytic normalization of commuting vector fields and the analytic normalization of Birkhoff normal form, respectively.

We now define the resonance degree for Poincaré-Dulac normal form. Let $x^p = x_1^{p_1} \cdot x_2^{p_2} \cdots x_n^{p_n}$ for $p = (p_1, \dots, p_n) \in \mathbb{Z}^n$ and $x = (x_1, \dots, x_n) \in \mathbb{C}^n$. We easily see that the normal form f can be expressed as

$$f(x) = Ax + \sum_{i=1}^{n} \sum_{p \in R_i} c_i(p) x^p x_i e_i, \tag{2.1.6}$$

where $c_i(p) \in \mathbb{C}$ is a constant and

$$R_i = \left\{ p \in \mathbb{Z}_i^n \;\middle|\; \sum_{j=1}^{n} \lambda_j p_j = 0 \right\}, \quad \mathbb{Z}_i^n = \{ p \in \mathbb{Z}^n \mid p_i \geq -1,\, p_j \geq 0,\, j \neq i,\, |p| \geq 1 \}$$

for $i = 1, \dots, n$ (see, e.g., [5, 12] and Lemma 2.3.2 below).

Definition 2.1.4. *Let R and R_{F} be subsets of $\mathbb{Z}^n$ given by*

$$R := \bigcup_{i=1}^{n} R_i \quad \text{and} \quad R_{\mathrm{F}} := \left\{ p \in \mathbb{Z}_{\geq 0}^n \;\middle|\; \sum_{i=1}^{n} \lambda_i p_i = 0,\, |p| \geq 1 \right\}.$$

We call R and R_{F} the resonance *and* F-resonance sets *of* (2.1.6)*, respectively, and refer to*

$$\gamma = \dim_{\mathbb{Q}} \mathrm{span}_{\mathbb{Q}} R \quad \text{and} \quad \gamma_{\mathrm{F}} = \dim_{\mathbb{Q}} \mathrm{span}_{\mathbb{Q}} R_{\mathrm{F}}$$

as the resonance *and* F-resonance degrees *of* (2.1.6)*, respectively.*

Note that $R_{\mathrm{F}} \subset R$ and $\gamma_{\mathrm{F}} \leq \gamma \leq n$. Moreover, we have $\gamma \leq n-1$ if A has at least one non-zero eigenvalue. As we will see later, the number of first integrals of the normal form is not greater than γ_{F} (see Corollary 2.3.4).

In this chapter, we study analytic integrability of Poincaré-Dulcac normal forms. We give a sufficient condition for the normal form (2.1.6) to be integrable, and prove that the condition is necessary for its $(n-\gamma_{\mathrm{F}}, \gamma_{\mathrm{F}})$-integrability under some condition. Moreover, we show that the normal form (2.1.6) with $\gamma < 2$ is analytically integrable and that there exists an n-dimensional, analytically non-integrable normal form with $n = \gamma + 1$ for $\gamma \geq 2$, as in Birkhoff normal form. We demonstrate the theoretical results for

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ i\omega x_2 \\ -i\omega x_3 \end{pmatrix} + \sum_{\substack{j,\,k \geq 0 \\ j+2k \geq 1}} x_1^j (x_2 x_3)^k \begin{pmatrix} a_{jk} x_1 \\ b_{jk} x_2 \\ c_{jk} x_3 \end{pmatrix} + \sum_{l \geq 1} (x_2 x_3)^l \begin{pmatrix} d_l \\ 0 \\ 0 \end{pmatrix}, \tag{2.1.7}$$

where $a_{jk}, b_{jk}, c_{jk}, d_l \in \mathbb{C}$ are constants. Equation (2.1.7) is a normal form appearing in the codimension-two fold-Hopf bifurcations [49] and satisfies $\gamma = \gamma_{\mathrm{F}} = 2$.

The paper is organized as follows. In Section 2.2, we precisely state our main results. We give preliminary results in Section 2.3 and prove the main theorems in Section 2.4. In Section 2.5, we apply our theory to (2.1.7).

2.2 Main results

We rewrite the normal form (2.1.6) as

$$f(x)=\sum_{i=1}^{n}\left(\lambda_i+\sum_{p\in R_{\mathrm{F}}}c_i(p)x^p+\sum_{q\in\hat{R}_i}c_i(q)x^q\right)x_ie_i+\sum_{i=2}^{n}\rho_ix_{i-1}e_i, \tag{2.2.1}$$

where $\rho_i\in\{0,\,1\}$, $i=2,\dots,n$, and $\hat{R}_i=\{q\in R_i\mid q_i=-1\}$, $i=1,\dots,n$. Note that $R_i=R_{\mathrm{F}}\cup\hat{R}_i, i=1,\dots,n$, and that the linear part of (2.2.1) is in Jordan normal form like (2.1.2).

Proposition 2.2.1. *A monomial x^r, $r\in R_{\mathrm{F}}$, is a first integral of the normal form* (2.2.1) *if and only if the following* condition *holds:*

$$(\mathrm{C})_r\qquad\begin{cases}\displaystyle\sum_{j=1}^{n}r_jc_j(p)=0 \quad \textit{for } p\in R_{\mathrm{F}};\\ r_ic_i(q)=0\quad \textit{for } q\in\hat{R}_i,\ i=1,\dots,n;\\ r_i\rho_i=0\quad \textit{for } i=2,\dots,n.\end{cases}$$

Proof. For $r\in R_{\mathrm{F}}$ we have

$$\mathscr{L}_fx^r=\sum_{i=1}^{n}\left(\sum_{p\in R_{\mathrm{F}}}r_ic_i(p)x^{p+r}+\sum_{q\in\hat{R}_i}r_ic_i(q)x^{q+r}+r_i\rho_ix^{r-e_i+e_{i-1}}\right).$$

Noting that the coefficient of each monomial in the above equation is zero if and only if condition $(\mathrm{C})_r$ holds, we obtain the desired result. □

Suppose that $\rho_i=0$, $i=2,\dots,n$, in the normal form (2.2.1), i.e., A is diagonal in (2.1.6). Let R' be a subset of R such that $c_i(p)=0$, $i=1,\dots,n$ for $p\in R\backslash R'$. Then equation (2.2.1) becomes

$$f(x)=\sum_{i=1}^{n}\left(\lambda_i+\sum_{p\in R_{\mathrm{F}}\cap R'}c_i(p)x^p+\sum_{q\in\hat{R}_i\cap R'}c_i(q)x^q\right)x_ie_i. \tag{2.2.2}$$

We state our first main result.

Theorem 2.2.2. *Let R' be a subset of R such that $c_i(p)=0$ for $p\in R\backslash R'$, $i=1,\dots,n$. Assume that* $\dim_{\mathbb{Q}}\mathrm{span}_{\mathbb{Q}}R'=\dim_{\mathbb{Q}}\mathrm{span}_{\mathbb{Q}}(R_{\mathrm{F}}\cap R')$ *and let* $\gamma'=\dim_{\mathbb{Q}}\mathrm{span}_{\mathbb{Q}}R'$. *If condition* $(\mathrm{C})_r$ *holds for any $r\in R_{\mathrm{F}}\cap R'$, then the normal form* (2.2.1) *is analytically* $(n-\gamma',\,\gamma')$-*integrable.*

Remark 2.2.1. Suppose that there exist $R_1'\subset R_2'\subset R$ such that $R'=R_1',R_2'$ satisfies the hypotheses of Theorem 2.2.2. Let $\gamma_j=\dim_{\mathbb{Q}}\mathrm{span}_{\mathbb{Q}}R_j'$, $j=1,2$. Then the normal form (2.2.1) is analytically $(n-\gamma',\gamma')$-integrable for $\gamma_1\le\gamma'\le\gamma_2$.

Suppose that equation (2.1.1) is in case 2 and satisfies conditions (A_2) and (ω). Then by Theorem 2.1.1 it is analytically transformed to the normal form (2.1.5), which is rewritten as

$$\dot{y} = \sum_{i=1}^{n} \left(\lambda_i + \sum_{p \in R_{\mathrm{F}}} (\lambda_i d_p + \bar{\lambda}_i d'_p) y^p \right) e_i, \tag{2.2.3}$$

where $\alpha(y) = 1 + \sum_{p \in \mathbb{Z}^n_{\geq 0} \setminus \{0\}} d_p y^p, \beta(y) = \sum_{p \in \mathbb{Z}^n_{\geq 0} \setminus \{0\}} d'_p y^p$ and $d_p, d'_p \in \mathbb{C}$ are constants. For $r \in R_{\mathrm{F}}$ equation (2.2.3) satisfies condition $(\mathrm{C})_r$ since $\sum_{i=1}^{n} r_i (\lambda_i d_p + \bar{\lambda}_i d'_p) = d_p \sum_{i=1}^{n} r_i \lambda_i + d'_p \overline{\sum_{i=1}^{n} r_i \lambda_i} = 0$. We take $R' = R_{\mathrm{F}}$ and apply Theorem 2.2.2 to obtain the following.

Corollary 2.2.3. *If equation* (2.1.1) *is in case 2 and satisfies conditions (A_2) and (ω), then it is analytically $(n - \gamma_{\mathrm{F}}, \gamma_{\mathrm{F}})$-integrable in a neighborhood of the origin.*

This corollary means the condition of Theorem 2.1.2 is weaker than that of Theorem 2.1.1.

Let P_{s} be the set of polynomial first integrals of f^{s}. Since P_{s} is a finitely generated $\mathbb{C}$-algebra, we can write $P_{\mathrm{s}} = \mathbb{C}[\phi_1, \ldots, \phi_{\gamma_{\mathrm{P}}}]$ for some $\gamma_{\mathrm{P}} \geq \gamma_{\mathrm{F}}$ [78, 85]. We take the smallest one as γ_{P}. As stated in Corollary 2.3.4 below, the normal form f generally has γ_{F} functionally independent, analytic first integrals at most.

Theorem 2.2.4. *Suppose that the normal form* (2.2.1) *satisfies $\gamma_{\mathrm{P}} = \gamma_{\mathrm{F}}$. If it has γ_{F} first integrals, then condition* $(\mathrm{C})_r$ *holds for any $r \in R_{\mathrm{F}}$. In particular, if it is $(n - \gamma_{\mathrm{F}}, \gamma_{\mathrm{F}})$-integrable, then condition* $(\mathrm{C})_r$ *holds for any $r \in R_{\mathrm{F}}$.*

Thus, the condition of Theorem 2.2.2 with $R' = R$ is necessary for the normal form (2.2.2) to be analytically $(n - \gamma_{\mathrm{F}}, \gamma_{\mathrm{F}})$-integrable if $\gamma_{\mathrm{P}} = \gamma_{\mathrm{F}} = \gamma$. Note that the condition $\gamma_{\mathrm{F}} = \gamma$ is equivalent to $\dim_{\mathbb{Q}} \mathrm{span}_{\mathbb{Q}} R' = \dim_{\mathbb{Q}} \mathrm{span}_{\mathbb{Q}} (R_{\mathrm{F}} \cap R')$ for $R' = R$.

Finally, we give the following result on a relationship between resonance degrees and integrability of Poincaré-Dulac normal forms as in Birkhoff normal form.

Theorem 2.2.5. *If the resonance degree γ is less than two, then the normal form* (2.2.1) *is integrable. For $\gamma \geq 2$, there exists an n-dimensional, analytically non-integrable normal form with $n = \gamma + 1$.*

Proofs of Theorem 2.2.2, 2.2.4 and 2.2.5 are given in Section 2.4.

2.3 Preliminary results

In this section, we give some preliminary results on analytic commutative vector fields and analytic first integrals for normal forms. These results play essential roles in the proofs of the main theorems in Section 2.4.

The following result on a relationship between the normal form f and its semisimple linear part f^{s} for commutative vector fields and first integrals has often been used in previous related studies (e.g., [85, 86]).

Proposition 2.3.1 (Proposition 1.8 of [85] and Proposition 2 of [86])**.** *The following hold when f is in normal form:*

(i) If an analytic vector field g is commutative with f, i.e., $[f, g] = 0$, then it is commutative with the semisimple linear part f^s of f.

(ii) If an analytic function H is a first integral of f, then it is a first integral of f^s.

Suppose that f^{s} has the expression (2.1.2).

Lemma 2.3.2. *The vector field $x^p x_i e_i$, $p + e_i \in \mathbb{Z}^n_{\geq 0}$, is commutative with $f^{\mathrm{s}}(x)$ if and only if $p \in R_i$. Moreover, the monomial x^q, $q \in \mathbb{Z}^n_{\geq 0}$, is a first integral of f^{s} if and only if $q \in R_{\mathrm{F}}$.*

Proof. It is easy to check that

$$[f^s(x),\, x^p x_i e_i] = \left(\sum_{j=1}^{n} p_j \lambda_j\right) x^p x_i e_i \tag{2.3.1}$$

and

$$\mathscr{L}_{f^s(x)}(x^p) = \left(\sum_{i=1}^{n} p_i \lambda_i\right) x^p$$

for $p \in Z^n_{\geq 0}$. Hence, we obtain the result. □

Let $H(x) = \sum_{p\in\mathbb{Z}^n_{\geq 0}} h_p x^p$, $h_p \in \mathbb{C}$, be an analytic first integral of f^{s}. Then we have

$$(\mathscr{L}_{f^{\mathrm{s}}} H)(x) = \sum_{p\in\mathbb{Z}^n_{\geq 0}} h_p \left(\sum_{i=1}^{n} p_i \lambda_i\right) x^p = 0$$

so that $h_p = 0$ for $p \in \mathbb{Z}^n_{\geq 0} \backslash R_{\mathrm{F}}$. Thus, we obtain the following proposition.

Proposition 2.3.3. *Any analytic first integral $H(x)$ of f^{s} is written as*

$$H(x) = \sum_{p\in R_{\mathrm{F}}} h_p x^p,$$

where $h_p \in \mathbb{C}$ is a constant.

From the definition of γ_{F} we immediately have the following corollary.

Corollary 2.3.4. *The normal form f has γ_{F} functionally independent, analytic first integrals at most.*

We write A in Jordan normal form as

$$A = \begin{pmatrix} J_{n_1}(\bar{\lambda}_1) & 0 & \dots & 0 \\ 0 & J_{n_2}(\bar{\lambda}_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{n_r}(\bar{\lambda}_r) \end{pmatrix},$$

where $\bar{\lambda}_1, \dots, \bar{\lambda}_r$ are eigenvalues of A and $J_i(\lambda)$, $\lambda \in \mathbb{C}$, represents the $i \times i$ Jordan block

$$J_i(\lambda) = \begin{pmatrix} \lambda & 0 & 0 & \dots & 0 & 0 \\ 1 & \lambda & 0 & \dots & 0 & 0 \\ 0 & 1 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 0 \\ 0 & 0 & 0 & \dots & 1 & \lambda \end{pmatrix}$$

with $r \leq n$ and $n_1 + \cdots + n_r = n$. Define the $i \times i$ matrix

$$\mu_i(\bar{c}) = \begin{pmatrix} c_1 & 0 & 0 & \dots & 0 & 0 \\ c_2 & c_1 & 0 & \dots & 0 & 0 \\ c_3 & c_2 & c_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{i-1} & c_{i-2} & c_{i-3} & \dots & c_1 & 0 \\ c_i & c_{i-1} & c_{i-2} & \dots & c_2 & c_1 \end{pmatrix}$$

for $\bar{c} = (c_1, \dots, c_i) \in \mathbb{C}^i$ and the $n \times n$ matrix

$$\mu_A(c) = \begin{pmatrix} \mu_{n_1}(\bar{c}^{(1)}) & 0 & \dots & 0 \\ 0 & \mu_{n_2}(\bar{c}^{(2)}) & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \mu_{n_r}(\bar{c}^{(r)}) \end{pmatrix}$$

for $\bar{c}^{(i)} \in \mathbb{C}^{n_i}$ and $c = (\bar{c}^{(1)}, \bar{c}^{(2)}, \dots, \bar{c}^{(r)}) \in \mathbb{C}^n$. For example, we have

$$\mu_A(c) = \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 \\ c_2 & c_1 & 0 & 0 & 0 \\ c_3 & c_2 & c_1 & 0 & 0 \\ 0 & 0 & 0 & c_4 & 0 \\ 0 & 0 & 0 & c_5 & c_4 \end{pmatrix}$$

for

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}.$$

Note that A itself can be written as $\mu_A(a)$ with some $a \in \mathbb{C}^n$. We easily see that the matrices $\mu_A(c)$ and $\mu_A(c')$ are commutative and $\mu_A(\alpha c + \alpha' c') = \alpha \mu_A(c) + \alpha' \mu_A(c')$ for any $\alpha, \alpha' \in \mathbb{C}$ and c, $c' \in \mathbb{C}^n$.

Proposition 2.3.5. *If $b^{(1)}$, $b^{(2)}, \dots,$ $b^{(k)} \in \mathbb{C}^n$, $k \leq n$, are linearly independent, then $\mu_A(b^{(1)})x$, $\mu_A(b^{(2)})x$, $\dots$, $\mu_A(b^{(k)})x$ are so for almost all $x \in \mathbb{C}^n$.*

Proof. Since $\mu_A(b)x = \mu_A(x)b$ for any $b, x \in \mathbb{C}^n$, we have

$$(\mu_A(b^{(1)})x, \mu_A(b^{(2)})x, \ldots, \mu_A(b^{(k)})x) = \mu_A(x)(b^{(1)}, b^{(2)}, \ldots, b^{(k)}).$$

If $x_1 x_2 \cdots x_n \neq 0$ for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$, then $\det \mu_A(x) \neq 0$. Hence the rank of the matrix $(\mu_A(b^{(1)})x, \mu_A(b^{(2)})x, \ldots, \mu_A(b^{(k)})x)$ is the same as that of $(b^{(1)}, b^{(2)}, \ldots, b^{(k)})$. Thus we obtain the result. □

In general, we consider (2.1.1). Zung [97] introduced the concept of a torus action and proved Theorem 2.1.2. Define

$$\mathscr{Q} = \left\{ b \in \mathbb{Z}^n \;\middle|\; \sum_{i=1} \lambda_i b_i = 0 \text{ and } b_i = b_j \text{ for } \lambda_i = \lambda_j \right\}$$

and let $(b^{(1)}, \ldots, b^{(d)}) \in \mathscr{Q}^d$ be a basis of $\mathrm{Span}_{\mathbb{Q}} \mathscr{Q}$. Let

$$Z_\ell(x) = \sum_{j=1}^{n} b_j^{(\ell)} x_j e_j.$$

We refer to an action generated by the diagonal linear vector fields $iZ_1(x), \ldots, iZ_d(x)$ as a *torus action* for f. The number $d = \dim \mathrm{Span}_{\mathbb{Q}} \mathcal{Q}$ is also called the *toric degree* of f. For any ℓ there exists a vector $c \in \mathbb{Z}^n$ such that $Z_\ell(x) = \mu_A(c)x$. Hence, the vector fields $\mu_A(c)x$, $c \in \mathbb{C}^n$, commute with the torus action. However, $\mu_A(c)x$ may not be tangent to any orbits of the torus action if the toric degree is less than n.

Consider the normal form (2.2.1). The coefficient matrix of its linear part is expressed as

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots & 0 & 0 \\ \rho_2 & \lambda_2 & 0 & \ldots & 0 & 0 \\ 0 & \rho_3 & \lambda_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & \ldots & \rho_n & \lambda_n \end{pmatrix}, \tag{2.3.2}$$

where $\rho_i \in \{0, 1\}$, $i = 2, \ldots, n$. We have the following result on commutative linear vector fields and first integrals for the linear part Ax.

Proposition 2.3.6. *Let $R' \subset R$, $\gamma' = \dim_{\mathbb{Q}} \mathrm{span}_{\mathbb{Q}} R'$ and $n' = n - \gamma'$, and assume that $A \neq 0$. Then there exist $n' - 1$ matrices $B^{(1)}, \ldots, B^{(n'-1)}$ such that $Ax, B^{(1)}x, \ldots, B^{(n'-1)}x$ are linearly independent for almost all $x \in \mathbb{C}^n$ and commutative, and the following hold:*

- *(i) The monomial x^p is a first integral of the n' linear vector fields $Ax, B^{(1)}x, \ldots,$ $B^{(n'-1)}x$ if $p \in R_{\mathrm{F}}$ and $\rho_i = 0$ when $p_i \neq 0$ for $i = 2, \ldots, n$.*
- *(ii) The n' linear vector fields $Ax, B^{(1)}x, \ldots, B^{(n'-1)}x$ commute with $x^q x_k e_k$ for some $k \in \{1, \ldots, n\}$ if $q \in R_k$, $\rho_{k+1} = 0$ and $\rho_i = 0$ when $q_i + \delta_{ik} \neq 0$ for $i = 2, \ldots, n$, where δ_{ik} denotes the Kronecker delta.*

Proof. For the proof of Proposition 2.3.6 we need the following three lemmas.

Lemma 2.3.7. *Under the hypotheses of Proposition 2.3.6, there exist $n'-1$ linearly independent $b^{(1)},\ldots,b^{(n'-1)}$ such that $Ax, \mu_A(b^{(1)})x,\ \ldots,\ \mu_A(b^{(n'-1)})x$ are linearly independent for almost all $x \in \mathbb{C}^n$ and*

$$p \cdot \Delta(\mu_A(b^{(i)})) = 0 \quad \text{for } p \in R',\ i = 1,\ldots,n'-1, \tag{2.3.3}$$

where $p \cdot q = \sum_{i=1}^n p_i q_i$ for $p, q \in \mathbb{C}^n$ and $\Delta(B) = (b_{11},\ b_{22},\ \ldots,\ b_{nn})$ for an $n \times n$ matrix $B = (b_{ij})$.

Proof. Let $p^{(1)},\ldots,p^{(\gamma')}$ be linearly independent in the vector space $V = \mathrm{span}_{\mathbb{C}} R' \subset \mathbb{C}^n$ and let $\pi(c) = (p^{(1)} \cdot c,\ldots,p^{(\gamma')} \cdot c) \in \mathbb{C}^{\gamma'}$ for $c \in \mathbb{C}^n$. Since the rank of the linear map $\pi \circ \Delta \circ \mu_A \colon \mathbb{C}^n \to \mathbb{C}^{\gamma'}$ is not greater than γ', there exist n' independent solutions of a system of linear equations $(\pi \circ \Delta \circ \mu_A)(x) = 0$. The condition $(\pi \circ \Delta \circ \mu_A)(x) = 0$ means that $p \cdot \Delta(\mu_A(x)) = 0$ for any $p \in R'$. Obviously, a vector $a \in \mathbb{C}^n$ satisfying $A = \mu_A(a)$ becomes a solution of $(\pi \circ \Delta \circ \mu_A)(x) = 0$. Thus, we get linearly independent vectors $b^{(1)},\ldots,b^{(n'-1)}$ satisfying (2.3.3). □

Lemma 2.3.8. *Let $p \in \mathbb{Z}^n_{\geq 0}$ and let $B = (b_{ij})$ be a lower triangular matrix such that $\sum_{i=1}^n p_i b_{ii} = 0$. A monomial x^p is a first integral of the vector field Bx if $b_{ij} p_i = 0$ for $i > j$, $j = 1,\ldots,n-1$.*

Proof. Since B is lower triangular, we have

$$Bx = \sum_{i=1}^n \left(\sum_{j=1}^i b_{ij} x_j \right) e_i.$$

Hence, we obtain

$$\mathscr{L}_{Bx}(x^p) = \sum_{i=1}^n \sum_{j=1}^i p_i b_{ij} x_j x^{p-\delta_i} = \left(\sum_{i=1}^n p_i b_{ii} \right) x^p = 0$$

under the assumptions. □

Lemma 2.3.9. *Let $q \in \mathbb{Z}^n_k$ for some $k \in \{1,\ \ldots,\ n\}$, and let $B = (b_{ij})$ be a lower triangular matrix such that $\sum_{i=1}^n q_i b_{ii} = 0$. The vector fields Bx and $x^q x_k e_k$ are commutative if*

$$(q_i + \delta_{ik}) b_{ij} = 0 \quad \text{for } i > j,\ j = 1,\ldots,n-1, \tag{2.3.4}$$

and

$$b_{ik} = 0 \quad \text{for } i > k. \tag{2.3.5}$$

Proof. From conditions (2.3.4) and (2.3.5) we obtain

$$\begin{aligned} D(x^q x_k e_k) Bx &= \sum_{i=1}^n \sum_{j=1}^n (q_i + \delta_{ik}) b_{ij} x^q x_k x_i^{-1} x_j e_k \\ &= \sum_{i=1}^n \sum_{j=1}^i (q_i + \delta_{ik}) b_{ij} x^q x_k x_i^{-1} x_j e_k = \sum_{i=1}^n (q_i + \delta_{ik}) b_{ii} x^q x_k e_k \end{aligned}$$

and

$$D(Bx)x^q x_k e_k = x^q x_k \sum_{i=1}^{n} b_{ik} e_i = x^q x_k \sum_{i=k}^{n} b_{ik} e_i = b_{kk} x^q x_k e_k,$$

respectively. Hence, we have

$$\begin{aligned}
[Bx,\, x^q x_k e_k] &= D(x^q e_k)Bx - D(Bx)x^q x_k e_k \\
&= \sum_{i=1}^{n} (q_i + \delta_{ik}) b_{ii} x^q x_k e_k - b_{kk} x^q x_k e_k \\
&= \sum_{i=1}^{n} q_i b_{ii} x^q x_k e_k = 0,
\end{aligned}$$

which yields the result. □

Remark 2.3.1. Letting $R' = \emptyset$ and using Lemma 2.3.7, we see that the linear vector field Ax is $(n, 0)$-integrable if $A \neq 0$.

We now prove Proposition 2.3.6. By Lemma 2.3.7, we get the $n' - 1$ vector fields $\mu_A(b^{(1)})x, \ldots, \mu_A(b^{(n'-1)})x$. By definition, if $\rho_i = 0$, then the $(i,\, j)$-element of the matrix $\mu_A(b)$, $b \in \mathbb{C}^n$, is zero for $j < i$. Under the hypotheses of Proposition 2.3.6 (i) and (ii), $\mu_A(b)$ satisfies the hypotheses of Lemmas 2.3.8 and 2.3.9 along with conditions (2.3.4) and (2.3.5), respectively. We complete the proof of the proposition. □

Finally, we give a necessary and sufficient condition for a normal form to have the maximal number γ_{F} of functionally independent, analytic first integrals (cf. Corollary 2.3.4).

Proposition 2.3.10. *The normal form f with $\gamma_{\mathrm{P}} = \gamma_{\mathrm{F}}$ has γ_{F} functionally independent analytic first integrals in a neighborhood of $x = 0$ if and only if all elements of P_{s} are first integrals of f.*

Proof. The sufficiency is obvious by the definition of P_{s}. So we only have to show the necessity. Let $\gamma' = \gamma_{\mathrm{F}} (= \gamma_{\mathrm{P}})$. Let $F_1(x),\, \ldots,\, F_{\gamma'}(x)$ be γ' functionally independent, analytic first integrals of f and let $\phi_1(x),\, \ldots,\, \phi_{\gamma'}(x)$ be γ' generators of P_{s}. From Propositions 2.3.1 and 2.3.3, any first integral of f is written as a formal summation of elements of P_{s}. Since P_{s} is generated by $\phi_1, \ldots, \phi_{\gamma'}$, we can write

$$F_i(x) = G_i(\phi_1(x),\, \ldots,\, \phi_{\gamma'}(x)),$$

where $G_i(y_1, \ldots, y_{\gamma'})$, $i = 1, \ldots, \gamma'$, are analytic power series. Letting

$$\begin{array}{rcl}
F(x) &=& (F_1(x),\, \ldots,\, F_{\gamma'}(x)), \\
G(y) &=& (G_1(y),\, \ldots,\, G_{\gamma'}(y)), \quad x \in \mathbb{C}^n,\ y = (y_1, \ldots, y_{\gamma'}) \in \mathbb{C}^{\gamma'}, \\
\phi(x) &=& (\phi_1(x),\, \ldots,\, \phi_{\gamma'}(x)),
\end{array}$$

we have $F(x) = (G \circ \phi)(x)$. Differentiating the above relation with respect to x, we get

$$DF(x) = DG(\phi(x))D\phi(x). \tag{2.3.6}$$

The rank of $DF(x)$ is γ' for almost all $x \in \mathbb{C}^n$ because $F_1, \ldots, F_{\gamma'}$ are functionally independent. Hence, by (2.3.6), the rank of the matrix $DG(\phi(x))$ is γ' and $DG(\phi(x))$ is invertible for almost all $x \in \mathbb{C}^n$. On the other hand, we have

$$DF(x)f(x) = 0$$

in a neighborhood of $x = 0$ because $F_1, \ldots, F_{\gamma'}$ are first integrals of f. From (2.3.6) we get

$$DG(\phi(x))D\phi(x)f(x) = 0.$$

Multiplying the above equation by the inverse of $DG(\phi(x))$ from the left, we have $D\phi(x)f(x) = 0$ for almost all $x \in \mathbb{C}^n$. Since $\phi_i(x)$, $i = 1, \ldots, \gamma'$, and $f(x)$ are analytic, we have $D\phi(x)f(x) = 0$ near $x = 0$. Thus, the generators $\phi_1, \ldots, \phi_{\gamma'}$ are first integrals of f and consequently all element of P_s are so. □

Remark 2.3.2. (i) Results similar to Proposition 2.3.10 for formal first integrals were given in [27, 52] although their proofs contained some unclear parts.

(ii) As stated just before Theorem 2.2.4, we generally have $\gamma_\mathrm{P} \geq \gamma_\mathrm{F}$. The inequality in this relation holds for some normal forms. Actually, consider the linear vector field Ax with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

We easily see that $R_\mathrm{F} = \{p \in \mathbb{Z}^3_{\geq 0} \mid p_1 + p_2 + 2p_3 = 0\}$ and $\gamma_\mathrm{F} = 2$. On the other hand, by Proposition 2.3.3, we have $P_\mathrm{s} = \mathbb{C}[x_1x_2x_3,\ x_1^2x_3,\ x_2^2x_3]$ so that $\gamma_\mathrm{P} = 3 > \gamma_\mathrm{F}$. This example was also discussed in [21].

2.4 Proofs of the main theorems

In this section we give the proofs of Theorems 2.2.2, 2.2.4 and 2.2.5. Now it is easy to prove Theorems 2.2.2 and 2.2.4.

Proof of Theorem 2.2.2. By Proposition 2.2.1, there exist γ' monomials $x^{p^{(i)}}$, $i = 1, \ldots, \gamma'$, which are first integrals of the normal form (2.2.2), where $p^{(i)} \in R'$, $i = 1, \ldots, \gamma'$. Let $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ denote the linear part of (2.2.2). By Proposition 2.3.6, there exist $n - \gamma' - 1$ commuting vector fields $B^{(1)}x, \ldots, B^{(n-\gamma'-1)}x$ such that they commute with Ax. Since A is semisimple, they commute with the nonlinear terms of f by Proposition 2.3.6 (ii). Moreover, the monomials $x^{p^{(i)}}, i = 1, \ldots, \gamma'$, are also first integrals of $B^{(1)}x, \ldots, B^{(n-\gamma'-1)}x$ by Proposition 2.3.6 (i). Thus, the normal form (2.2.2) is $(n - \gamma', \gamma')$-integrable. □

Proof of Theorem 2.2.4. By hypotheses, it follows from Proposition 2.3.10 that all monomials x^p, $p \in R_\mathrm{F}$, are first integrals of f. Hence, by Proposition 2.2.1, condition $(\mathrm{C})_p$ holds for any $p \in R_\mathrm{F}$. Since $(n - \gamma_\mathrm{F}, \gamma_\mathrm{F})$-integrable normal forms have γ_F independent first integrals, we complete the proof. □

In the rest of this section we prove Theorem 2.2.5.

of Theorem 2.2.5. If $\gamma = 0$, then the normal form (2.2.1) only contains the linear part so that it is $(n,\, 0)$-integrable by Remark 2.3.1. Thus, the conclusion holds for $\gamma = 0$.

We next prove the case of $\gamma = 1$. Note that $0 \le \gamma_{\mathrm{F}} \le \gamma = 1$. We may also assume that the normal form (2.2.1) is not linear since non-trivial linear vector fields are integrable as stated in Remark 2.3.1. Let A be the linear part of (2.2.1) given by (2.3.2). By Proposition 2.3.6, there exist $n-2$ commuting matrices $B^{(1)}, \dots, B^{(n-2)}$ that commute with A. By Lemma 2.3.7, we can write $A = \mu_A(a), B^{(1)} = \mu_A(b^{(1)}), \dots, B^{(n-2)} = \mu_A(b^{(n-2)})$ with linearly independent vectors $a, b^{(1)}, \dots, b^{(n-2)} \in \mathbb{C}^n$. Note that by (2.3.3)

$$r \cdot \Delta(\mu_A(b)) = 0 \quad \text{for any } r \in R \text{ and } b \in \mathrm{span}_{\mathbb{C}}\{a, b^{(1)}, \dots, b^{(n-2)}\}. \tag{2.4.1}$$

We first consider the case of $\gamma_{\mathrm{F}} = 1$.

Lemma 2.4.1. *If $\gamma_{\mathrm{F}} = \gamma = 1$, then A has distinct eigenvalues.*

Proof. Assume that $\lambda_i = \lambda_j$ for some $i \neq j$. For $p \in R_{\mathrm{F}} \subset R$ we have $\sum_{l=1}^n \lambda_l p_l + \lambda_i - \lambda_j = 0$ and $p + e_i - e_j \in R$ by definition. On the other hand, since any element of p is positive, p and $e_i - e_j$ and hence p and $p + e_i - e_j$ are linearly independent. This means that $\gamma \ge 2$ and gives a contradiction. $\square$

By Lemma 2.4.1, A is semisimple so that $\sum_{i=1}^n c_i(p) x_i e_i = \mu_A(c(p))x$ with $c(p) = (c_1(p), \dots, c_n(p))$. Since $\hat{R}_i = \emptyset$, $i = 1, \dots, n$, the normal form (2.2.1) is written as

$$f(x) = \sum_{i=1}^n \left(\lambda_i + \sum_{p \in R_{\mathrm{F}}} c_i(p) x^p \right) x_i e_i = \mu_A(a)x + \sum_{p \in R_{\mathrm{F}}} x^p \mu_A(c(p))x.$$

Using Proposition 2.3.6 (ii), we see that $B^{(1)}x, \dots, B^{(n-2)}x$ commute with f. If $\sum_{i=1}^n c_i(p) r_i = 0$ for all $r, p \in R_{\mathrm{F}}$, then by Propositions 2.2.1 and 2.3.6 (i) the monomial x^r with $r \in R_{\mathrm{F}}$ is a first integral of $f, B^{(1)}x, \dots,\, B^{(n-2)}x$, so that the normal form (2.2.1) is $(n-1,\, 1)$-integrable. So we now assume that $\sum_{i=1}^n c_i(p) r_i \neq 0$ for some $r, p \in R_{\mathrm{F}}$. Since

$$r \cdot \Delta(\mu_A(c(p))) = r \cdot c(p) \neq 0,$$

we see that $a, b^{(1)}, \dots, b^{(n-2)}, c(p)$ are linearly independent by (2.4.1) and consequently $Ax, B^{(1)}x, \dots,\, B^{(n-2)}x, f(x)$ are so for almost all $x \in \mathbb{C}^n$ by Proposition 2.3.5. Thus, the normal form (2.2.1) is $(n,\, 0)$-integrable.

We next consider the case of $\gamma_{\mathrm{F}} = 0$.

Lemma 2.4.2. *Suppose that $\gamma_{\mathrm{F}} = 0$ and $\gamma = 1$. Then R has only one element $\hat{p}$ and there exists an integer $k \in \{1, \dots, n\}$ such that $\hat{p} \in \hat{R}_k$. Moreover, the following two statements hold:*

(i) If $i \neq k$, then $\lambda_i \neq \lambda_k$;

(ii) If $\hat{p}_j + \delta_{jk} \neq 0$ for some j, then $\lambda_i \neq \lambda_j$ for $i \neq j$.

Proof. Suppose that $\gamma_{\rm F}=0$ and $\gamma=1$. Let $p,q\in R$. Then we see that $p=rq$ for some $r\in\mathbb{Q}$ and $p\in\hat{R}_j$, $q\in\hat{R}_k$ for $j,k\in\{1,\dots,n\}$ since $R_{\rm F}=\emptyset$. We see that $j=k$ and $p=q$.

Assume that $\lambda_i=\lambda_k$ for some $i\neq k$. Since $\hat{p}\in R$, we have $\sum_{l=1}^n\hat{p}_l\lambda_l+\lambda_k-\lambda_i=0$, so that $\hat{p}+e_k-e_i\in R_i\subset R$. This contradicts the fact that R contains only one element. We obtain part (i).

Assume that $\hat{p}_j+\delta_{jk}\neq 0$ for some $j\neq k$ and $\lambda_i=\lambda_j$ for $i\neq j$. We have $\hat{p}_j\geq 1$ and $\sum_{l=1}^n\hat{p}_l\lambda_l+\lambda_i-\lambda_j=0$, so that $\hat{p}+e_i-e_j\in R_k\subset R$. This yields a contradiction as in the above. Thus, we obtain part (ii). $\square$

It follows from Lemma 2.4.2 that we have $x_ke_k=\mu_A(e_k)x$ and write f as

$$f(x)=Ax+c_k(\hat{p})x^{\hat{p}}x_ke_k=\mu_A(a)x+c_k(\hat{p})x^{\hat{p}}\mu_A(e_k)x,\ c_k(\hat{p})\neq 0$$

for $\hat{p}\in\hat{R}_k$. By Proposition 2.3.6 (ii) and Lemma 2.4.2, $Ax,B^{(1)}x,\dots,B^{(n-2)}x$ commute with the nonlinear part $x^{\hat{p}}x_ke_k$ of f. Since

$$\hat{p}\cdot\Delta(\mu_A(e_k))=\hat{p}\cdot e_k=\hat{p}_k=-1\neq 0,$$

we see that $a,b^{(1)},\dots,b^{(n-2)},e_k$ are linearly independent by (2.4.1) and consequently $Ax,B^{(1)}x,\dots,$ $B^{(n-2)}x,f(x)$ are so for almost all $x\in\mathbb{C}^n$ by Proposition 2.3.5. Thus, the normal form (2.2.1) is $(n,0)$-integrable and the conclusion holds for $\gamma=1$.

Finally, we give nonintegrable normal forms for any $\gamma\geq 2$. Let $n\geq 3$ be an integer and let $\lambda_1=n$, $\lambda_2=n+1$, $\dots$, $\lambda_{n-1}=2n-2$ and $\lambda_n=\prod_{i=1}^{n-1}\lambda_i$. Consider the n-dimensional normal form

$$f(x)=c_0\begin{pmatrix}\lambda_1&\dots&0&0\\ \vdots&\ddots&\vdots&\vdots\\ 0&\dots&\lambda_{n-1}&0\\ 0&\dots&0&\lambda_n\end{pmatrix}x+\left(\sum_{i=1}^{n-1}c_ix_i^{\lambda_n/\lambda_i}x_n^{-1}\right)x_ne_n, \tag{2.4.2}$$

where $c_0,\ c_1,\dots,c_{n-1}\in\mathbb{C}\backslash\{0\}$ are constants. We easily see that $\sum_{i=1}^n p_i\lambda_i\neq 0$ for $p\in\mathbb{Z}_j^n$, $j=1,\dots,n-1$ with $|p|\geq 2$ so that $\hat{R}_1=\dots=\hat{R}_{n-1}=\emptyset$. Let

$$p^{(1)}=\left(\frac{\lambda_n}{\lambda_1},0,\dots,0,-1\right),p^{(2)}=\left(0,\frac{\lambda_n}{\lambda_2},0,\dots,0,-1\right),\dots,$$
$$p^{(n-1)}=\left(0,\dots,0,\frac{\lambda_n}{\lambda_{n-1}},-1\right)\in\hat{R}_n.$$

Noting that $p^{(1)},p^{(2)},\dots,p^{(n-1)}$ are linearly independent, we see that $\gamma=n-1$. Since $\lambda_1,\dots,\lambda_n$ are positive, we have $R_{\rm F}=\emptyset$ and $\gamma_{\rm F}=0$. Hence, there are no analytic first integrals of f except constants by Corollary 2.3.4. Thus, the normal form (2.4.2) is not $(m,n-m)$-integrable for $1\leq m<n$.

Let g be an analytic vector field commuting with f. By Proposition 2.3.1 (i)

$$[f^{\rm s},\,g]=0,\quad\text{ i.e.,}\quad [f^{\rm s},\,g^m]=0\quad\text{ for } m=1,\,2,\,\dots,$$

where g^m is a homogeneous part of degree m for g. Hence, the coefficient matrix $\tilde{A}$ of g^1 is diagonal and by Lemma 2.3.2 the nonlinear part of g is written as

$$\left(\sum_{q\in\hat{R}_n} \tilde{c}_q x^q\right) x_n e_n$$

for $\tilde{c}_q \in \mathbb{C}$, $q \in \hat{R}_n$.

From $[f, g] = 0$ we get

$$\left[\sum_{i=1}^{n-1} c_i x_i^{\lambda_n/\lambda_i} x_n^{-1} x_n e_n, g^1\right] = 0, \quad \text{i.e.,} \quad \left[x_i^{\lambda_n/\lambda_i} x_n^{-1} x_n e_n, g^1\right] = 0 \quad \text{for } i = 1, \ldots, n-1.$$

Using (2.3.1), we obtain $\sum_{j=1}^{n} p_j^{(i)} \tilde{a}_{jj} = 0$ for $i = 1, \ldots, n-1$, where $\tilde{a}_{jj}$ denotes the $(j,\, j)$-element of $\tilde{A}$ for $j = 1, \ldots, n$. So we have

$$\tilde{A} = \tilde{c}_0 \begin{pmatrix} \lambda_1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \lambda_{n-1} & 0 \\ 0 & \ldots & 0 & \lambda_n \end{pmatrix}$$

for some $\tilde{c}_o \in \mathbb{C}$, so that

$$g(x) = \tilde{c}_0 \begin{pmatrix} \lambda_1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \lambda_{n-1} & 0 \\ 0 & \ldots & 0 & \lambda_n \end{pmatrix} x + \left(\sum_{q\in\hat{R}_n} \tilde{c}_q x^q x_n\right) e_n.$$

This means that $g(x) \in \operatorname{span}_{\mathbb{C}}\{f^{\mathrm{s}}(x), e_n\}$ for $x \in \mathbb{C}^n$. Hence, the normal form (2.4.2) is not $(n, 0)$-integrable. For $\gamma \geq 2$, we take $n = \gamma + 1$ to obtain a nonintegrable normal form. Thus we complete the proof. □

Remark 2.4.1. (i) Although it is analytically nonintegrable for $n \geq 3$, the normal form (2.4.2) is rationally $(2, n-2)$-integrable since the semisimple part f^{s} commutes with f and $n-2$ functionally independent, rational functions $x_1^{\lambda_2}/x_2^{\lambda_1}$,$x_1^{\lambda_3}/x_3^{\lambda_1}, \ldots$, $x_1^{\lambda_{n-1}}/x_{n-1}^{\lambda_1}$ are first integrals for both f and f^{s}. Note that f and f^{s} are linearly independent almost everywhere.

(ii) Assume that equation (2.1.1) is $(m, n-m)$-integrable and let $f_1, \ldots, f_m$ and $F_1, \ldots,$ F_{n-m}, respectively, denote the m commutative vector fields and $n-m$ first integrals satisfying conditions (i)-(iii) of Definition 2.1.3. Let g_i be the linear part of f_i and let G_i be the lowest homogeneous part of G_i. Then we say that the vector fields and first integrals $(f_1, \ldots, f_m, F_1, \ldots, F_{n-m})$ are *non-degenerate* if the m vector fields $g_1, \ldots, g_m$ and $n-m$ functions $G_1, \ldots, G_{n-m}$ satisfy conditions (i)-(iii) of Definition 2.1.3 and $g_1, \ldots, g_m$ are semisimple. Zung [99] called such an n-tuple a non-degenerate integrable system and showed that it has several good properties. However, many integrable normal forms do not have non-degenerate vector fields and first integrals. For example, any linear non-semisimple vector field Ax, which is integrable as stated in Remark 2.3.1, does not satisfy the non-degenerate condition. Another example is the two-dimensional vector field $f(x) = x_1e_1 + (2x_2 + cx_1^2)e_2, c \in \mathbb{C} \setminus \{0\}$. Actually, although f is integrable by Theorem 2.2.5 since its resonance degree is one, it is shown to have no first integral by using Corollary 2.3.4. Moreover, if a vector filed f_2 commutes with f, then it is also shown to be written as $f_2(x) = c_1x_1e_1 + (2c_1x_2 + c_2x_1^2)e_2$ with $c_1, c_2 \in \mathbb{C}$ as in the proof of Theorem 2.2.5, so that the linear parts of f and f_2 are linearly dependent. Thus, f never has non-degenerate vector fields and first integrals.

2.5 Example

We now demonstrate the main results for the concreste example given by (2.1.7). We have $R_1 = \{(j,k,k) \in \mathbb{Z}^3 \mid j \geq -1, k \geq 0, j + 2k \geq 1\}$ and $R_2 = R_3 = \{(j,\ k,\ k) \in \mathbb{Z}^3_{\geq 0} \mid j + 2k \geq 1\}$, so that $R = R_1$ and $R_\mathrm{F} = R_2, R_3$. Thus, $\gamma = \gamma_\mathrm{F} = 2$ and the normal form (2.1.7) may not be integrable by Theorem 2.2.5. Applying Theorems 2.2.2 and 2.2.4, we obtain the following result.

Theorem 2.5.1. *The normal form* (2.1.7) *is* (1, 2)*-integrable if and only if*

$$a_{jk} = d_l = b_{jk} + c_{jk} = 0 \quad \textit{for } j + 2k \geq 1,\ j,k \geq 0 \textit{ and } l \geq 0. \tag{2.5.1}$$

Moreover, if condition (2.5.1) *holds, then equation* (2.1.7) *is* (2, 1)*-integrable.*

Proof. We first have $\hat{R}_1 = \{(-1,k,k) \in \mathbb{Z}^3 \mid k \geq 1\}$ and $\hat{R}_2 = \hat{R}_3 = \emptyset$. Since $c_1(j,k,k) = a_{jk}$, $c_2(j,k,k) = b_{jk}$, $c_3(j,k,k) = c_{jk}$ for $j + 2k \geq 1$ and $j,k \geq 0$ and $c_1(-1,l,l) = d_l$ for $l \geq 0$ in the normal form (2.1.7), condition (2.5.1) holds if and only if $(\mathrm{C})_r$ holds for $r \in R_\mathrm{F}$. Taking $R' = R$ and using Theorem 2.2.2, we see that under condition (2.5.1), equation (2.1.7) is $(1,2)$-integrable. On the other hand, it follows from Proposition 2.3.3 that $P_s = \mathbb{C}[x_1, x_2x_3]$ so that $\gamma_\mathrm{P} = 2$. Using Theorem 2.2.4, we see that if equation (2.1.7) is $(1,2)$-integrable, then condition (2.5.1) holds.

On the other hand, assume that condition (2.5.1) holds and take $R' = \{(0,\ k,\ k) \in \mathbb{Z}^3 \mid k \geq 1\}$. We have $\dim_\mathbb{Q}\mathrm{span}_\mathbb{Q}R' = \dim_\mathbb{Q}\mathrm{span}_\mathbb{Q}(R_\mathrm{F} \cap R') = 1$. Hence, applying Theorem 2.2.4, we show that equation (2.1.7) is $(2,1)$-integrable. □

Note that Theorem 2.5.1 does not necessarily mean analytic nonintegrability of (2.1.7) when condition (2.5.1) does not hold: It may be analytically $(2,1)$- or $(3,0)$-integrable. Moreover, even though it is not analytically integrable, it may be meromorphically integrable. However, Yagasaki [93] used an extension of the Morales-Ramis-Simo theory [66] due to Ayoul and Zung [6] to show that a similar system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \mu \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ (\nu + i\omega)x_2 \\ (\nu - i\omega)x_3 \end{pmatrix} + \begin{pmatrix} ax_1^2 \\ bx_1x_2 \\ cx_1x_3 \end{pmatrix} + \begin{pmatrix} dx_2x_3 \\ 0 \\ 0 \end{pmatrix}, \tag{2.5.2}$$

where μ, ν, a, b, c, d are constants, is meromorphically nonintegrable near the x_1-axis for almost all parameter values. This result suggests that equation (2.1.7) may be meromorphically or rationally nonintegrable for almost all parameter unless if condition (2.5.1) does not hold.

Chapter 3

Nonintegrability of dynamical systems with homo- and heteroclinic orbits

3.1 Introduction

Nonintegrability of differential equations is one of the most important topics in dynamical systems [31, 33, 62]. It is believed that complicated dynamical behavior such as chaos may occur in general if differential equations are nonintegrable [33], although there are several notions of integrability. However, it is difficult in general to determine whether given differential equations are integrable or not.

For Hamiltonian systems, the notion of integrability has been established and called the Liuoville integrability [51]: an n-degree-of-freedom Hamiltonian system is said to be *Liuoville integrable* if it possesses n independent first integrals 'in involution'. It is a well-known result as the Liouville-Arnold theorem that if an n-degree-of-freedom Hamiltonian system is integrable in this sense and a level set of n first integrals is compact, then the flow on the level set is diffeomorphic to a linear flow on the n-dimensional torus $\mathbb{T}^n$, i.e., it is quasiperiodic [4]. Much research has been done on nonintegrability of Hamiltonian systems [31, 33, 62, 63], and two powerful techniques to prove their nonintegrability have been developed: the Ziglin analysis [94] and Morales-Ramis theory [62, 65]. Both the techniques rely on some properties of *variational equations* (VEs), i.e., linearized equations, along particular solutions of the Hamiltonian systems. The former uses their monodromy matrices and the latter uses their differential Galois groups [42, 23, 81].

For general differential equations which may be non-Hamiltonian, Bogoyavlenskij [9] introduced a general striking definition of integrability, among several notions (see, e.g., [63]). Consider differential equations of the form

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{3.1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is analytic. His definition of integrability is stated for (3.1.1) as follows.

Definition 3.1.1 (Bogoyavlenskij)**.** *Equation* (3.1.1) *is called* $(q, n-q)$-integrable *or just* integrable *if there exist* q (≥ 1) *vector fields* $f_1(x)(:= f(x)), f_2(x), \dots, f_q(x)$ *and* $n-q$ *scalar-valued functions* $H_1(x), \dots, H_{n-q}(x)$ *such that the following three conditions hold:*

(i) $f_1, \dots, f_q$ *are linearly independent and* $DH_1, \dots, DH_{n-q}$ *are linearly independent almost everywhere;*

(ii) $f_1, f_2, \dots, f_q$ *commute, i.e.,* $[f_j, f_k] := (Df_k)f_j - (Df_j)f_k = 0$*, for* $j, k = 1, \dots, q$*;*

(iii) $H_1, \dots, H_{n-q}$ *are first integrals of* $f_1, f_2, \dots, f_q$*, i.e.,* $(DH_k)f_j = 0$ *for* $j = 1, \dots, q$ *and* $k = 1, \dots, n-q$*.*

If $f_1, f_2, \dots, f_q$ *and* $H_1, \dots, H_{n-q}$ *are real-meromorphic, then equation* (3.1.1) *is said to be* real-meromorphically integrable*.*

Definition 3.1.1 is thought to be the most general at present since equation (3.1.1) can be integrable even when it has only $n - q$ ($< n - 1$) first integrals for some $q > 1$. In particular, if a Hamiltonian system is Liouville integrable, then it is also integrable in the meaning of Bogoyavlenskij. Ayoul and Zung [6] extended the Morales-Ramis theory to the nonintegrability of non-Hamiltonian differential equations in this meaning. Their method was successfully applied to prove the nonintegrability of the five-dimensional Karabut system, which is non-Hamiltonian and appears in relation to a fluid of finite depth (e.g., [44]), in [19].

On the other hand, Morales-Ruiz and Peris [64] studied a class of two-degree-of-freedom Hamiltonian systems with saddle-center equilibria connected by homoclinic orbits, and applied the Morales-Ramis theory to obtain a sufficient condition for their nonintegrability. Moreover, they used the results of [34, 50] to show that chaotic dynamics occurs if the condition holds. Their result was extended to more general two-degree-of-freedom Hamiltonian systems with saddle-centers connected by homoclinic orbits in [91], based on the result of [89] as well as [64].

In this chapter we extend the result of the first part of [64, 91] to (3.1.1) with $n \geq 3$ in a general situation. Equation (3.1.1) is assumed to have equilibria connected by heteroclinic orbits on an $(n-2)$-dimensional, locally invariant manifold $\mathscr{N}$, but it may be non-Hamiltonian and the equilibria do not have to be saddle-centers. Here "local invariance" means that the trajectory $x(t)$ may pass through $\partial\mathscr{N}$ and escape $\mathscr{N}$ even if $x(0) \in \mathscr{N}$. The equilibria are also allowed to be the same and connected by a homoclinic orbit. Under additional assumptions, we prove that the monodromy group for the *normal variational equation* (NVE) of (3.1.1), which is represented by components of the VE normal to $\mathscr{N}$ and defined on a Riemann surface is diagonalizable or infinitely cyclic if equation (3.1.1) is real-meromorphically integrable. Our assumptions may seem to be more or less limited but similar ones were assumed in the previous work on homoclinic orbits [64, 91] (see also Remark 3.4.1). See Section 3.4 for precise statements of our assumptions and main result. Our approach relies on a classification of algebraic subgroups of $\mathrm{SL}(2, \mathbb{C})$ [67] like [64] and on the extension of the Morales-Ramis theory due to Ayoul and Zung [6]. Generalizations of the second part of [64, 91] to non-Hamiltonian systems with homoclinic and heteroclinic orbits will be pursued in subsequent work.

We apply the theory to a three-dimensional volume-preserving system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha x_3 - 8x_1x_2 \\ 11x_1^2 + 3x_2^2 + x_3^2 + \beta x_1 x_3 - 3 \\ -\alpha x_1 + 2x_2x_3 - \beta x_1 x_2 \end{pmatrix}, \tag{3.1.2}$$

which was introduced by Bajer and Moffatt [7] as a mathematical model which describes the streamline of a steady incompressible flow, where $\alpha, \beta \in \mathbb{R}$ are constants. The right hand side of (3.1.2) satisfies the Stokes equation for a viscous fluid, and the unit sphere is invariant under the flow of (3.1.2). Bajer and Moffatt [7] also carried out numerical simulations and demonstrated that chaotic dynamics occurs for $\alpha = 0.01$ with $\beta = 0$ and for several values of α with $\beta = 1$. When $\alpha, \beta = 0$, it has two first integrals $H_1 = x_1 x_3^4$ and $H_2 = (x_1^2 + x_2^2 + x_3^2 - 1)/x_3^3$ [7], so that it is $(1, 2)$-integrable. Neishtadt et al. [69] and Vainshtein et al. [82] explained its chaotic dynamics by jumps in adiabatic invariants when $\alpha > 0$ is sufficiently small and $\beta = 0$. See also [70]. On the other hand, Nishiyama [71, 72] showed that it has no meromorphic first integral when $\alpha > 0$ and $\beta = 1$, or $\alpha = 1$, $\beta > 0$ and $\beta \notin \{2\sqrt{23}, 8\sqrt{5}, 16\sqrt{2}\}$. His approach was similar to that of Maciejewski and Przybylska [53] for the so-called ABC flow, based on the Ziglin analysis [94]. It is still open to prove whether it is nonintegrable or exhibits chaotic dynamics for the other parameter values. Here we show that it is real-meromorphically nonintegrable for almost all values of the pair (α, β) (see Section 3.5).

The outline of this chapter is as follows: In Section 3.2, we state necessary information on the differential Galois theory for linear differential equations, including their monodromy groups. In Section 3.3, a key result for two-dimensional linear systems, which is used to prove our main result, is given. In Section 3.4, we present our assumptions and the main result. Finally, we illustrate our theory for the example (3.1.2) in Section 3.5. In Appendix A, we give definitions of VEs and NVEs in a more general setting and outline the extension of the Morales-Ramis theory due to Ayoul and Zung [6].

3.2 Differential Galois theory

Differential Galois theory deals with the problem of integrability by quadratures for differential equations. In this section we briefly review a part of differential Galois theory for linear differential equations, which is often referred to as the Picard-Vessiot theory, including monodromy groups.

3.2.1 Picard-Vessiot extensions

Consider a system of linear differential equations

$$\dot{y} = Ay, \quad A \in \mathrm{gl}(n, \mathbb{K}), \tag{3.2.1}$$

where $\mathbb{K}$ is a differential field and $\mathrm{gl}(n, \mathbb{K})$ denotes the ring of $n \times n$ matrices with entries in $\mathbb{K}$. We recall that a *differential field* is a field endowed with a derivation ∂, which is an additive endomorphism satisfying the Leibniz rule. By abuse of notation we write $\dot{y}$ instead of ∂y. The set $\mathrm{C}_{\mathbb{K}}$ of elements of $\mathbb{K}$ for which ∂ vanishes is a subfield of $\mathbb{K}$ and called the *field of constants of* $\mathbb{K}$. In our application of the theory in this chapter, the differential field $\mathbb{K}$ is the field of meromorphic functions on a Riemann surface Γ endowed with a meromorphic vector field, so that the field of constants becomes the field of complex numbers $\mathbb{C}$.

A *differential field extension* $\mathbb{L} \supset \mathbb{K}$ is a field extension such that $\mathbb{L}$ is also a differential field and the derivations on $\mathbb{L}$ and $\mathbb{K}$ coincide on $\mathbb{K}$. A differential field extension $\mathbb{L} \supset \mathbb{K}$ satisfying the following conditions is called a *Picard-Vessiot extension* for (3.2.1).

(PV1) There is a fundamental matrix Φ of (3.2.1) with entries in $\mathbb{L}$.

(PV2) The field $\mathbb{L}$ is generated by $\mathbb{K}$ and entries of the fundamental matrix Φ.

(PV3) The fields of constants for $\mathbb{L}$ and $\mathbb{K}$ coincide.

The system (3.2.1) admits a Picard-Vessiot extension which is unique up to isomorphism. An algebraic construction of the Picard-Vessiot extension was given in a general situation by Kolchin (see, e.g., [47]).

We now fix a Picard-Vessiot extension $\mathbb{L} \supset \mathbb{K}$ and fundamental matrix Φ with entries in $\mathbb{L}$ for (3.2.1). Let σ be a $\mathbb{K}$-*automorphism* of $\mathbb{L}$, which is a field automorphism of $\mathbb{L}$ that commutes with the derivation of $\mathbb{L}$ and leaves $\mathbb{K}$ pointwise fixed. Obviously, $\sigma(\Phi)$ is also a fundamental matrix of (3.2.1) and consequently there is a matrix M_σ with constant entries such that $\sigma(\Phi) = \Phi M_\sigma$. This relation gives a faithful representation of the group of $\mathbb{K}$-automorphisms of $\mathbb{L}$ on the general linear group as

$$R\colon \mathrm{Aut}_{\mathbb{K}}(\mathbb{L}) \to \mathrm{GL}(n, \mathrm{C}_{\mathbb{L}}), \quad \sigma \mapsto M_\sigma,$$

where $\mathrm{GL}(n, \mathrm{C}_{\mathbb{L}})$ is the group of $n \times n$ invertible matrices with entries in $\mathrm{C}_{\mathbb{L}}$. The image of R is a linear algebraic subgroup of $\mathrm{GL}(n, \mathrm{C}_{\mathbb{L}})$, which is called the *differential Galois group* of (3.2.1) and denoted by $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$. This representation is not unique and depends on the choice of the fundamental matrix Φ, but a different fundamental matrix only gives rise to a conjugated representation. Thus, the differential Galois group is unique up to conjugation as an algebraic subgroup of the general linear group.

Definition 3.2.1. *A differential field extension* $\mathbb{L} \supset \mathbb{K}$ *is called*

(i) an integral extension *if there exists* $a \in \mathbb{L}$ *such that* $\dot{a} \in \mathbb{K}$ *and* $\mathbb{L} = \mathbb{K}(a)$*, where* $\mathbb{K}(a)$ *is the smallest extension of* $\mathbb{K}$ *containing* a*;*

(ii) an exponential extension *if there exists* $a \in \mathbb{L}$ *such that* $\dot{a}/a \in \mathbb{K}$ *and* $\mathbb{L} = \mathbb{K}(a)$*;*

(iii) an algebraic extension *if there exists* $a \in \mathbb{L}$ *such that it is algebraic over* $\mathbb{K}$ *and* $\mathbb{L} = \mathbb{K}(a)$.

Definition 3.2.2. *A differential field extension* $\mathbb{L} \supset \mathbb{K}$ *is called a* Liouvillian extension *if it can be decomposed as a tower of extensions,*

$$\mathbb{L} = \mathbb{K}_n \supset \ldots \supset \mathbb{K}_1 \supset \mathbb{K}_0 = \mathbb{K},$$

such that each extension $\mathbb{K}_{j+1} \supset \mathbb{K}_j$ *is either integral, exponential or algebraic.*

Let $G \subset \mathrm{GL}(n, \mathrm{C}_{\mathbb{L}})$ be an algebraic group. Then it contains a unique maximal connected algebraic subgroup G^0, which is called the *connected component of the identity* or *connected identity component*. The connected identity component $G^0 \subset G$ is a normal algebraic subgroup and the smallest subgroup of finite index, i.e., the quotient group G/G^0 is finite. By the Lie-Kolchin Theorem [42, 23, 81], a connected solvable linear algebraic group is triangularizable. Here a subgroup of $\mathrm{GL}(n, \mathrm{C}_{\mathbb{L}})$ is said to be *triangularizable* if it is conjugated to a subgroup of the group of (lower) triangular matrices. The following theorem relates the solvability of the differential Galois group with a Liouvillian Picard-Vessiot extension (see [42, 23, 81] for the proof).

Theorem 3.2.1. *Let* $\mathbb{L} \supset \mathbb{K}$ *be a Picard-Vessiot extension of* (3.2.1)*. The connected identity component of the differential Galois group* $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$ *is solvable if and only if the extension* $\mathbb{L} \supset \mathbb{K}$ *is Liouvillian.*

3.2.2 Monodromy group and Fuchsian equations

Let $\mathbb{K}$ be the field of meromorphic functions on a Riemann surface Γ. So the set of singularitie in the entries of A is a discrete subset of Γ, which is denoted by S. We also refer to a singularity of the entries of A as that of (3.2.1). Let $t_0 \in \Gamma \setminus S$. We prolong the fundamental matrix $\Phi(t)$ analytically along any loop γ based at t_0 and containing no singular point, and obtain another fundamental matrix $\gamma * \Phi(t)$. So there exists a constant nonsingular matrix $M_{[\gamma]}$ such that

$$\gamma * \Phi(t) = \Phi(t) M_{[\gamma]}.$$

The matrix $M_{[\gamma]}$ depends on the homotopy class $[\gamma]$ of the loop γ and is called the *monodromy matrix* of $[\gamma]$.

Let $\pi_1(\Gamma \setminus S, t_0)$ be the fundamental group of homotopy classes of loops based at t_0. We have a representation

$$\tilde{R} \colon \pi_1(\Gamma \setminus S, t_0) \to \mathrm{GL}(n, \mathbb{C}), \quad [\gamma] \mapsto M_{[\gamma]}.$$

The image of $\tilde{R}$ is called the *monodromy group* of (3.2.1). As in the differential Galois group, the representation $\tilde{R}$ depends on the choice of the fundamental matrix, but the monodromy group is defined as a group of matrices up to conjugation. In general, a monodromy transformation defines an automorphism of the corresponding Picard-Vessiot extension. We also just write M_γ for $M_{[\gamma]}$ below.

A singular point $t = \bar{t}$ of (3.2.1) is called *regular* if for any sector $a < \arg(t - \bar{t}) < b$ with $a < b$ there exists a fundamental matrix $\Phi(t) = (\phi_{ij}(t))$ such that for some $c > 0$ and integer N, $|\phi_{ij}(t)| < c|t - \bar{t}|^N$ as $t \to \bar{t}$ in the sector; otherwise it is called *irregular*. Equation (3.2.1) is said to be *Fuchsian* if all singularities are regular. Any univalued solution of a Fuchsian equation is meromorphic. This gives us the following result along with the normality of the Picard-Vessiot extensions (see, e.g., Theorem 5.8 in [81] for the proof).

Theorem 3.2.2 (Schlessinger)**.** *Assume that Equation* (3.2.1) *is Fuchsian. Then the differential Galois group of* (3.2.1) *is the Zariski closure of the monodromy group.*

3.3 Key result for two-dimensional linear systems

Now we give a key result for two-dimensional Fuchsian systems, which is used in the proof of our main result in Section 3.4. Let $\mathbb{K}$ be the field of meromorphic functions on a Riemann surface Γ, as in Section 3.2.2, and let $n = 2$. We assume that equation (3.2.1) is Fuchsian and has regular singularities at $t = t_j \in \Gamma$, $j = 1, \dots, k$. Let $S = \{t_1, \dots, t_k\}$ be the set of singular points of (3.2.1) and let $\gamma_j \subset \Gamma \setminus S$ be an infinitesimal loop around $t_j \in S$ for $j = 1, \dots, k$. Let G and M, respectively, denote the differential Galois group and monodromy group of (3.2.1), and let G^0 be the connected identity component of G. If $\mathrm{M} \subset \mathrm{SL}(2, \mathbb{C})$, then we can use the following lemma directly to discuss the relation between G^0 and M, as in [64] (see, e.g., [62] for the proof).

Lemma 3.3.1. *Any algebraic group $G \subset \mathrm{SL}(2, \mathbb{C})$ is similar to one of the following types.*

(i) G is finite and $G^0 = \{I_2\}$;

(ii) $G = G^0 = \left\{ \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} \middle| \mu \in \mathbb{C} \right\}$;

(iii) $G = \left\{ \begin{pmatrix} \lambda & 0 \\ \mu & \lambda^{-1} \end{pmatrix} \middle| \lambda \text{ is a root of } 1,\ \mu \in \mathbb{C} \right\}$ *and* $G^0 = \left\{ \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} \middle| \mu \in \mathbb{C} \right\}$;

(iv) $G = G^0 = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \middle| \lambda \in \mathbb{C}^* \right\}$;

(v) $G = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & 0 \end{pmatrix} \middle| \lambda, \beta \in \mathbb{C}^* \right\}$ *and* $G^0 = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \middle| \lambda \in \mathbb{C}^* \right\}$;

(vi) $G = G^0 = \left\{ \begin{pmatrix} \lambda & 0 \\ \mu & \lambda^{-1} \end{pmatrix} \middle| \lambda \in \mathbb{C}^*,\ \mu \in \mathbb{C} \right\}$;

(vii) $G = G^0 = \mathrm{SL}(2,\, \mathbb{C})$,

where I_2 *is the* 2×2 *identity matrix.*

However, M may not be an algebraic subgroup of $\mathrm{SL}(2, \mathbb{C})$ in general. Instead, we prove the following result for (3.2.1). Let $M_j = M_{\gamma_j}$, $j = 1, \ldots, k$.

Proposition 3.3.2. *Suppose that* M_j *has* $\lambda_j, \lambda_j' \neq 0$ *as its eigenvalues for* $j = 1, \ldots, k$ *such that* $(\lambda_j/\lambda_j')^{q_j} \neq 1$ *for any* $q_j \in \mathbb{Z}$. *If* G^0 *is commutative, then so is* M.

To prove this proposition, we use the following trick. Substituting

$$z = \exp\left(-\frac{1}{2} \int \operatorname{tr} A(t) dt \right) y$$

into (3.2.1), we have

$$\dot{z} = \tilde{A}(t) z, \quad \tilde{A}(t) = A(t) - \frac{1}{2} \operatorname{tr} A(t) I_2, \tag{3.3.1}$$

which is also Fuchsian. Note that $\operatorname{tr} \tilde{A}(t) = 0$. Let $\tilde{\mathrm{G}}$ and $\tilde{\mathrm{M}}$, respectively, denote the differential Galois and monodromy groups of (3.3.1), and let $\tilde{\mathrm{G}}^0$ be the connected identity component of $\tilde{\mathrm{G}}$.

Lemma 3.3.3. $\tilde{\mathrm{G}}$ *is an algebraic subgroup of* $\mathrm{SL}(2, \mathbb{C})$.

Proof. Let $Z(t)$ be a fundamental matrix of (3.3.1). Since $\det Z(t) \neq 0$ and

$$\frac{d}{dt} \det Z(t) = \operatorname{tr} \tilde{A}(z) \det Z(z) = 0,$$

we see that $\det Z(z)$ is a nonzero constant. On the other hand, analytic continuation of $Z(t)$ along a loop $\gamma \subset \Gamma \setminus S$ yields

$$\gamma * Z(t) = Z(t) \tilde{M}_\gamma.$$

Taking the determinant of the above identity and noting that $\det(\gamma * Z(t)) = \det Z(t) \neq 0$, we obtain $\det \tilde{M}_\gamma = 1$, so that $\tilde{\mathrm{M}} \subset \mathrm{SL}(2, \mathbb{C})$. Hence, it follows from Theorem 3.2.2 that $\tilde{\mathrm{G}} \subset \mathrm{SL}(2, \mathbb{C})$, since equation (3.3.1) is Fuchsian. □

Thus, equation (3.3.1) is so nice that Lemma 3.3.1 can be applicable to it. So we replace arguments of (3.2.1) with those of (3.3.1). We have the following two lemmas on the relationship between them.

Lemma 3.3.4. G^0 *is triangularizable if and only if so is* $\tilde{\mathrm{G}}^0$*.*

Proof. Obviously, the Picard-Vessiot extension of (3.2.1) is Liouvillian if and only if so is that of (3.3.1). Using Theorem 3.2.1, we immediately obtain the result. □

Lemma 3.3.5. *We have*

$$M_j = m_j \tilde{M}_j, \quad j = 1, \ldots, k, \tag{3.3.2}$$

where $\tilde{M}_j$ *is a monodromy matrix of* (3.3.1) *along* γ_j *and*

$$m_j = \exp\left(\frac{1}{2}\int_{\gamma_j} \operatorname{tr} A(t)dt\right) \neq 0, \quad j = 1, \ldots, k.$$

Moreover, M *is commutative if and only if so is* $\tilde{\mathrm{M}}$*.*

Proof. Choose fundamental matrices $Y(t)$ and $Z(t)$ of (3.2.1) and (3.3.1), respectively, such that

$$Y(t) = \kappa(t)Z(t), \tag{3.3.3}$$

where

$$\kappa(t) = \exp\left(\frac{1}{2}\int \operatorname{tr} A(t)dt\right).$$

We prolong $Y(t)$, $Z(t)$ and $\kappa(t)$ analytically along γ_j to obtain

$$\gamma_j * Y(t) = Y(t)M_j, \quad \gamma_j * Z(t) = Z(t)\tilde{M}_j, \quad \gamma_j * \kappa(t) = \kappa(t)m_j,$$

which yield

$$\gamma_j * Y(t) = (\gamma_j * \kappa(t))(\gamma_j * Z(t)) = \kappa(t)Z(t)m_j\tilde{M}_j = Y(t)m_j\tilde{M}_j$$

along with (3.3.3). Hence, we have

$$Y(t)M_j = Y(t)m_j\tilde{M}_j,$$

which results in the first part since $Y(t)$ is nonsingular. The second part immediately follows from the first part since M is generated by M_j, $j = 1, \ldots, k$. □

Proof of Proposition 3.3.2. Let j be any integer such that $1 \leq j \leq k$, and let $\lambda_j = e^{2\pi i\rho_j}$ and $\lambda'_j = e^{2\pi i\rho'_j}$. Let $\tilde{M}_j$ denote a monodromy matrix of (3.3.1) along γ_j. By the hypothesis, we have $\rho_j - \rho'_j \notin \mathbb{Q}$. Recall that $\tilde{\mathrm{M}} \subset \tilde{\mathrm{G}} \subset \mathrm{SL}(2, \mathbb{C})$ by Theorem 3.2.2 and Lemma 3.3.3. Hence, we have $\det \tilde{M}_j = 1$, so that by Lemma 3.3.5 $m_j^2 = \det M_j = e^{2\pi j(\rho_j + \rho'_j)}$, i.e., $m_j = e^{\pi j(\rho_j + \rho'_j)}$, and $\tilde{M}_j$ has $e^{2\pi i\rho}/m_j = e^{\pi i(\rho_j - \rho'_j)}$ and $e^{2\pi i\rho'}/m_j = e^{-\pi i(\rho_j - \rho'_j)}$ as its eigenvalues.

Suppose that G^0 is commutative. Then it is triangularizable and by Lemma 3.3.4 so is $\tilde{\mathrm{G}}^0$. Thus, $\tilde{\mathrm{G}}$ is not of type (vii) in Lemma 3.3.1. Since $\rho_j - \rho'_j \notin \mathbb{Q}$, $\tilde{M}_j$ has no root of 1 as its eigenvalue for $j = 1, \ldots, k$. Hence, $\tilde{\mathrm{G}}$ is of type (iv), (v) or (vi) since its element

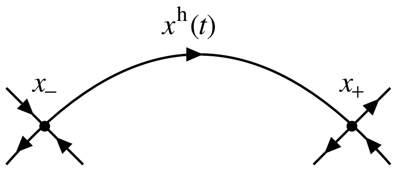

Figure 3.1: Assumptions (A1) and (A2).

would have a root of 1 as its eigenvalue if it was of type (i), (ii) or (iii). If $\tilde{\mathrm{G}}$ is of type (v), then $\tilde{\mathrm{M}} \subset \tilde{\mathrm{G}}^0$ since $\tilde{M}_j$, $j = 1, \ldots, k$, have no roots of 1 as their eigenvalues, so that the Zariski closure of $\tilde{\mathrm{M}}$ does not coincide with $\tilde{\mathrm{G}}$. This implies by Theorem 3.2.2 that $\tilde{\mathrm{G}}$ is not of type (v). If $\tilde{\mathrm{G}}$ is of type (iv), then $\tilde{\mathrm{M}}\,(\subset \tilde{\mathrm{G}})$ is also commutative, and by the second part of Lemma 3.3.5 so is M.

To complete the proof, we show that $\tilde{\mathrm{G}}$ is not of type (vi) in Lemma 3.3.1 if G^0 is commutative. Suppose that G^0 is commutative and $\tilde{\mathrm{G}}$ is of type (vi). Then by Theorem 3.2.2 the monodromy matrices of (3.2.1) can be represented by

$$M_l = m_l \begin{pmatrix} \tilde{\lambda}_l & 0 \\ \tilde{\mu}_l & \tilde{\lambda}_l^{-1} \end{pmatrix}, \quad l = 1, \ldots, k,$$

where $\tilde{\lambda}_l = \lambda_l / m_l$ is not a root of 1 and $\tilde{\mu}_l \in \mathbb{C}$ for $l = 1, \ldots, k$. One of $\tilde{\mu}_l$, $l = 1, \ldots, k$, is not zero at least since $\tilde{\mathrm{G}}$ would not be of type (vi) by Theorem 3.2.2 if it is not true. We assume that $\tilde{\mu}_j \neq 0$. So the cyclic group M_j generated by M_j becomes

$$\left\{ m_j^l \begin{pmatrix} \tilde{\lambda}_j^l & 0 \\ \tilde{\mu}_j p_l(\tilde{\lambda}_j) & \tilde{\lambda}_j^{-l} \end{pmatrix} \middle| \, l \in \mathbb{Z} \right\}$$

and infinite, where

$$p_l(\lambda) = \frac{\lambda^l - \lambda^{-l}}{\lambda - \lambda^{-1}}.$$

We see that M_j is infinite since $p_l(\lambda) = 0$ if and only if $l = 0$. In particular, the nondiagonal entries of its elements, $\tilde{\mu}_j m_j^l p_l(\tilde{\lambda}_j)$, $l \in \mathbb{Z}$, take an infinite number of values. Moreover, it follows from Theorem 3.2.2 that

$$\mathrm{G}^0 \subset \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix} \middle| \, \lambda, \lambda' \in \mathbb{C}^* \right\} \quad \text{or} \quad \left\{ \begin{pmatrix} \lambda & 0 \\ \mu & \lambda \end{pmatrix} \middle| \, \lambda \in \mathbb{C}^*, \mu \in \mathbb{C} \right\}$$

since it is commutative. If the former is true, then G^0 is a subgroup of finite index in $\mathrm{G} \supset \mathrm{M}_j$ (see, e.g., Proposition 3.2.1 of [23]), which contradicts to the fact that the nondiagonal entries of elements of M_j take infinitely many values. If the latter is true, then we also have a contradiction since the ratios between two diagonal entries for elements of M_j, $\tilde{\lambda}_j^{2l}$, $l \in \mathbb{Z}$, also take infinitely many values and G^0 is a subgroup of finite index in $\mathrm{G} \supset \mathrm{M}_j$. Hence, $\tilde{\mathrm{G}}$ is not of type (vi). Thus, we complete the proof. □

3.4 Main result

In this section we give our main result. We consider (3.1.1) for $n \geq 3$ and make the following assumptions (see Fig. 3.1):

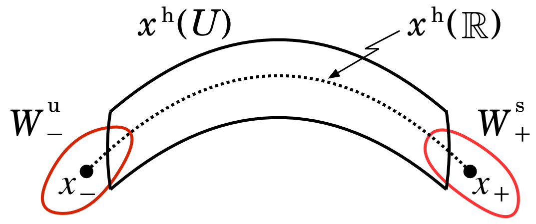

Figure 3.2: Riemann surface $\Gamma = \hat{x}(U) \cup \tilde{\mathscr{C}}_+ \cup \tilde{\mathscr{C}}_-$.

(A1) There exists an $(n-2)$ dimensional analytic locally invariant manifold with boundary, $\mathscr{N}$, such that its interior $\mathscr{N} \setminus \partial\mathscr{N}$ contains a pair of equilibria $x_\pm$ and a heteroclinic orbit $\hat{x}(t)$ with $\lim_{t\to\pm\infty} \hat{x}(t) = x_\pm$.

(A2) There exist one-dimensional analytic locally invariant manifolds with boundaries, $\mathscr{C}_\pm$, such that $x_\pm \in \mathscr{C}_\pm \setminus \partial\mathscr{C}_\pm$ and $\hat{x}(t) \in \mathscr{C}_\pm$ for $|t| \geq T$ with $T > 0$ sufficiently large.

When $n = 3$, assumption (A2) immediately follows from (A1) with $\mathscr{C}_\pm = \mathscr{N}$. We also allow that $x_+ = x_-$ and $\hat{x}(t)$ is a homoclinic orbit.

Let $\lambda_\pm$, $\mu_\pm$ and $\nu_\pm$ be eigenvalues of $Df(x_\pm)$ such that the eigenvectors associated with $\lambda_\pm$ belong to the one-dimensional tangent spaces $T_{x_\pm}\mathscr{C}_\pm$ and the eigenvectors associated with $\mu_\pm$ and $\nu_\pm$ belong to the two-dimensional complementary spaces of $T_{x_\pm}\mathscr{N}$. From (A1) and (A2) we see that $\lambda_\pm \in \mathbb{R}$. We also assume some of the following:

(A3) $\lambda_\pm \neq 0$ and $\dfrac{\mu_\pm - \nu_\pm}{\lambda_\pm} \notin \mathbb{Q}$;

(A4) $\dfrac{\mu_\pm + \nu_\pm}{\lambda_\pm} \in \mathbb{Z}$;

(A5) $\dfrac{\mu_+}{\lambda+} - \dfrac{\mu_-}{\lambda_-} \in \mathbb{Z}$.

Consider the complexification of (3.1.1) which is defined in a neighborhood of $\mathbb{R}^n$. Suppose that (A1)-(A3) hold. Let $\tilde{\mathscr{C}}_\pm$ and $\tilde{\mathscr{N}}$ be the complexifications of $\mathscr{C}_\pm$ and $\mathscr{N}$, respectively, such that $\tilde{\mathscr{C}}_\pm$ contain no other equilibria than $x_\pm$. Let $R > 0$ be sufficiently large and let U be a neighborhood of $(-R, R)$ in $\mathbb{C}$ such that $\hat{x}(U)$ contains no equilibrium and $\tilde{\mathscr{C}}_\pm \cap \hat{x}(U) \neq \emptyset$. Obviously, $\hat{x}(U)$ is a one-dimensional complex manifold with boundary. We take $\Gamma = \hat{x}(U) \cup \tilde{\mathscr{C}}_+ \cup \tilde{\mathscr{C}}_-$ and the inclusion map as immersion $i : \Gamma \to \mathbb{C}^n$. See Fig. 3.2. If $x_+ = x_-$ and $\hat{x}(t)$ is a homoclinic orbit, then small modifications are needed in the definitions of Γ and i. Let $0_\pm \in \Gamma$ denote points corresponding to the equilibria $x_\pm$.

We now define the VE and NVE of (3.1.1) along Γ. See Appendix A.1 for a more general treatment. Since $i(\Gamma)$ is locally invariant, the vector field f can be written as

$$f|_{i(\Gamma)} = h(s)\frac{d}{ds},$$

where s is a local coordinate on Γ and $h(s)$ is a holomorphic function. Let $\hat{A}(s) = Df(i(s))$ for $s \in \Gamma$. We take t and $s_\pm$ with $s_\pm = 0$ at $0_\pm$ as the local coordinates on $\hat{x}(U)$ and

$\tilde{\mathscr{C}}_\pm$, respectively. Here t is the original independent variable of (3.1.1). The VE of (3.1.1) along Γ is given by

$$\frac{d\xi}{dt} = \hat{A}(t)\xi, \quad \xi \in \mathbb{C}^n, \tag{3.4.1}$$

on $\hat{x}(U)$ and

$$h_\pm(s_\pm)\frac{d\xi}{ds_\pm} = \hat{A}(s_\pm)\xi, \quad \xi \in \mathbb{C}^n, \tag{3.4.2}$$

on $\tilde{\mathscr{C}}_\pm$, where $h_\pm(s_\pm)$ are holomorphic and satisfy $h_\pm(0_\pm) = 0$. Since $\tilde{\mathscr{N}}$ is of dimension $n-2$, there exist local coordinates $(y_1, \dots, y_n) = T(x) \in \mathbb{C}^n$ for $x \in \tilde{\mathscr{N}}$ such that $\tilde{\mathscr{N}}$ is represented by $y_{n-1} = y_n = 0$ and $T(x)$ is holomorphic on $\tilde{\mathscr{N}}$. Using the coordinates, we have

$$D\hat{f}(i(s)) = \begin{pmatrix} A_{11}(s) & A_{12}(s) \\ 0 & A_{22}(s) \end{pmatrix},$$

where $\hat{f}(y) = DT(T^{-1}(y))f(T^{-1}(y))$. So the components of the VE given by (3.4.1) and (3.4.2) normal to $\tilde{\mathscr{N}}$, i.e., the NVE of (3.1.1) to $\tilde{\mathscr{N}}$ along Γ, are written as

$$\frac{d\eta}{ds} = A(s)\eta, \quad \eta \in \mathbb{C}^2,$$

on $\hat{x}(U)$ and

$$h_\pm(s_\pm)\frac{d\eta}{ds_\pm} = A_\pm(s_\pm)\eta, \quad \eta \in \mathbb{C}^2, \tag{3.4.3}$$

on $\tilde{\mathscr{C}}_\pm$, where $A(t) = A_{22}(t)$ on $\hat{x}(U)$ and $A_\pm(s_\pm) = A_{22}(s_\pm)$ on $\tilde{\mathscr{C}}_\pm$. These VE and NVE of (3.1.1) along Γ, which are the same as those given by (A.1.3) and (A.1.7) in Appendix A.1 for $\mathscr{M} = \mathscr{N}$ and $m = 2$ (cf. Equations (A.1.4), (A.1.5), (A.1.8) and (A.1.9)), have only two singular points $0_\pm$ since there is no other equilibrium than $x_\pm$ on $i(\Gamma)$.

Lemma 3.4.1. *The NVE of* (3.1.1) *along* Γ *is Fuchsian with regular singularities at* $0_\pm$. *Moreover the eigenvalues of monodromy matrices* $M_\pm$ *around* $0_\pm$ *are given by* $e^{2\pi i\mu_\pm/\lambda_\pm}$ *and* $e^{2\pi i\nu_\pm/\lambda_\pm}$.

Proof. By (A1) and (A2), $h_\pm(s_\pm)$ in (3.4.3) are written as $h_\pm(s_\pm) = \lambda_\pm s_\pm + O(|s_\pm|^2)$ with $\lambda_\pm \neq 0$. Recall that $\lambda_\pm$ are eigenvalues of $Df(x_\pm)$ such that the associated eigenvectors belong to $T_{x_\pm}\mathscr{C}_\pm$. Thus, the singular points $0_\pm \in \Gamma$ are regular and the NVE is Fuchsian. Since the eigenvalues of $A_\pm(0)$ are $\mu_\pm$ and $\nu_\pm$, the characteristic exponents of (3.4.3) are given by $\mu_\pm/\lambda_\pm$ and $\nu_\pm/\lambda_\pm$ and their difference is not an integer by (A3). Hence, we compute the local monodromy matrices of (3.4.3) around $s_\pm = 0$ as

$$\exp\left(\frac{2\pi i}{\lambda_\pm}A_\pm(0)\right),$$

which means that the eigenvalues of $M_\pm$ are given by $e^{2\pi i\mu_\pm/\lambda_\pm}$ and $e^{2\pi i\nu_\pm/\lambda_\pm}$. □

Let M be the monodromy group of the NVE of (3.1.1) along Γ, which is generated by two elements $M_\pm$. Now we state our main result.

Theorem 3.4.2. *Suppose that (A1) - (A3) hold and equation* (3.1.1) *is real-meromorphically integrable. Then the monodromy matrices* $M_\pm$ *are simultaneously diagonalizable, i.e.,* M *is diagonalizable, and their eigenvalues are not roots of* 1*. Moreover, if (A4) and (A5) hold, then*

$$M_+ = M_-^{-1} \quad \text{or} \quad M_+ = M_-, \tag{3.4.4}$$

so that M *is infinitely cyclic.*

Proof. To prove this theorem, we use the extension of the Morales-Ramis theory due to Ayoul and Zung [6], which is briefly reviewed in Appendix A.2 (especially Corollary A.2.2).

Suppose that (A1)-(A3) hold and equation (3.1.1) is real-meromorphically integrable. Then the complexification of (3.1.1) is meromorphically integrable on $i(\Gamma)$. Let G be the differential Galois group of the NVE of (3.1.1) along Γ. From Corollary A.2.2 we see that the connected identity component G^0 of G is commutative. By Lemma 3.4.1 the monodromy matrices $M_\pm$ have $e^{2\pi i(\mu_\pm/\lambda_\pm)}$ and $e^{2\pi i(\nu_\pm/\lambda_\pm)}$ as their eigenvalues, and by (A3) $\mu_\pm/\lambda_\pm - \nu_\pm/\lambda_\pm = (\mu_\pm - \nu_\pm)/\lambda_\pm \notin \mathbb{Q}$, which means that the eigenvalues of $M_\pm$ are not roots of 1. Using Proposition 3.3.2, we obtain the first part.

We turn to the second part. Suppose additionally that (A4) and (A5) hold. Then by (A4)

$$\det M_\pm = e^{2\pi i(\mu_\pm/\lambda_\pm)} \cdot e^{2\pi i(\nu_\pm/\lambda_\pm)} = e^{2\pi(\mu_\pm+\nu_\pm)/\lambda_\pm} = 1.$$

Thus, $\mathrm{M} \subset \mathrm{SL}(2,\mathbb{C})$, so that $\mathrm{G} \subset \mathrm{SL}(2,\mathbb{C})$ by Theorem 3.2.2. Since G has an element whose eigenvalues are not roots of 1 and G^0 is commutative, G is of type (iv) in Lemma 3.3.1, as in the proof of Proposition 3.3.2. Hence, we can write

$$M_+ = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \quad M_- = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix},$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$ are some constants which are not roots of 1. Since $e^{2\pi i(\mu_+/\lambda_+)} = e^{2\pi i(\mu_-/\lambda_-)}$ by (A5), M_+ and M_- have common eigenvalues. This means that $\lambda_1 = \lambda_2$ or $\lambda_1 = \lambda_2^{-1}$, i.e., the second part holds. □

Remark 3.4.1. Consider the situation in which equation (3.1.1) is a two-degree-of-freedom Hamiltonian system and has a saddle-center equilibrium with a homoclinic orbit, as in [64, 91]. So we take $\lambda_- = -\lambda_+ = \lambda$, $\mu_\pm = \pm i\omega$ and $\nu_\pm = \mp i\omega$ with $\lambda, \omega > 0$, so that

$$\frac{\mu_\pm - \nu_\pm}{\lambda_\pm} = \mp\frac{2i\omega}{\lambda} \notin \mathbb{Q}, \quad \frac{\mu_\pm + \nu_\pm}{\lambda_\pm} = 0, \quad \frac{\mu_+}{\lambda_+} - \frac{\mu_-}{\lambda_-} = 0.$$

Hence, we can apply Theorem 3.4.2 to conclude that condition (3.4.4) holds if equation (3.1.1) is real-meromorphically integrable in the meaning of Liuoville. The second case of (3.4.4) was overlooked in [64, 91]. Theorem 3.4.2 also says that for a large class of integrable systems which may be non-Hamiltonian and have homo- or heteroclinic orbits to equilibria which may not be saddle-centers, the same condition (3.4.4) holds.

3.5 Application

We now demonstrate the theoretical result of Section 3.4 for the three-dimensional system (3.1.2). We first see that the x_2-axis $\{(0, x_2, 0) | x_2 \in \mathbb{R}\}$ is invariant under the flow of

(3.1.2), and that it contains equilibria at $x_\pm = (0, \mp 1, 0)$ and a heteroclinic orbit to them,

$$\hat{x}(t) = (0, -\tanh 3t, 0).$$

Thus, (A1) and (A2) hold with $\mathscr{N} = \mathscr{C}_\pm = \{(0, x_2, 0) | -2 \le x_2 \le 2\}$. Moreover, the eigenvalues of $Df(x_\pm)$ are computed as

$$\lambda_\pm = \mp 6, \quad \mu_\pm = \pm 3 - \sqrt{25 - \alpha(\alpha \mp \beta)}, \quad \nu_\pm = \pm 3 + \sqrt{25 - \alpha(\alpha \mp \beta)}.$$

Hence, (A3) is equivalent to

$$\frac{\sqrt{25 - \alpha(\alpha \pm \beta)}}{3} \notin \mathbb{Q} \tag{3.5.1}$$

since $\lambda_\pm = \mp 6$. We also note that (A4) holds but (A5) does not necessarily since

$$\frac{\mu_\pm + \nu_\pm}{\lambda_\pm} = -1, \quad \frac{\mu_+}{\lambda_+} - \frac{\mu_-}{\lambda_-} = \frac{\sqrt{25 - \alpha(\alpha - \beta)} + \sqrt{25 - \alpha(\alpha + \beta)}}{6}.$$

The formal NVE along $\hat{x}(t)$ is given by

$$\dot{\eta} = \begin{pmatrix} 8 \tanh 3t & \alpha \\ -\alpha + \beta \tanh 3t & -2 \tanh 3t \end{pmatrix} \eta, \quad \eta \in \mathbb{C}^2,$$

which is transformed to the NVE on the (trivial) Riemann surface $\mathbb{C}$,

$$\frac{d\eta}{ds} = \frac{1}{3(s^2 - 1)} \begin{pmatrix} -8s & \alpha \\ -\alpha - \beta s & 2s \end{pmatrix} \eta, \tag{3.5.2}$$

by the transformation $s = -\tanh 3t$. Letting $\zeta = \eta_1$, we rewrite (3.5.2) as

$$\frac{d^2\zeta}{ds^2} + \frac{4s}{s^2 - 1} \frac{d\zeta}{ds} + \frac{\alpha(\alpha + \beta s) + 8s^2 - 24}{9(s^2 - 1)^2} \zeta = 0, \tag{3.5.3}$$

which has regular singularities at $s = \pm 1, \infty$ on the Riemann sphere $\mathbb{P}^1$. Let $M_{\pm 1}$ be the monodromy matrices of (3.5.3) around $s = \pm 1$. The monodromy matrices $M_\pm$ of (3.5.2) aroud $s = \mp 1$ are simultaneously similar to $M_{\mp 1}$.

We compute the characteristic exponents of (3.5.3) at $s = -1, 1$ and ∞ as (ϕ_+, ϕ_-), (ψ_+, ψ_-) and $(1/3, 8/3)$, respectively, where

$$\phi_\pm = -\frac{1}{2} \pm \frac{1}{6}\sqrt{25 - \alpha(\alpha - \beta)}, \quad \psi_\pm = -\frac{1}{2} \mp \frac{1}{6}\sqrt{25 - \alpha(\alpha + \beta)}.$$

Solutions of (3.5.3) are expressed by a Riemann P function (see, e.g.,[41]) as

$$P \begin{Bmatrix} -1 & 1 & \infty & \\ \phi_- & \psi_- & \frac{1}{3} & s \\ \phi_+ & \psi_+ & \frac{8}{3} & \end{Bmatrix},$$

which are transformed to

$$P \begin{Bmatrix} 0 & 1 & \infty & \\ \phi_- & \psi_- & \frac{1}{3} & z \\ \phi_+ & \psi_+ & \frac{8}{3} & \end{Bmatrix}$$

by the transformation $z=(s+1)/2$. Using the identity

$$
\begin{aligned}
& z^{-\phi_-}(z-1)^{-\psi_-} P\begin{Bmatrix} 0 & 1 & \infty & \\ \phi_- & \psi_- & \frac{1}{3} & z \\ \phi_+ & \psi_+ & \frac{8}{3} & \end{Bmatrix} \\
& \quad = P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \frac{1}{3}+\phi_-+\psi_- & z \\ \phi_+-\phi_- & \psi_+-\psi_- & \frac{8}{3}+\phi_-+\psi_- & \end{Bmatrix},
\end{aligned}
$$

we rerwite (3.5.3) as the hypergeometric equation

$$
\frac{d^2\zeta}{dz^2}+\frac{(a+b+1)z-c}{z(z-1)}\frac{d\zeta}{dz}+\frac{ab}{z(z-1)}\zeta=0, \tag{3.5.4}
$$

where

$$
a=\frac{1}{3}+\phi_-+\psi_-, \quad b=\frac{8}{3}+\phi_-+\psi_-, \quad 1-c=\phi_+-\phi_-.
$$

Hence, $M_{-1}=e(\phi_-)M_0$ and $M_{+1}=e(\psi_-)M_1$, where M_0, M_1 are monodromy matrices of (3.5.4) at $z=0,1$ and $e(a)=e^{2\pi ia}$. Thus, M_{-1} and M_{+1} are simultaneously diagonalizable if and only if so are M_0 and M_1. We have the following lemma on monodromy matrices of the hypergeometric equation (3.5.4).

Lemma 3.5.1. *If $c, c-a-b\notin\mathbb{Z}$, then M_0 and M_1 are not simultaneously diagonalizable.*

Proof. Assume that $c, c-a-b\notin\mathbb{Z}$. By Theorem 4.7.2 of Chapter 2 of [41], we obtain

$$
\begin{aligned}
M_0 &= \begin{pmatrix} 1 & 0 \\ 0 & e(-c) \end{pmatrix}, \\
M_1 &= \frac{1}{\ell_{11}\ell_{22}-\ell_{12}\ell_{21}}\begin{pmatrix} \ell_{11}\ell_{22}-\ell_{12}\ell_{21}e(c-a-b) & \ell_{12}\ell_{22}(e(c-a-b)-1) \\ \ell_{11}\ell_{21}(1-e(c-a-b)) & \ell_{11}\ell_{22}e(c-a-b)-\ell_{12}\ell_{21} \end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
\ell_{11} &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \ell_{12}=\frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}, \\
\ell_{21} &= \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \quad \ell_{22}=\frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}.
\end{aligned}
$$

Here $\Gamma(a)$ represents the gamma function. If M_0 and M_1 are simultaneously diagonalizable, then $\ell_{11}\ell_{21}=0$ and $\ell_{12}\ell_{22}=0$ because $e(c), e(c-a-b)\neq 1$. Note that $1/\Gamma(x)=0$ if and only if $x\in\mathbb{Z}$ and $x\leq 0$. Easy calculations yield that $\ell_{11}\ell_{21}\neq 0$ or $\ell_{12}\ell_{22}\neq 0$ under the assumptions. □

Applying Theorem 3.4.2, we obtain the following result.

Proposition 3.5.2. *If condition* (3.5.1) *holds, then the system* (3.1.2) *is not real-meromorphically integrable.*

Proof. Assume that condition (3.5.1) holds. It follows from the relations

$$\phi_+ = \frac{\mu_+}{\lambda_+}, \quad \phi_- = \frac{\nu_+}{\lambda_+}, \quad \psi_+ = \frac{\mu_-}{\lambda_-}, \quad \psi_- = \frac{\nu_-}{\lambda_-}$$

that

$$\phi_+ - \phi_- \notin \mathbb{Q}, \quad \psi_+ - \psi_- \notin \mathbb{Q}.$$

Hence, we use Fuchs' relation $\phi_+ + \psi_+ + \phi_- + \psi_- + 3 = 1$ to show that

$$c = 1 - (\phi_+ - \phi_-) \notin \mathbb{Z}, \quad c - a - b = \psi_+ - \psi_- \notin \mathbb{Z}.$$

This means that M_{-1} and M_{+1} are not simultaneously diagonalizable since so are M_0 and M_1 by Lemma 3.5.1. Applying Theorem 3.4.2, we complete the proof. □

Condition (3.5.1) holds when $\alpha = 1$ and $\beta \in \{2\sqrt{23}, 8\sqrt{5}, 16\sqrt{2}\}$ or when $\beta = 0$ and $\alpha^2 \neq 24 - 9q^2$ for some $q \in \mathbb{Q}$. The former case was excluded in [71] and the latter case was studied in [82, 69]. Proposition 3.5.2 says that the system (3.1.2) is real-meromorphically non-integrable in both the cases.

Chapter 4

Heteroclinic orbits and nonintegrability in two-degree-of-freedom Hamiltonian systems with saddle-centers

4.1 Introduction

Chaotic dynamics and nonintegrability of Hamiltonian systems are classical and fundamental topics in dynamical systems, as seen in the famous work of Poincaré [73], and they have attracted much attention [48, 59, 62, 68, 76]. A Hamiltonian system is nonintegrable if it exhibits chaotic dynamics (see, e.g., [68]), but the converse is not always true: it may not exhibit chaotic dynamics even if it is nonintegrable. Chaotic dynamics is also very often closely related to the existence of transverse homo- and heteroclnic orbits. For example, if there exist transverse homoclinic orbits to periodic orbits, then a Poincaré map appropriately defined is topologically conjugated to a horseshoe map, which has an invariant set consisting of orbits characterized by the Bernoulli shift, i.e., chaotic dynamics occurs [35, 68, 88]. Morales-Ruiz and Peris [64] and Yagasaki [91] discussed a relationship between nonintegrability and chaos for a class of two-degree-of-freedom Hamiltonian systems with saddle centers having homoclinic orbits. They showed that if a sufficient condition for nonintegrability holds, then there exist transverse homoclnic orbits to periodic orbits. Here we extend their results to a similar class of Hamiltonian systems with saddle centers connected by heteroclinic orbits.

More concretely, we consider two-degree-of-freedom Hamiltonian systems of the form

$$\dot{x} = J\mathrm{D}_x H(x, y), \qquad \dot{y} = J\mathrm{D}_y H(x, y), \qquad (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2, \tag{4.1.1}$$

where $H\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is analytic and J represents the 2×2 symplectic matrix,

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We make the following assumptions.

(B1) The x-plane, $\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid y = 0\}$, is invariant under the flow of (4.1.1), i.e., $\mathrm{D}_y H(x, 0) = 0$ for any $x \in \mathbb{R}^2$.

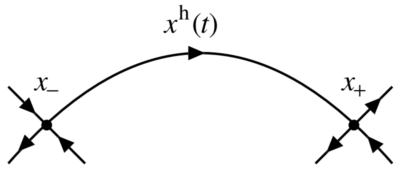

Figure 4.1: Assumptions (B2) and (B3).

(B2) There exist two saddle-centers at $(x, y) = (x_\pm, 0)$ on the x-plane such that the matrix $J\mathrm{D}_x^2 H(x_\pm, 0)$ has a pair of real eigenvalues $\lambda_\pm$, $-\lambda_\pm$ and the matrix $J\mathrm{D}_y^2 H(x_\pm, 0)$ has a pair of purely imaginary eigenvalues $\mathrm{i}\omega_\pm$, $-\mathrm{i}\omega_\pm$ $(\lambda_\pm, \omega_\pm > 0)$, where the upper and lower signs in the subscripts are taken simultaneously.

Assumption (B2) implies that there exist one-parameter families of periodic orbits near the saddle-centers $(x_\pm, 0)$ by the Lyapunov center theorem (see, e.g., [59]). In addition, the system restricted on the x-plane,

$$\dot{x} = J\mathrm{D}_x H(x, 0), \tag{4.1.2}$$

has saddles at $x = x_\pm$. The reader may think that assumption (B1) is too restrictive but quite a few important Hamiltonian systems satisfy this assumption. See, e.g., [75, 90] for such examples.

(B3) The two saddles $x = x_\pm$ are connected by a heteroclinic orbit $x^{\mathrm{h}}(t)$ in (4.1.2), as shown in Fig. 4.1.

In (B3), if $x_- = x_+$, then $x^{\mathrm{h}}(t)$ becomes a homoclinic orbit.

In [75] a Melnikov-type technique (see, e.g., [35, 58] for its original version) was developed for (4.1.1) to detect the existence of transverse heteroclinic orbits connecting periodic orbits near the saddle-centers $(x, y) = (x_\pm, 0)$, when $H(x, y)$ is only C^{r+1} $(r \geq 2)$. The Melnikov function was defined in terms of a fundamental matrix to the normal variational equation (NVE) along the heteroclinic orbit $(x, y) = \big(x^{\mathrm{h}}(t), 0\big)$,

$$\dot{\eta} = J\mathrm{D}_y^2 H\left(x^{\mathrm{h}}(t), 0\right)\eta, \qquad \eta \in \mathbb{R}^2, \tag{4.1.3}$$

and such transverse heteroclinic orbits were detected if it has a simple zero. See Section 4.2.1 for more details. This is an extension of a technique developed in [89], which enables us to show that there exist transverse homoclinic orbits to such periodic orbits and chaotic dynamics occurs [35, 88], when $x_- = x_+$ and $x^{\mathrm{h}}(t)$ becomes a homoclinic orbit. Moreover, if there exist transverse heteroclinic orbits from periodic orbits near $(x_-, 0)$ to those near $(x_+, 0)$ and vice versa, i.e., transverse heteroclinic cycles between the periodic orbits, then so do transverse homoclinic orbits to those near $(x_+, 0)$ and $(x_-, 0)$, so that the Hamiltonian system (4.1.1) exhibits chaotic dynamics and is nonintegrable. We also point out that Grotta Ragazzo [34] obtained a concrete sufficient condition for the occurrence of chaotic dynamics in a special class of (4.1.1) with $x_- = x_+$, using a general result of [50], a little earlier.

On the other hand, Morales-Ruiz and Ramis [65] presented a sufficient condition for meromorphic nonintegrability of general complex Hamiltonian systems. Their theory, which is now called the Morales-Ramis theory, states that complex Hamiltonian systems are meromorphically nonintegrable if the identity components of the differential Galois groups [23, 81] for their variational equations (VEs) or NVEs around particular nonconstant solutions such as periodic, homoclinic and heteroclinic orbits are not commutative. See also [62]. Ayoul and Zung [6] used a simple trick called the *cotangent lifting* to show that the Morales-Ramis theory is also valid for detection of meromorphic nonintegrability of non-Hamiltonian systems in the meaning of Bogoyavlenskijj [9]. Moreover, Morales-Ruiz and Peris [64] studied a special class of (4.1.1) with $x_- = x_+$ and showed that if the Hamiltonian system (4.1.1) is determined by the Morales-Ramis theory to be real-meromorphically nonintegrable, then chaotic dynamics occurs, using the results of [34]. See also [62]. Their result was extended to (4.1.1) with $x_- = x_+$ in [91], based on the result of [89]. A further extension on sufficient conditions for real-meromorphic nonintegrability to general dynamical systems having homo- or heteroclinic orbits was accomplished in Chapter 3. See Section 4.2.2 for more details.

In this chapter, based on [75] and Chapter 3, we extend the results of [64, 91] and show the following for (4.1.1) under assumptions (B1)–(B3).

- Assume that $\omega_+ = \omega_-$. If sufficient conditions obtained in Chapter 3 for real-meromorphic nonintegrability near the heteroclinic orbit hold, then the stable and unstable manifolds of periodic orbits on the same Hamiltonian energy surface near the saddle-centers $(x_\pm, 0)$ intersect transversely, i.e., there exist transverse heteroclinic orbits connecting the periodic orbits.
- Assume that $\omega_+ \neq \omega_-$. Then these manifolds intersect transversely, have quadratic tangencies or do not intersect whether the sufficient conditions hold or not. Moreover, under an additional condition, if the sufficient condition does not hold, i.e., a necessary condition for real-meromorphic integrability holds, then these manifolds do not intersect. This may be surprising for the reader since they do not coincide even if the Hamiltonian systems are integrable.

Here the associated Hessian matrices of the Hamiltonian are assumed to have the same number of positive eigenvalues: otherwise there exist no periodic orbits near $(x_\pm, 0)$ on the same energy surface, as shown in Proposition 4.3.1 below. Our theory is illustrated for a system with quartic single-well potential and some numerical results by using the computer software `AUTO` [24] are given to support the theoretical results.

The above results are remarkable since a relationship between the existence of transverse heteroclinic orbits and nonintegrability for Hamiltonian systems, both of which are important properties of dynamical systems, is addressed for the first time, to the authors' knowledge. If not only transverse heteroclinic orbits but also heteroclinic cycles exist, then chaotic dynamics occurs (see the last paragraph of Section 4.2.1), so that the Hamiltonian systems are nonintegrable. However, if transverse heteroclinic orbits exist but heteroclinic cycles are not formed, then chaotic dynamics may not occur and it is not clear that the systems are nonintegrable. See, e.g., an example in [95, Section 1.1.2]. We remark that in different settings the non-existence of first integrals when transverse heteroclinic orbits to hyperbolic periodic orbits exist was discussed in [26, 95]. Moreover, transverse heteroclinic orbits may not exist even if the systems are nonintegrable. Thus, our problem is more subtle, so that our conclusions are more complicated as stated above, compared with the previous one discussed for homoclinic orbits in [64, 91].

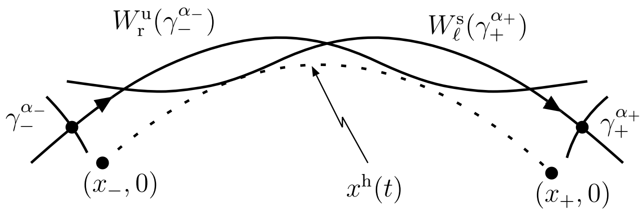

Figure 4.2: The right branch of the unstable manifold of $\gamma_-^{\alpha_-}$ and the left branch of the stable manifold of $\gamma_+^{\alpha_+}$, denoted by $W_{\mathrm{r}}^{\mathrm{u}}(\gamma_-^{\alpha_-})$ and $W_\ell^{\mathrm{s}}(\gamma_+^{\alpha_+})$, on a Poincaré section.

The outline of this chapter is as follows. In Section 4.2 we briefly review the previous results of [75] and Chapter 3 on the existence of transverse heteroclinic orbits to periodic orbits near $(x_\pm,0)$ and on necessary conditions for real-meromorphic integrability, i.e., sufficient conditions for real-meromorphic nonintegrability. We state the main theorems and prove them in Section 4.3, and give the example stated above along with numerical results in Section 4.4.

4.2 Previous results

4.2.1 Melnikov-type technique

We first review the result of [75] for the existence of transverse heteroclinic orbits in (4.1.1).

Suppose that assumptions (B1)–(B3) hold. As stated in Section 4.1, near the saddle-centers $(x_\pm,0)$, there exist one-parameter families of periodic orbits, which are denoted by $\gamma_\pm^{\alpha_\pm}, \alpha_\pm\in(0,\bar{\alpha}_\pm]$, with $\bar{\alpha}_\pm>0$. As $\alpha_\pm\to0$, they approach $(x_\pm,0)$ and their periods approach $2\pi/\omega_\pm$. Let $W_{\mathrm{r}}^{\mathrm{u}}(\gamma_-^{\alpha_-})$ (resp. $W_\ell^{\mathrm{s}}(\gamma_+^{\alpha_+})$) denote the right branch of the unstable manifold of $\gamma_-^{\alpha_-}$ (resp. the left branch of the stable manifold of $\gamma_+^{\alpha_+}$) near the heteroclinic orbit $\big(x^{\mathrm{h}}(t),0\big)$. See Fig. 4.2.

Let $\Psi(t)$ denote the fundamental matrix of the NVE (4.1.3) along $\big(x^{\mathrm{h}}(t),0\big)$. Let $\Phi_\pm(t)$ be the fundamental matrices of the NVEs around the saddle-centers $(x_\pm,0)$,

$$\dot{\eta}=J\mathrm{D}_y^2H(x_\pm,0)\eta, \tag{4.2.1}$$

with $\Phi_\pm(0)=\mathrm{id}_2$, where id_2 represents the 2×2 identity matrix. We easily show that the limits

$$B_-=\lim_{t\to-\infty}\Phi_-(-t)\Psi(t),\qquad B_+=\lim_{t\to+\infty}\Phi_+(-t)\Psi(t) \tag{4.2.2}$$

exist (cf. [89, Lemma 3.1]) and set $B_0=B_+B_-^{-1}$. We define the *Melnikov function* $M(t_0)$ as

$$M(t_0)=m_-(\eta_0)-m_+(B_0\Phi_-(t_0)\eta_0), \tag{4.2.3}$$

where $\eta_0\in\mathbb{R}^2$ with $|\eta_0|=1$ and

$$m_\pm(\eta)=\frac{1}{2}\eta\cdot\mathrm{D}_y^2H(x_\pm,0)\eta. \tag{4.2.4}$$

We have the following theorem (see [75, Appendix A] for the proof).

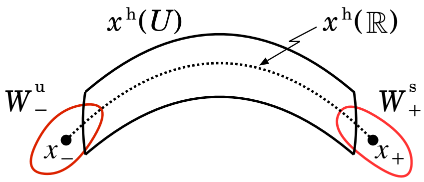

Figure 4.3: Riemann surface $\Gamma = x^{\rm h}(U) \cup W_+^{\rm s} \cup W_-^{\rm u}$.

Theorem 4.2.1. *For some $\alpha_\pm \in (0, \bar{\alpha}_\pm]$, let $\gamma_\pm^{\alpha_\pm}$ be periodic orbits sufficiently close to $(x_\pm, 0)$ on the same energy surface. Suppose that $M(t_0)$ has a simple zero. Then the right branch of the unstable manifold $W_{\rm r}^{\rm u}(\gamma_-^{\alpha_-})$ and the left branch of the stable manifold $W_\ell^{\rm s}(\gamma_+^{\alpha_+})$ intersect transversely on the energy surface, i.e., there exist transverse heteroclinic orbits from $\gamma_-^{\alpha_-}$ to $\gamma_+^{\alpha_+}$.*

Remark 4.2.1. Theorem 4.2.1 is also valid when $x_+ = x_-$. In this situation, if $M(t_0)$ has a simple zero, then the stable and unstable manifolds of periodic orbits near the corresponding saddle-center intersect transversely on the energy surface, i.e., there exist transverse homoclinic orbits to the periodic orbits and consequently chaotic dynamics occurs (e.g., [35, 88]). See also [89].

Suppose that there also exists a heteroclinic orbit $\hat{x}^{\rm h}(t)$ from x_+ to x_- on the x-plane and that the hypothesis of Theorem 4.2.1 holds for both of $x^{\rm h}(t)$ and $\hat{x}^{\rm h}(t)$. Then the unstable manifolds of $\gamma_\mp^{\alpha_\mp}$ intersect the stable manifolds of $\gamma_\pm^{\alpha_\pm}$ transversely on the energy surface and these manifolds form a heteroclinic cycle. This implies that there exist transverse homoclinic orbits to $\gamma_\pm$ (see, e.g., [88, Section 26.1]), so that chaotic dynamics occurs in (4.1.1).

4.2.2 Necessary conditions for integrability

We next briefly describe the result of Chapter 3 for integrability of (4.1.1) in our setting.

Suppose that (B1)–(B3) hold. Let $\Gamma_\mathbb{R} = \big\{\big(x^{\rm h}(t), 0\big) \in \mathbb{R}^2 \times \mathbb{R}^2 \,|\, t \in \mathbb{R}\big\} \cup \{(x_\pm, 0)\}$. Consider the complexification of (4.1.1) in a neighborhood of $\Gamma_\mathbb{R}$ in $\mathbb{C}^4$. Let $W_\pm^{\rm s,u}$ be the one-dimensional local holomorphic stable and unstable manifolds of $(x_\pm, 0)$ on the x-plane. See [36] for the existence of such holomorphic stable and unstable manifolds. Let $R > 0$ be sufficiently large and let U be a neighborhood of the open interval $(-R, R) \subset \mathbb{R}$ in $\mathbb{C}$ such that $x^{\rm h}(U)$ contains no equilibrium and intersects both $W_+^{\rm s}$ and $W_-^{\rm u}$. Here for simplicity we have identified $x^{\rm h}(U) \subset \mathbb{C}^2$ with $x^{\rm h}(U) \times \{0\}$ in $\mathbb{C}^4$. Obviously, $x^{\rm h}(U)$ is a one-dimensional complex manifold with boundary. We take $\Gamma = x^{\rm h}(U) \cup W_+^{\rm s} \cup W_-^{\rm u}$ and the inclusion map as immersion $i\colon \Gamma \to \mathbb{C}^4$. See Fig. 4.3. If $x_+ = x_-$ and $x^{\rm h}(t)$ is a homoclinic orbit, then small modifications are needed in the definitions of Γ and i. Let $0_\pm \in \Gamma$ denote points corresponding to the equilibria $x_\pm$. Taking three charts, $W_\pm^{\rm s,u}$ and $x^{\rm h}(U)$, we rewrite the NVE (4.1.3) along Γ as follows (see Section 3.4 for the details).

In $x^{\rm h}(U)$ we use the complex variable $t \in U$ as the coordinate and rewrite the NVE (4.1.3) as

$$\frac{{\rm d}\eta}{{\rm d}t} = J{\rm D}_y^2 H(i(t))\eta, \tag{4.2.5}$$

which has no singularity there. In W_+^{s} and W_-^{u} there exist local coordinates s_+ and s_-, respectively, such that $s_\pm(0_\pm)=0$ and $\mathrm{d}/\mathrm{d}t=h_\pm(s_\pm)\mathrm{d}/\mathrm{d}s_\pm$, where $h_\pm(s_\pm)=\mp\lambda_\pm s_\pm+O\big(|s_\pm|^2\big)$ are holomorphic functions. We use the coordinates $s_\pm$ and rewrite the NVE (4.1.3) as

$$\frac{\mathrm{d}\eta}{\mathrm{d}s_\pm}=\frac{1}{h_\pm(s_\pm)}J\mathrm{D}_y^2H(i(s_\pm))\eta, \tag{4.2.6}$$

which have regular singularities at $s_\pm=0$. Let $M_\pm$ be monodromy matrices of the NVE along Γ around $s_\pm=0$.

Let $\lambda'_+=-\lambda_+$ and $\lambda'_-=\lambda_-$, and let $\mu_\pm=\pm\mathrm{i}\omega_\pm$ and $\nu_\pm=\mp\mathrm{i}\omega_\pm$ be eigenvalues of $J\mathrm{D}_y^2H(x_\pm,0)$. Then we have

$$\frac{\mu_\pm-\nu_\pm}{\lambda'_\pm}=\mp\frac{2\mathrm{i}\omega_\pm}{\lambda_\pm}\notin\mathbb{Q},\qquad \frac{\mu_\pm+\nu_\pm}{\lambda'_\pm}=0\in\mathbb{Z},$$

which mean that conditions (A3) and (A4) of Chapter 3 hold. Applying Theorem 3.4.2 of Chapter 3, we obtain the following result.

Theorem 4.2.2. *Suppose that assumptions* (B1)–(B3) *hold and the Hamiltonian system* (4.1.1) *is real-meromorphically integrable near* $\Gamma_\mathbb{R}$*. Then the monodromy matrices* $M_\pm$ *are commutative. Moreover, if*

$$\frac{\mu_+}{\lambda'_+}-\frac{\mu_-}{\lambda'_-}=-\frac{\mathrm{i}\omega_+}{\lambda_+}+\frac{\mathrm{i}\omega_-}{\lambda_-}=0, \tag{4.2.7}$$

then

$$M_+=M_-^{-1}\qquad \text{or}\qquad M_+=M_-. \tag{4.2.8}$$

Remark 4.2.2.

(i) Let $U_\mathbb{R}$ and $U_\mathbb{C}$ be, respectively, neighborhoods of $\Gamma_\mathbb{R}$ in $\mathbb{R}^4$ and in $\mathbb{C}^4$. By real-meromorphic integrablity we mean that the real Hamiltonian system (4.1.1) has an additional first integral which is a restriction of some meromorphic function defined in $U_\mathbb{C}$ onto $U_\mathbb{R}$. If the Hamiltonian system (4.1.1) is real-meromorphically integrable in $U_\mathbb{R}$, then its complexification is also meromorphically integrable in $U_\mathbb{C}$. Such real-meromorphically nonintegrable Hamiltonian systems were also discussed by using a different approach in [54, 55, 96].

(ii) Under the hypothesis of Theorem 4.2.2, the identity component G^0 of the differential Galois group for the NVE (4.1.3) along Γ is commutative if and only if so is $M_\pm$. Moreover, if condition (4.2.7) holds, then condition (4.2.8) is necessary and sufficient for G^0 to be commutative.

(iii) If $x_+=x_-$, then condition (4.2.7) automatically holds, so that conclusion (4.2.8) is necessary for the real-meromorphic integrability of (4.1.1). We also note that the latter case in (4.2.8) was overlooked in the early results of [64, 91].

4.3 Main results

Let $\sigma_1^\pm$ and $\sigma_2^\pm$ be eigenvalues of $\mathrm{D}_y^2 H(x_\pm, 0)$. We have $\sigma_1^\pm \sigma_2^\pm = \omega_\pm^2$, so that $\sigma_1^\pm$ and $\sigma_2^\pm$ are of the same sign, where the upper and lower signs in super- and subscripts are taken simultaneously. Recall that there are one-parameter families of periodic orbits $\gamma_\pm^{\alpha\pm}$ near the saddle-centers $(x_\pm, 0)$, as stated in Section 4.2.1.

Proposition 4.3.1. *If $\sigma_1^\pm$ have the opposite signs, then there does not exist a pair (α_+, α_-) with $0 < \alpha_\pm \ll 1$ such that the periodic orbits $\gamma_\pm^{\alpha\pm}$ around $(x_\pm, 0)$ are on the same energy surface.*

Proof. Since the saddle-centers $(x_\pm, 0)$ are connected by the heteroclinic orbit $\big(x^{\mathrm{h}}(t), 0\big)$, we assume that $H(x_+, 0) = H(x_-, 0) = 0$ without loss of generality. Using the center manifold theorem [35, 88], we see that there exist center manifolds of $(x_\pm, 0)$ on which $\gamma_\pm^{\alpha\pm} = \big(x_\pm^{\alpha\pm}(t), y_\pm^{\alpha\pm}(t)\big)$ lie. Moreover, on the center manifolds, the relations $x - x_\pm = O\big(|y|^2\big)$ hold near $(x_\pm, 0)$. Hence,

$$H\big(\gamma_\pm^{\alpha\pm}\big) = \frac{1}{2} y_\pm^{\alpha\pm}(t) \cdot \mathrm{D}_y^2 H(x_\pm, 0) y_\pm^{\alpha\pm}(t) + O\big(|y^{\alpha\pm}(t)|^3\big),$$

which implies that for $\alpha_\pm > 0$ sufficiently small there does not exist a pair (α_+, α_-) with $H\big(\gamma_+^{\alpha+}\big) = H\big(\gamma_-^{\alpha-}\big)$ if σ_1^+ and σ_1^- have the opposite signs. $\square$

Henceforth we assume that $\sigma_1^\pm$ have the same sign. From the proof of Proposition 4.3.1 we can take $\alpha_+ \in (0, \bar{\alpha}_+)$ for $\alpha_- \in (0, \bar{\alpha}_-)$ sufficiently small such that $H\big(\gamma_+^{\alpha+}\big) = H\big(\gamma_-^{\alpha-}\big)$, i.e., there exist periodic orbits $\gamma_\pm^{\alpha\pm}$ near $(x_\pm, 0)$ on the same energy surface. Let $M_\pm$ be the monodromy matrices of the transformed NVE (4.2.5) and (4.2.6) around $s_\pm = 0$, as defined in Section 4.2.1. We state our main theorems as follows.

Theorem 4.3.2. *Assume that $\sigma_1^\pm$ are of the same sign. Let $\alpha_\pm > 0$ be sufficiently small and satisfy $H\big(\gamma_+^{\alpha+}\big) = H\big(\gamma_-^{\alpha-}\big)$. Then the following hold:*

- *(i) If $\omega_+ = \omega_-$ and the monodromy matrices $M_\pm$ are not commutative, then the right branch of the unstable manifold $W_{\mathrm{r}}^{\mathrm{u}}\big(\gamma_-^{\alpha-}\big)$ intersects the left branch of the stable manifold $W_\ell^{\mathrm{s}}\big(\gamma_+^{\alpha+}\big)$ transversely on the energy surface, i.e., transverse heteroclinic orbits from $\gamma_-^{\alpha-}$ to $\gamma_+^{\alpha+}$ exist.*
- *(ii) If $\omega_+ \neq \omega_-$, then $W_{\mathrm{r}}^{\mathrm{u}}\big(\gamma_-^{\alpha-}\big)$ and $W_\ell^{\mathrm{s}}\big(\gamma_+^{\alpha+}\big)$ intersect transversely on the energy surface, have quadric tangencies or do not intersect. In particular, they do not coincide.*

Theorem 4.3.3. *Assume that $\sigma_1^\pm$ are of the same sign and $\omega_+/\lambda_+ = \omega_-/\lambda_-$. Let $\alpha_\pm > 0$ be sufficiently small and satisfy $H\big(\gamma_+^{\alpha+}\big) = H\big(\gamma_-^{\alpha-}\big)$. Then the following hold:*

- *(i) If $\omega_+ = \omega_-$ and $M_+ \neq M_-^{-1}$, then $W_{\mathrm{r}}^{\mathrm{u}}\big(\gamma_-^{\alpha-}\big)$ intersects $W_\ell^{\mathrm{s}}\big(\gamma_+^{\alpha+}\big)$ transversely on the energy surface.*
- *(ii) If $\omega_+ \neq \omega_-$ and $M_+ = M_-^{-1}$, then $W_{\mathrm{r}}^{\mathrm{u}}\big(\gamma_-^{\alpha-}\big)$ does not intersect $W_\ell^{\mathrm{s}}\big(\gamma_+^{\alpha+}\big)$.*

Remark 4.3.1.

- (i) The hypothesis of Theorem 4.3.3(i) does not coincide with the sufficient condition given in Theorem 4.2.2 for real-meromorphic nonintegrability while the hypothesis of Theorem 4.3.2(i) does. Similarly, the hypothesis of Theorem 4.3.3(ii) does not coincide with the necessary condition for real-meromorphic integrability.

(ii) Assume that $x_- = x_+$ and $x^{\mathrm{h}}(t)$ is a homoclinic orbit. Then $\omega_+ = \omega_-$ and $\lambda_+ = \lambda_-$. Hence, we apply Theorem 4.3.2(i) to recover the result of [91] with a necessary correction stated in Remark 4.2.2(iii): If $M_+ \neq M_-^{-1}$, then the stable and unstable manifolds intersect transversely on the energy surface. In particular, by Theorem 4.2.2 and Remark 4.2.2(iii), we see that under the sufficient condition for real-meromorphical nonintegrability, the same conclusion holds.

(iii) In Theorem 4.3.2(ii) and Theorem 4.3.3(ii), by saying that $W_{\mathrm{r}}^{\mathrm{u}}(\gamma_-^{\alpha-})$ and $W_\ell^{\mathrm{s}}(\gamma_+^{\alpha+})$ do not intersect, we mean that $W_{\mathrm{r}}^{\mathrm{u}}(\gamma_-^{\alpha-})$ and $W_\ell^{\mathrm{s}}(\gamma_+^{\alpha+})$ do not intersect near the heteroclinic orbit $(x^{\mathrm{h}}(t), 0)$ before going away from it. It is difficult to generally exclude the case in which they intersect after that.

In the rest of this section we prove the main theorems. We first provide some necessary properties of the Melnikov function $M(t_0)$. Using (4.2.4), we can rewrite (4.2.3) as

$$M(t_0) = \frac{1}{2}(\Phi_-(t_0)\eta_0)^{\mathrm{T}}\big(\mathrm{D}_y^2 H(x_-,0) - B_0^{\mathrm{T}}\mathrm{D}_y^2 H(x_+,0)B_0\big)(\Phi_-(t_0)\eta_0), \tag{4.3.1}$$

where the superscript T represents the transpose operator. Since the matrix $\mathrm{D}_y^2 H(x_\pm,0)$ is symmetric, there exist a pair of orthogonal matrices $P_\pm$ such that

$$P_\pm^{\mathrm{T}}\mathrm{D}_y^2 H(x_\pm,0)P_\pm = \begin{pmatrix} \sigma_1^\pm & 0 \\ 0 & \sigma_2^\pm \end{pmatrix} \tag{4.3.2}$$

and $\det P_\pm = 1$. Hence, we have

$$\begin{aligned} M(t_0) &= \frac{1}{2}\big(P_-^{\mathrm{T}}\Phi_-(t_0)\eta_0\big)^{\mathrm{T}}\left[\begin{pmatrix} \sigma_1^- & 0 \\ 0 & \sigma_2^- \end{pmatrix} - \tilde{B}_0^{\mathrm{T}}\begin{pmatrix} \sigma_1^+ & 0 \\ 0 & \sigma_2^+ \end{pmatrix}\tilde{B}_0\right]\big(P_-^{\mathrm{T}}\Phi_-(t_0)\eta_0\big) \\ &= \frac{1}{2}\tilde{\eta}(t_0)^{\mathrm{T}}R\tilde{\eta}(t_0), \end{aligned} \tag{4.3.3}$$

where $\tilde{B}_0 = P_+^{\mathrm{T}}B_0P_-$, $\tilde{\eta}(t_0) = P_-^{\mathrm{T}}\Phi_-(t_0)\eta_0$ and

$$R = \begin{pmatrix} \sigma_1^- & 0 \\ 0 & \sigma_2^- \end{pmatrix} - \tilde{B}_0^{\mathrm{T}}\begin{pmatrix} \sigma_1^+ & 0 \\ 0 & \sigma_2^+ \end{pmatrix}\tilde{B}_0.$$

On the other hand, there exist a pair of nonsingular matrices $Q_\pm$ such that

$$Q_\pm^{-1}J\mathrm{D}_y^2 H(x_\pm,0)Q_\pm = \begin{pmatrix} \mathrm{i}\omega_\pm & 0 \\ 0 & -\mathrm{i}\omega_\pm \end{pmatrix}.$$

So we have

$$\Phi_\pm(t) = \exp\big(J\mathrm{D}_y^2 H(x_\pm,0)t\big) = Q_\pm\begin{pmatrix} \mathrm{e}^{\mathrm{i}\omega_\pm t} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\omega_\pm t} \end{pmatrix}Q_\pm^{-1}. \tag{4.3.4}$$

Noting that R is symmetric and using (4.3.3) and (4.3.4), we immediately obtain the following result.

Lemma 4.3.4.

(*i*) $M(t_0)$ *has a simple zero if and only if* $\det R < 0$.

(*ii*) *$M(t_0)$ has no zero if and only if* $\det R > 0$.

(*iii*) *$M(t_0)$ is not identically zero but has double zeros if and only if* $\det R = 0$ *and* $\operatorname{tr} R \neq 0$.

(*iv*) *$M(t_0)$ is identically zero if and only if* $\det R = 0$ *and* $\operatorname{tr} R = 0$.

This lemma enables us to easily determine by $\det R$ and $\operatorname{tr} R$ whether $M(t_0)$ is not identically zero or not, whether it has a zero or not, and whether its zero is simple or double if it has.

Denote

$$\tilde{B}_0 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Since $\Phi_\pm(t)$ and $\Psi(t)$ are fundamental matrices of linear Hamiltonian systems and $\Phi_\pm(0) = \mathrm{id}_2$ (see Section 4.2.1), we have $\det B_\pm = \det \Psi(0)$ by (4.2.2), so that

$$\det \tilde{B}_0 = \det B_0 = 1. \tag{4.3.5}$$

Hence, we compute

$$\operatorname{tr} R = -\left(\sigma_1^+ b_{11}^2 + \sigma_1^+ b_{12}^2 + \sigma_2^+ b_{21}^2 + \sigma_2^+ b_{22}^2\right) + \sigma_1^- + \sigma_2^-$$

and

$$\det R = (\omega_+ - \omega_-)^2 - \left(b_{11}\sqrt{\sigma_1^+\sigma_2^-} - b_{22}\sqrt{\sigma_2^+\sigma_1^-}\right)^2 - \left(b_{12}\sqrt{\sigma_1^+\sigma_1^-} + b_{21}\sqrt{\sigma_2^+\sigma_2^-}\right)^2. \tag{4.3.6}$$

Here we have used the relations $\sigma_1^\pm \sigma_2^\pm = \omega_\pm^2$.

Lemma 4.3.5. *If $\omega_+ = \omega_-$, then $M(t_0)$ is identically zero or it has a simple zero.*

Proof. Assume that $\omega_+ = \omega_-$. Obviously, $\det R \leq 0$ by (4.3.6). If $\det R = 0$, then

$$b_{11}\sqrt{\sigma_1^+\sigma_2^-} = b_{22}\sqrt{\sigma_2^+\sigma_1^-}, \qquad b_{12}\sqrt{\sigma_1^+\sigma_1^-} = -b_{21}\sqrt{\sigma_2^+\sigma_2^-},$$

so that

$$\begin{aligned} \operatorname{tr} R &= -\sqrt{\sigma_1^+\sigma_2^+}\left(\sqrt{\frac{\sigma_1^-}{\sigma_2^-}} + \sqrt{\frac{\sigma_2^-}{\sigma_1^-}}\right)(b_{11}b_{22} - b_{12}b_{21}) + \sigma_1^- + \sigma_2^- \\ &= -\sqrt{\sigma_1^-\sigma_2^-}\left(\sqrt{\frac{\sigma_1^-}{\sigma_2^-}} + \sqrt{\frac{\sigma_2^-}{\sigma_1^-}}\right) + \sigma_1^- + \sigma_2^- = 0. \end{aligned}$$

Here we have used the relations $\sigma_1^+\sigma_2^+ = \sigma_1^-\sigma_2^-$ and $\det \tilde{B}_0 = b_{11}b_{22} - b_{12}b_{21} = 1$. Using parts (i) and (iv) of Lemma 4.3.4 we obtain the result. □

We also need the following result on the monodromy matrices $M_\pm$ defined in Section 4.2.2.

Lemma 4.3.6. *The monodromy matrices can be expressed as*

$$M_+ = B_0^{-1} \exp\left(-\frac{2\pi \mathrm{i}}{\lambda_+} J\mathrm{D}_y^2 H(x_+, 0)\right) B_0, \qquad M_- = \exp\left(\frac{2\pi \mathrm{i}}{\lambda_-} J\mathrm{D}_y^2 H(x_-, 0)\right) \tag{4.3.7}$$

for a common fundamental matrix.

Proof. Let

$$\tilde{\Psi}(t) = \Psi(t) B_-^{-1}.$$

Then $\tilde{\Psi}(t)$ is a fundamental matrix of (4.1.3) such that

$$\lim_{t\to-\infty} \Phi_-(-t)\tilde{\Psi}(t) = \mathrm{id}_2 \qquad \text{and} \qquad \lim_{t\to+\infty} \Phi_+(-t)\tilde{\Psi}(t) = B_0.$$

For the transformed NVE on Γ, we take a fundamental matrix corresponding to $\tilde{\Psi}(t)$. Since by (4.3.4) its analytic continuation yields the monodromy matrices

$$\exp\left(\mp\frac{2\pi \mathrm{i}}{\lambda_\pm} J\mathrm{D}_y^2 H(x_\pm, 0)\right)$$

along small loops around $0_\pm$, we choose the base point near 0_- to obtain (4.3.7). □

Now we prove the main theorems.

Proof of Theorem 4.3.2. Assume that $M(t_0)$ is identically zero. It follows from (4.3.1) that

$$\mathrm{D}_y^2 H(x_-, 0) = B_0^{\mathrm{T}} \mathrm{D}_y^2 H(x_+, 0) B_0.$$

Since $\det B_0 = 1$, we have $B_0 J B_0^{\mathrm{T}} = J$, so that

$$J\mathrm{D}_y^2 H(x_-, 0) = B_0^{-1} J\mathrm{D}_y^2 H(x_+, 0) B_0. \tag{4.3.8}$$

Hence, $J\mathrm{D}_y^2 H(x_-, 0)$ and $J\mathrm{D}_y^2 H(x_+, 0)$ have the same eigenvalues, i.e., $\omega_+ = \omega_-$. This implies that if $\omega_+ \neq \omega_-$, then $M(t_0)$ is not identically zero. Using Lemma 4.3.4 and Theorem 4.2.1, we obtain part (ii).

On the other hand, using Lemma 4.3.6 and (4.3.8), we see that if $M(t_0)$ is identically zero, then

$$M_+ = \exp\left(-\frac{2\pi \mathrm{i}}{\lambda_+} J\mathrm{D}_y^2 H(x_-, 0)\right),$$

so that $M_\pm$ are commutative. Hence, if $M_\pm$ are not commutative, then $M(t_0)$ is not identically zero. This yields part (i) by Lemma 4.3.5 and Theorem 4.2.1. □

Proof of Theorem 4.3.3. Assume that $\omega_+/\lambda_+ = \omega_-/\lambda_-$. From Lemma 4.3.6 and (4.3.2) we have

$$M_+ = B_0^{-1} P_+ \exp\left(-\frac{2\pi \mathrm{i}}{\lambda_+}\begin{pmatrix} 0 & \sigma_2^+ \\ -\sigma_1^+ & 0 \end{pmatrix}\right) P_+^{-1} B_0,$$
$$M_- = P_- \exp\left(-\frac{2\pi \mathrm{i}}{\lambda_-}\begin{pmatrix} 0 & \sigma_2^- \\ -\sigma_1^- & 0 \end{pmatrix}\right) P_-^{-1}.$$

Using the relations $\sigma_1^\pm\sigma_2^\pm=\omega_\pm^2$, we easily compute

$$\exp\left(-\frac{2\pi\mathrm{i}}{\lambda_\pm}\begin{pmatrix}0 & \sigma_2^\pm\\ -\sigma_1^\pm & 0\end{pmatrix}\right)=\begin{pmatrix}\cosh 2\pi\mu & \mathrm{i}\sqrt{\sigma_2^\pm/\sigma_1^\pm}\sinh 2\pi\mu\\ -\mathrm{i}\sqrt{\sigma_1^\pm/\sigma_2^\pm}\sinh 2\pi\mu & \cosh 2\pi\mu\end{pmatrix},$$

where $\mu=\omega_+/\lambda_+=\omega_-/\lambda_-$. So the condition $M_+=M_-^{-1}$ is equivalent to

$$\begin{aligned}&\tilde{B}_0\begin{pmatrix}\cosh 2\pi\mu & \mathrm{i}\sqrt{\sigma_2^-/\sigma_1^-}\sinh 2\pi\mu\\ -\mathrm{i}\sqrt{\sigma_1^-/\sigma_2^-}\sinh 2\pi\mu & \cosh 2\pi\mu\end{pmatrix}\\ &=\begin{pmatrix}\cosh 2\pi\mu & \mathrm{i}\sqrt{\sigma_2^+/\sigma_1^+}\sinh 2\pi\mu\\ -\mathrm{i}\sqrt{\sigma_1^+/\sigma_2^+}\sinh 2\pi\mu & \cosh 2\pi\mu\end{pmatrix}\tilde{B}_0,\end{aligned}$$

so that

$$b_{11}\sqrt{\sigma_1^+\sigma_2^-}-b_{22}\sqrt{\sigma_2^+\sigma_1^-}=0,\qquad b_{21}\sqrt{\sigma_2^+\sigma_2^-}+b_{12}\sqrt{\sigma_1^-\sigma_1^+}=0.$$

Hence, if $M_+=M_-^{-1}$, then by (4.3.6)

$$\det R=(\omega_+-\omega_-)^2.$$

Thus, we obtain part (ii) by Theorem 4.2.1 and Lemma 4.3.4. Moreover, when $\omega_+=\omega_-$, the above observation along with (4.3.6) shows that $\det R=0$ (if and) only if $M_+=M_-^{-1}$. This implies part (i) by Theorem 4.2.1 and Lemma 4.3.5. □

4.4 Example

To illustrate our theory, we consider the two-degree-of-freedom Hamiltonian system

$$\begin{aligned}\dot{x}_1&=x_2, & \dot{x}_2&=-x_1+x_1^3+\tfrac{1}{2}\beta_1y_1^2+\beta_2x_1y_1^2,\\ \dot{y}_1&=y_2, & \dot{y}_2&=-\omega^2y_1+\beta_1x_1y_1+\beta_2x_1^2y_1-y_1^3\end{aligned}\tag{4.4.1}$$

with the Hamiltonian

$$H=\tfrac{1}{2}\big(x_2^2+y_2^2\big)+\tfrac{1}{2}\big(x_1^2+\omega^2y_1^2\big)-\tfrac{1}{4}\big(x_1^4+y_1^4\big)-\tfrac{1}{2}\beta_1x_1y_1^2-\tfrac{1}{2}\beta_2x_1^2y_1^2,$$

where $\beta_1,\beta_2,\omega\in\mathbb{R}$ are constants such that

$$\omega^2-\beta_2>|\beta_1|.\tag{4.4.2}$$

We easily see that assumption (B1) holds, i.e., the x-plane is invariant under the flow of (4.4.1). On the x-plane, the Hamiltonian system (4.4.1) has two saddles at $x=(\pm1,0)$ with $\lambda_\pm=\sqrt{2}$, and they are connected by a pair of heteroclinic orbits,

$$x_\pm^{\mathrm{h}}(t)=\left(\pm\tanh\left(\frac{t}{\sqrt{2}}\right),\pm\frac{1}{\sqrt{2}}\operatorname{sech}^2\left(\frac{t}{\sqrt{2}}\right)\right),$$

satisfying

$$\lim_{t\to+\infty}x_\pm^{\mathrm{h}}(t)=(\pm1,0)\qquad\text{and}\qquad\lim_{t\to-\infty}x_\pm^{\mathrm{h}}(t)=(\mp1,0).$$

Thus, assumption (A3) holds for $x_\pm = (\pm 1, 0)$ or $(\mp 1, 0)$, where the upper and lower signs are taken simultaneously. Moreover, by (4.4.2), the two equilibria in (4.4.1) are saddle-centers, so that assumption (A2) holds. In the following, we describe the details of computations for $x_\pm = (\pm 1, 0)$ and $x_+^{\mathrm{h}}(t)$, from which the corresponding results for $x_\pm = (\mp 1, 0)$ and $x_-^{\mathrm{h}}(t)$ also follow immediately.

Let $x_\pm = (\pm 1, 0)$. Then

$$\omega_\pm = \sqrt{\omega^2 \mp \beta_1 - \beta_2}, \qquad \sigma_1^\pm = 1, \qquad \sigma_2^\pm = \omega^2 \mp \beta_1 - \beta_2 > 0.$$

We see that $\omega_+ = \omega_-$ if and only if $\beta_1 = 0$ and that $\sigma_1^\pm$ are of the same sign. The NVE (4.1.3) becomes

$$\dot\eta_1 = \eta_2, \qquad \dot\eta_2 = -\big(\omega^2 - \beta_1 x_{1+}^{\mathrm{h}}(t) - \beta_2 x_{1+}^{\mathrm{h}}(t)^2\big)\eta_1, \tag{4.4.3}$$

which reduces to the second-order differential equation

$$\ddot\eta_1 + \big(\omega^2 - \beta_1 x_{1+}^{\mathrm{h}}(t) - \beta_2 x_{1+}^{\mathrm{h}}(t)^2\big)\eta_1 = 0, \tag{4.4.4}$$

where $x_{1+}^{\mathrm{h}}(t)$ represents the x_1-component of $x_+^{\mathrm{h}}(t)$, i.e., $x_{1+}^{\mathrm{h}}(t) = \tanh\big(t/\sqrt{2}\big)$. Letting $\rho_\pm = -\mathrm{i}\omega_\pm/\sqrt{2}$ and using the transformation

$$\tau = \frac{x_{1+}^{\mathrm{h}}(t) + 1}{2}, \qquad \eta_1 = \tau^{\rho_-}(1-\tau)^{\rho_+}\xi, \tag{4.4.5}$$

we rewrite (4.4.4) as the Gauss hypergeometric equation [41, 87]

$$\tau(1-\tau)\frac{\mathrm{d}^2\xi}{\mathrm{d}\tau^2} + (c_3 - (c_1 + c_2 + 1)\tau)\frac{\mathrm{d}\xi}{\mathrm{d}\tau} - c_1 c_2 \xi = 0, \tag{4.4.6}$$

where

$$c_1 = \chi_+ + \rho_+ + \rho_-, \qquad c_2 = \chi_- + \rho_+ + \rho_-, \qquad c_3 = 2\rho_- + 1$$

with $\chi_\pm = \frac{1}{2}\big(1 \pm \sqrt{1 + 8\beta_2}\big)$. The equilibria x_- and x_+ correspond to $\tau = 0$ and 1, respectively. Singular points of (4.4.6) are $\tau = 0, 1, \infty$ and all of them are regular.

The necessary condition for real-meromorphic integrability given by Theorem 4.2.2 holds only in a limited case for (4.4.1) as follows.

Lemma 4.4.1. *If the monodromy matrices $M_\pm$ are commutative, then*

$$\beta_1 = 0, \qquad \beta_2 = \tfrac{1}{2}n(n-1) \qquad \text{for some} \quad n \in \mathbb{N} \tag{4.4.7}$$

and $M_+ = M_-^{-1}$.

Proof. Let M_0 and M_1 be the monodromy matrices of (4.4.6) around $\tau = 0$ and $\tau = 1$, respectively. Using (4.4.5), we compute $M_- = e(\rho_-)M_0$ and $M_+ = e(\rho_+)M_1$, where $e(\rho) = \mathrm{e}^{2\pi\mathrm{i}\rho}$ for $\rho \in \mathbb{C}$. It is a well known fact (see, e.g., [41, Chapter 2, Theorem 4.7.2]) that the monodromy matrices of (4.4.6) are given by

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & e(-c_3) \end{pmatrix},$$

$$M_1 = \frac{1}{\ell_0}\begin{pmatrix} \ell_{11}\ell_{22} - \ell_{12}\ell_{21}e(c_3 - c_1 - c_2) & \ell_{12}\ell_{22}(e(c_3 - c_1 - c_2) - 1) \\ \ell_{11}\ell_{21}(1 - e(c_3 - c_1 - c_2)) & \ell_{11}\ell_{22}e(c_3 - c_1 - c_2) - \ell_{12}\ell_{21} \end{pmatrix},$$

where $\ell_0 = \ell_{11}\ell_{22} - \ell_{12}\ell_{21}$,

$$\ell_{11} = \frac{\Gamma(c_3)\Gamma(c_3 - c_1 - c_2)}{\Gamma(c_3 - c_1)\Gamma(c_3 - c_2)}, \qquad \ell_{12} = \frac{\Gamma(2 - c_3)\Gamma(c_3 - c_1 - c_2)}{\Gamma(1 - c_1)\Gamma(1 - c_2)},$$
$$\ell_{21} = \frac{\Gamma(c_3)\Gamma(c_1 + c_2 - c_3)}{\Gamma(c_1)\Gamma(c_2)}, \qquad \ell_{22} = \frac{\Gamma(2 - c_3)\Gamma(c_1 + c_2 - c_3)}{\Gamma(c_1 - c_3 + 1)\Gamma(c_2 - c_3 + 1)},$$

and $\Gamma(\rho)$ represents the gamma function. Since $c_3 = 2\rho_- + 1$ and $c_3 - c_1 - c_2 = 1 - \rho_+$ are not integers, we see that if M_0 and M_1 are commutative, then M_1 must be diagonal and consequently $\ell_{12}\ell_{22} = \ell_{11}\ell_{21} = 0$. Moreover, c_1 and c_2 are not integers, so that $\ell_{12}, \ell_{21} \neq 0$, since $1/\Gamma(\rho) = 0$ if and only if $\rho \in \mathbb{Z}$ and $\rho \leq 0$. Hence, if $M_\pm$ are commutative, then $\ell_{11}, \ell_{22} = 0$.

If $\beta_1 \neq 0$, then $c_3 - c_1$ and $c_3 - c_2$ are not integers, so that $\ell_{11}, \ell_{22} \neq 0$ and consequently $M_\pm$ are not commutative. On the other hand, if $\beta_1 = 0$, then $c_3 - c_1 = 1 - \chi_+ = c_2 - c_3 + 1$ and $c_3 - c_2 = \chi_+ = c_1 - c_3 + 1$, so that $\ell_{11}, \ell_{22} = 0$ if and only if $\chi_+ \in \mathbb{N}$. Hence, if $M_\pm$ are commutative, then $\beta_1 = 0$ and $\chi_+ \in \mathbb{N}$, so that the second condition of (4.4.7) holds. Moreover, if condition (4.4.7) holds, then $\ell_{11}, \ell_{22} = 0$ and $\rho_+ + \rho_- = 0$, so that

$$M_+ = \begin{pmatrix} e(1 - \rho_+) & 0 \\ 0 & e(\rho_+) \end{pmatrix} = \begin{pmatrix} e(\rho_-) & 0 \\ 0 & e(1 - \rho_-) \end{pmatrix}^{-1} = M_-^{-1}.$$

Thus, we obtain the desired result. □

Obviously, the statement of Lemma 4.4.1 is also true for $x_\pm = (\mp 1, 0)$ and $x_-^{\mathrm{h}}(t)$. Let $\gamma_\pm^{\alpha\pm}$ denote periodic orbits around the saddle-centers at $x = (\pm 1, 0)$ and let $W_{\mathrm{r}}^{\mathrm{s}}(\gamma_-^{\alpha-})$ and $W_\ell^{\mathrm{u}}(\gamma_+^{\alpha+})$ be the right and left branches of the stable and unstable manifolds of $\gamma_-^{\alpha-}$ and $\gamma_+^{\alpha+}$, respectively. Note that $\omega_+/\lambda_+ = \omega_-/\lambda_-$ holds if and only if $\beta_1 = 0$. Using Theorems 4.2.2, 4.3.2 and 4.3.3 and Lemma 4.4.1, we obtain the following proposition.

Proposition 4.4.2. *Suppose that condition* (4.4.7) *does not hold. Then the Hamiltonian system* (4.4.1) *is real-meromorphically nonintegrable near the heteroclinic orbits* $(x, y) = (x_\pm^{\mathrm{h}}(t), 0)$. *Moreover, let* $\alpha_\pm > 0$ *be sufficiently small and satisfy* $H(\gamma_+^{\alpha+}) = H(\gamma_-^{\alpha-})$. *If* $\beta_1 = 0$, *then* $W_{\mathrm{r}}^{\mathrm{u}}(\gamma_-^{\alpha-})$ *and* $W_\ell^{\mathrm{u}}(\gamma_+^{\alpha+})$, *respectively, intersect* $W_\ell^{\mathrm{s}}(\gamma_+^{\alpha+})$ *and* $W_{\mathrm{r}}^{\mathrm{s}}(\gamma_-^{\alpha-})$ *transversely on the energy surface, i.e., there exists a heteroclinic cycle. If* $\beta_1 \neq 0$, *then* $W_{\mathrm{r}}^{\mathrm{u}}(\gamma_-^{\alpha-})$ *and* $W_\ell^{\mathrm{u}}(\gamma_+^{\alpha+})$, *respectively, intersect* $W_\ell^{\mathrm{s}}(\gamma_+^{\alpha+})$ *and* $W_{\mathrm{r}}^{\mathrm{s}}(\gamma_-^{\alpha-})$ *transversely on the energy surface, or these manifolds have quadratic tangencies or do not intersect.*

Remark 4.4.1. The existence of such a heteroclinic cycle implies that chaotic dynamics occurs in (4.4.1), as stated at the end of Section 4.2.1. From Proposition 4.4.2 we immediately see that when $\beta_1 \neq 0$, the system (4.4.1) is real-meromorphically nonintegrable near the heteroclinic orbits although there may not exist a heteroclinic cycle.

We next compute the Melnikov function $M(t_0)$ for (4.4.1). Let $x_\pm = (\pm 1, 0)$. The NVE (4.2.1) becomes

$$\dot{\eta}_1 = \eta_2, \qquad \dot{\eta}_2 = -(\omega^2 \mp \beta_1 - \beta_2)\eta_1,$$

of which the fundamental matrix with $\Phi_\pm(0) = \mathrm{id}_2$ are given by

$$\Phi_\pm(t) = \begin{pmatrix} \cos\omega_\pm t & \sin\omega_\pm t/\omega_\pm \\ -\omega_\pm \sin\omega_\pm t & \cos\omega_\pm t \end{pmatrix}. \tag{4.4.8}$$

Let $F(c_1,c_2,c_3;\tau)$ be the Gauss hypergeometric function,

$$F(c_1,c_2,c_3;\tau)=\sum_{k=0}^{\infty}\frac{c_1(c_1+1)\cdots(c_1+k-1)c_2(c_2+1)\cdots(c_2+k-1)}{k!c_3(c_3+1)\cdots(c_3+k-1)}\tau^k.$$

Then

$$\xi=\tau^{1-c_3}F(c_1-c_3+1,c_2-c_3+1,2-c_3;\tau)$$

is a solutions to (4.4.6) as well as $\xi=F(c_1,c_2,c_3;\tau)$ (see, e.g., [41, Chapter 2, Section 1.3] or [87, Section 14.4]). So we obtain the complex valued solution to (4.4.4),

$$\begin{aligned}\eta=\bar{\eta}(t):=&\left(\frac{1+\tanh(t/\sqrt{2})}{2}\right)^{-\rho_-}\left(\frac{1-\tanh(t/\sqrt{2})}{2}\right)^{\rho_+}\\&\times F\left(c_1-c_3+1,c_2-c_3+1,2-c_3;\frac{1+\tanh(t/\sqrt{2})}{2}\right),\end{aligned}$$

and the fundamental matrix of (4.4.3),

$$\Psi(t)=\begin{pmatrix}\operatorname{Re}\bar{\eta}(t) & \operatorname{Im}\bar{\eta}(t)/\omega_-\\ \operatorname{Re}\dot{\bar{\eta}}(t) & \operatorname{Im}\dot{\bar{\eta}}(t)/\omega_-\end{pmatrix}. \tag{4.4.9}$$

We easily see that

$$\left(\frac{1+\tanh(t/\sqrt{2})}{2}\right)^{-\rho_-}\to 1\qquad\text{and}\qquad\left(\frac{1-\tanh(t/\sqrt{2})}{2}\right)^{\rho_+}\to e^{i\omega_+t}$$

as $t\to\infty$ and

$$\left(\frac{1+\tanh(t/\sqrt{2})}{2}\right)^{-\rho_-}\to e^{i\omega_-t}\qquad\text{and}\qquad\left(\frac{1-\tanh(t/\sqrt{2})}{2}\right)^{\rho_+}\to 1$$

as $t\to-\infty$. Thus, we have

$$\bar{\eta}(t)\to e^{i\omega_-t}\qquad\text{as}\quad t\to-\infty \tag{4.4.10}$$

since

$$\lim_{\tau\to 0}F(c_1-c_3+1,c_2-c_3+1,2-c_3;\tau)=1.$$

Using a well-known formula of the hypergeometric function (see, e.g., [41, Chapter 2, equation (4.7.9)]), we obtain

$$\begin{aligned}\tau^{1-c_3}F(c_1-c_3+1,c_2-c_3+1,2-c_3;\tau)=&\,\ell_{12}F(c_1,c_2,c_1+c_2-c_3+1;1-\tau)\\&+\ell_{22}(1-\tau)^{c_3-c_1-c_2}F(c_3-c_1,c_3-c_2,c_3-c_1-c_2+1;1-\tau),\end{aligned}$$

so that

$$\bar{\eta}(t)\to\ell_{12}e^{i\omega_+t}+\ell_{22}e^{-i\omega_+t}\qquad\text{as}\quad t\to\infty. \tag{4.4.11}$$

Substituting (4.4.8) and (4.4.9) into (4.2.2) and using (4.4.10) and (4.4.11), we compute

$$B_+ = \begin{pmatrix} \operatorname{Re}\ell_{12} + \operatorname{Re}\ell_{22} & (\operatorname{Im}\ell_{12} + \operatorname{Im}\ell_{22})/\omega_- \\ -\omega_+(\operatorname{Im}\ell_{12} + \operatorname{Im}\ell_{22}) & \omega_+(\operatorname{Re}\ell_{12} - \operatorname{Re}\ell_{22})/\omega_- \end{pmatrix}, \qquad B_- = \mathrm{id}_2,$$

which yields

$$B_0 = B_-^{-1}B_+ = \begin{pmatrix} \operatorname{Re}\ell_{12} + \operatorname{Re}\ell_{22} & (\operatorname{Im}\ell_{12} + \operatorname{Im}\ell_{22})/\omega_- \\ -\omega_+(\operatorname{Im}\ell_{12} + \operatorname{Im}\ell_{22}) & \omega_+(\operatorname{Re}\ell_{12} - \operatorname{Re}\ell_{22})/\omega_- \end{pmatrix}. \tag{4.4.12}$$

Equation (4.2.4) becomes

$$m_\pm(\eta) = \tfrac{1}{2}\big((\omega^2 \mp \beta_1 - \beta_2)\eta_1^2 + \eta_2^2\big).$$

Using (4.2.3) and (4.4.12), we obtain the Melnikov function as

$$\begin{aligned} M(t_0) &= (-\operatorname{Re}\ell_{12}\operatorname{Re}\ell_{22} + \operatorname{Im}\ell_{12}\operatorname{Im}\ell_{22})\omega_+^2 \cos 2\omega_- t_0 \\ &\quad + (\operatorname{Re}\ell_{12}\operatorname{Im}\ell_{22} + \operatorname{Im}\ell_{12}\operatorname{Re}\ell_{22})\omega_+^2 \sin 2\omega_- t_0 + \tfrac{1}{2}\big(\omega_-^2 - \big(|\ell_{12}|^2 + |\ell_{22}|^2\big)\omega_+^2\big) \\ &= \omega_+^2|\ell_{12}||\ell_{22}|\cos(2\omega_- t_0 - \phi_0) + \tfrac{1}{2}\big(\omega_-^2 - (|\ell_{12}|^2 + |\ell_{22}|^2)\omega_+^2\big), \end{aligned}$$

where

$$\tan\phi_0 = \frac{\operatorname{Re}\ell_{12}\operatorname{Im}\ell_{22} + \operatorname{Im}\ell_{12}\operatorname{Re}\ell_{22}}{-\operatorname{Re}\ell_{12}\operatorname{Re}\ell_{22} + \operatorname{Im}\ell_{12}\operatorname{Im}\ell_{22}}.$$

Let

$$\begin{aligned} G(\beta_1, \beta_2, \omega) &:= \big(\omega_+^2|\ell_{12}||\ell_{22}|\big)^2 - \tfrac{1}{4}\big(\omega_-^2 - \big(|\ell_{12}|^2 + |\ell_{22}|^2\big)\omega_+^2\big)^2 \\ &= \omega_+^2\omega_-^2|\ell_{22}|^2 - \tfrac{1}{4}\omega_-^2(\omega_+ - \omega_-)^2. \end{aligned}$$

Here we have used the relation $|\ell_{12}|^2 - |\ell_{22}|^2 = \omega_-/\omega_+$ obtained from (4.3.5) and (4.4.12). The Melnikov function $M(t_0)$ has a simple zero (resp. no zero) if and only if $G(\beta_1, \beta_2, \omega) > 0$ (resp. $G(\beta_1, \beta_2, \omega) < 0$). Obviously, the above arguments are valid for $x_\pm = (\mp 1, 0)$ and $x_-^{\mathrm{h}}(t)$. Applying Theorem 4.2.1, we obtain the following proposition.

Proposition 4.4.3. *Let $\alpha_\pm > 0$ be sufficiently small and satisfy $H(\gamma_+^{\alpha_+}) = H(\gamma_-^{\alpha_-})$. If $G(\beta_1, \beta_2, \omega) > 0$, then $W_{\mathrm{r}}^{\mathrm{u}}(\gamma_-^{\alpha_-})$ and $W_\ell^{\mathrm{u}}(\gamma_+^{\alpha_+})$, respectively, intersect $W_\ell^{\mathrm{s}}(\gamma_+^{\alpha_+})$ and $W_{\mathrm{r}}^{\mathrm{s}}(\gamma_-^{\alpha_-})$ transversely on the energy surface, i.e., there exists a heteroclinic cycles. If $G(\beta_1, \beta_2, \omega) < 0$, then $W_{\mathrm{r}}^{\mathrm{u}}(\gamma_-^{\alpha_-})$ and $W_\ell^{\mathrm{u}}(\gamma_+^{\alpha_+})$, respectively, do not intersect $W_\ell^{\mathrm{s}}(\gamma_+^{\alpha_+})$ and $W_{\mathrm{r}}^{\mathrm{s}}(\gamma_-^{\alpha_-})$.*

Remark 4.4.2.

(i) As expected from Proposition 4.4.2, when $\beta_1 = 0$, we see that $G(\beta_1, \beta_2, \omega) > 0$ if and only if the second condition of (4.4.7) does not hold, i.e.,

$$\beta_2 \neq \tfrac{1}{2}n(n-1) \qquad \text{for any} \quad n \in \mathbb{N}. \tag{4.4.13}$$

This follows from the fact that $\ell_{22} \neq 0$ if and only if condition (4.4.13) holds (see the proof of Lemma 4.4.1).

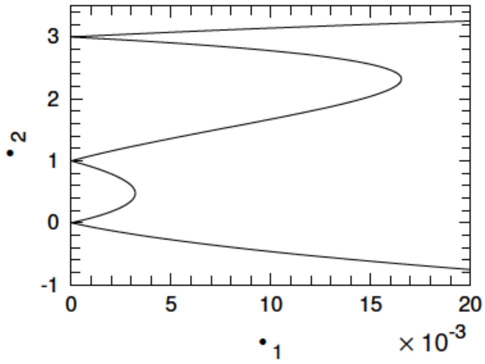

Figure 4.4: Numerical computation of the curve given by $G(\beta_1, \beta_2, \omega) = 0$ with $\omega = 2$ in the (β_1, β_2)-plane.

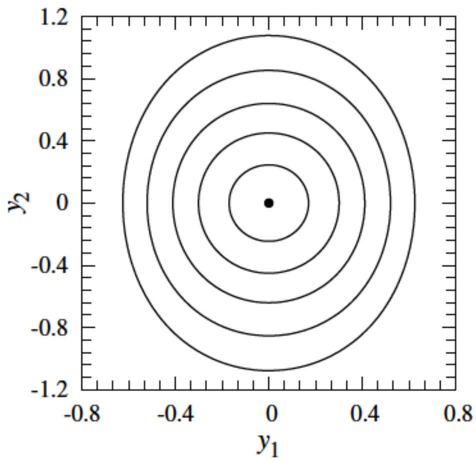

Figure 4.5: Periodic orbits near the saddle-center with $x = (-1, 0)$ for $\beta_1 = 5.0 \times 10^{-3}$, $\beta_2 = 2$ and $\omega = 2$. Their projections to the y-plane are plotted, and their energy values are $H = 0.28, 0.35, 0.45, 0.6, 0.8$ from the inside.

(ii) When $\beta_1 \neq 0$, Proposition 4.4.2 means that the Hamiltonian system (4.4.1) is always real-meromorphically nonintegrable as stated in Remark 4.4.1, but there may not exist heteroclinic cycles for periodic orbits: the function $G(\beta_1, \beta_2, \omega)$ may be negative.

In Fig. 4.4 we plot the curve given by $G(\beta_1, \beta_2, \omega) = 0$ in the (β_1, β_2)-parameter plane for $\omega = 2$. Here we have used the function `fsolve` of `Maple` to numerically solve $G(\beta_1, \beta_2, 2) = 0$ for β_2 varied. By Proposition 4.4.3, heteroclinic cycles on energy surfaces near the saddle-centers exist (resp. do not exist) for the parameter values of β_1, β_2 in the left (resp. right) side of the curve since $G(\beta_1, \beta_2, 2) > 0$ (resp. < 0) there.

To support the above theoretical results, we give numerical computations of the stable and unstable manifolds of periodic orbits near the saddle-centers with $x = (\pm 1, 0)$ for the Hamiltonian system (4.4.1). Our numerical approach was described in [75, Section 4.3] and similar to that of [15]. The calculations were carried out by using the numerical computation tool `AUTO` [24], as in [15, 75], although the monodromy matrix (the derivative of the Poincaré map) was computed by numerically solving the variational equation around the corresponding periodic orbit directly.

Fig. 4.5 shows numerically computed periodic orbits near the saddle-center with $x = (-1, 0)$ for $\beta_1 = 5 \times 10^{-3}$, $\beta_2 = 2$ and $\omega = 2$. Similar pictures for periodic orbits were also obtained for the other cases, and periodic orbits far from the saddle-centers could be computed like Fig. 4.5 although the Lyapunov center theorem only guarantees their existence near the saddle-centers.

Fig. 4.6 shows numerically computed the stable and unstable manifolds, $W^{\mathrm{s}}(\gamma_{\pm}^{\alpha\pm})$ and $W^{\mathrm{u}}(\gamma_{\pm}^{\alpha\pm})$, of periodic orbits $\gamma_{\pm}^{\alpha\pm}$ near the saddle-centers on the Poincaré section $\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \,|\, y_1 = 0\}$ for $\beta_2 = 2$, $\omega = 2$ and $H = 0.28$. In Fig. 4.6(a) for $\beta_1 = 5 \times 10^{-3}$, we observe that $W_{\mathrm{r}}^{\mathrm{u}}(\gamma_{-}^{\alpha-})$ and $W_{\ell}^{\mathrm{u}}(\gamma_{+}^{\alpha+})$, respectively, intersect $W_{\ell}^{\mathrm{s}}(\gamma_{+}^{\alpha+})$ and $W_{\mathrm{r}}^{\mathrm{s}}(\gamma_{-}^{\alpha-})$

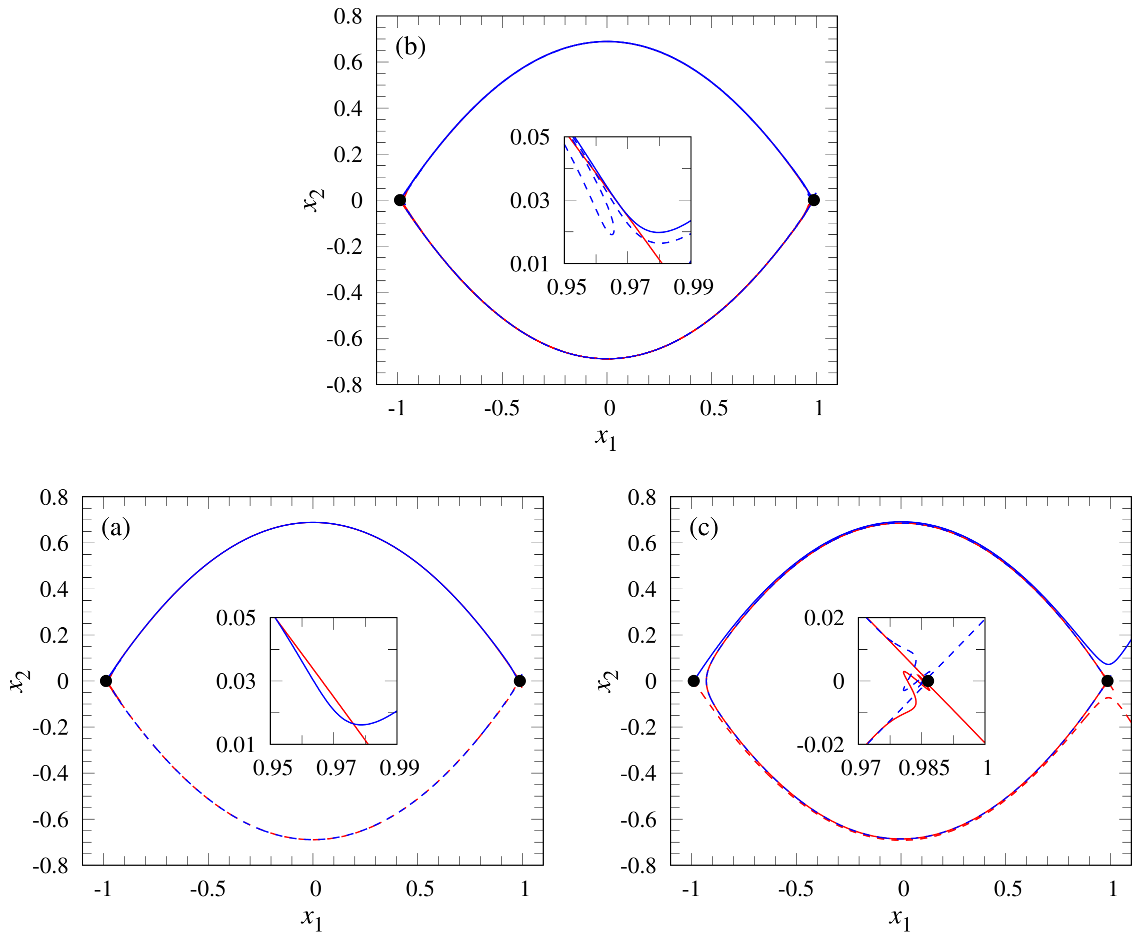

Figure 4.6: Stable and unstable manifolds of periodic orbits near the saddle-centers with $x = (\pm 1, 0)$ on the Poincaré section $\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \,|\, y_1 = 0\}$ for $\beta_2 = 2$, $\omega = 2$ and $H = 0.28$: (a) $\beta_1 = 5\times10^{-3}$; (b) 1.32×10^{-2}; (c) 2×10^{-1}. These manifolds near $\left(x_+^{\mathrm{h}}(t), 0\right)$ and $\left(x_-^{\mathrm{h}}(t), 0\right)$ are plotted as solid and dashed lines, respectively, and blue and red colors are used for the stable and unstable manifolds, respectively.

transversely, and there exists a heteroclinic cycle. In Fig. 4.6(b) for $\beta_1 = 1.32 \times 10^{-2}$, $W_{\mathrm{r}}^{\mathrm{u}}\left(\gamma_-^{\alpha-}\right)$ and $W_\ell^{\mathrm{u}}\left(\gamma_+^{\alpha+}\right)$, respectively, seem to be quadratically tangent to $W_\ell^{\mathrm{s}}\left(\gamma_+^{\alpha+}\right)$ and $W_{\mathrm{r}}^{\mathrm{s}}\left(\gamma_-^{\alpha-}\right)$. In Fig. 4.6(c) for $\beta_1 = 2 \times 10^{-1}$, $W_{\mathrm{r}}^{\mathrm{u}}\left(\gamma_-^{\alpha-}\right)$ and $W_\ell^{\mathrm{u}}\left(\gamma_+^{\alpha+}\right)$, respectively, do not intersect $W_\ell^{\mathrm{s}}\left(\gamma_+^{\alpha+}\right)$ and $W_{\mathrm{r}}^{\mathrm{s}}\left(\gamma_-^{\alpha-}\right)$. We see that for $(\beta_2, \omega) = (2, 2)$, $G(\beta_1, \beta_2, \omega) = 0$ at $\beta_1 \approx 1.5\times10^{-2}$ in Fig. 4.4, and predict by Proposition 4.4.3 that a heteroclinic cycle exists or not, depending on whether β_1 is less or greater than the value. Thus, the theoretical prediction fairly agrees with the numerical observation in Fig. 4.6. The agreement becomes better when the periodic orbits $\gamma_\pm^{\alpha\pm}$ are closer to the saddle-centers. In Fig. 4.6(c) we also observe that $W^{\mathrm{s}}(\gamma_+^{\alpha+})$ and $W^{\mathrm{u}}\left(\gamma_+^{\alpha+}\right)$ still intersect transversely. Hence, the Hamiltonian system (4.4.1) exhibits chaotic dynamics and it is nonintegrable. This consists with the results of Proposition 4.4.2.

Chapter 5

Conclusions

5.1 Concluding remarks

In this thesis, we first studied integrability and nonintegrability of Poincaré-Dulac normal forms around equilibria. We next considered general systems with heteroclinic orbits and used the Morales-Ramis theory [6] to obtain sufficient conditions for their nonintegrability by the monodromy groups of variational equations along the heteroclinic orbits. Finally, we considered two-degree-of-freedom Hamiltonian systems with heteroclinic orbits and used the extended Melnikov method [75] along with the result of Chapter 3 to obtain some relationships between nonintegrability and chaos.

In Chapter 2, we considered dynamical systems in Poincaré-Dulac normal form having an equilibrium at the origin. We introduced condition $(\mathrm{C})_r$ and proved that it is a sufficient condition for their integrability as in Theorem 2.2.2. This implies that systems satisfying conditions (A_2) and (ω) for having convergent normalizations are integrable and that the assumptions of Theorem 2.1.2 is weaker than those of Theorem 2.1.1. We also proved that condition $(\mathrm{C})_r$ is also necessary for existence of the maximal number of first integrals under some condition. Finally, we proved an analogous relationships between resonance degrees and integrablity already known in Birkhoff normal forms for Hamiltonian systems. We demonstrated the theoretical results for a normal form appearing in the codimension-two fold-Hopf bifurcation.

In Chapter 3, we considered general n-dimensional systems of differential equations having $(n-2)$-dimensional, locally invariant manifolds on which there exist equilibria connected by heteroclinic orbits for $n \geq 3$. The system may be non-Hamiltonian and have no saddle-centers, and the equilibria are allowed to be the same and connected by a homoclinic orbit. Under assumption (A3), we proved that the monodromy group for the normal variational equation, which is represented by components of the variational equation normal to the locally invariant manifold and defined on a Riemann surface, is diagonalizable if the system is real-meromorphically integrable in the meaning of Bogoyavlenskij. We can reagard the result as an extension of the Ziglin analysis [94] for general dynamical systems. In fact, assumption (A3) corresponds to a non-resonant condition of the Ziglin analysis. We applied the theory to a three-dimensional volume-preserving system describing the streamline of a steady incompressible flow with two parameters, and showed that it is real-meromorphically nonintegrable for almost all values of the two parameters.

In Chapter 4, we considered a class of two-degree-of-freedom Hamiltonian systems with saddle-centers connected by heteroclinic orbits and discussed some relationships between the existence of transverse heteroclinic orbits and nonintegrability. By the Lyapunov center theorem there is a family of periodic orbits near each of the saddle-centers, and the Hessian matrices of the Hamiltonian at the two saddle-centers are assumed to have the same number of positive eigenvalues. We showed that if the associated Jacobian matrices have the same pair of purely imaginary eigenvalues, then the stable and unstable manifolds of the periodic orbits intersect transversely on the same Hamiltonian energy surface when the sufficient conditions obtained in Chapter 3 for real-meromorphic nonintegrability of the Hamiltonian systems hold; if not, then these manifolds intersect transversely on the same energy surface, have quadratic tangencies or do not intersect whether the sufficient conditions hold or not. Our theory was illustrated for a system with quartic single-well potential and some numerical results are given to support the theoretical results.

5.2 Future work

In Chapter 2, we discussed integrability and nonintegrability of Poincaré-Dulac normal forms. To complete the study of integrability for general systems around equilibria, we need to show the existence of convergent normalizations. Especially, if all normalizations for a given dynamical system are divergent, then it is analytically nonintegrable by Zung's result [97]. As stated in Chapter 1, there are only few studies on systems whose normalizations are all divergent. It is well known that Écalle's theory [29, 74] is useful to study normalizations for non-resonant cases. On the other hand, normalizations of resonant systems are well studied only when its dimensions are two [57, 74]. So it is expected to extend Écalle's theory so that we can discuss normalizations for resonant systems whose dimensions are greater than two.

To clarify a relationship between the divergence of normalizations and the Écalle theory, we consider

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^2 \\ x_2 - F(x_1) \end{pmatrix}, \tag{5.2.1}$$

where $F(x_1) = x_1 \sum_{n=0}^{\infty} a_n x_1^n \in x_1\mathbb{C}\{x_1\}$. When $F(x_1) = 0$, (5.2.1) is one of the normal forms. When $F(x_1) = x_1$, (5.2.1) was discussed in some references [12, 22], and it is well known that it has no convergent normalizations. Following the approach of [22], we can easily show that (5.2.1) has no convergent normalization if $\sum_{n=0}^{\infty} a_n/n! \neq 0$. Moreover, using the result of Chapter 2, we can show that (5.2.1) is analytically integrable if and only if $\sum_{k=0}^{\infty} a_n/n! = 0$ because its resonance degree is one.

On the other hand, Martinet and Ramis [57] studied analytic classification of differential equations of the form

$$\begin{gathered} \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^2 \\ A(x_1, x_2) \end{pmatrix}, \\ A(0, x_2) = x_2, \quad \frac{\partial^2 A}{\partial x_1 \partial x_2}(0,0) = 0. \end{gathered} \tag{5.2.2}$$

They also demonstrated their theory for Riccati-type equations, i.e., $A(x_1, x_2)$ is a polynomial over $\mathbb{C}\{x_1\}$ of degree 2 with respect to x_2 in (5.2.2). Sauzin [74] reformulated their result using the mould calculus and resurgence theory established by Écalle [29]. The mould calculus enables us to calculate normalizations explicitly. The resurgence theory deals with singular points of the Borel transformations of normalizations. He determined an infinite number of constants $\{C_j\}, j = -1, 1, 2, \ldots$, called Écalle invariants for (5.2.2). For $A(x_1, x_2) = x_2 - x_1 \sum_{n=0}^{\infty} a_n x_1^n$, the invariants are computed as $C_{-1} = -2\pi i \sum_{j=0}^{\infty} a_n/n!$ and $C_j = 0, j \geq 1$. He also proved that two vector fields written in the form (5.2.2) are analytically conjugate if and only if they have the same invariants. Hence we see that (5.2.1) is analytically conjugate to the normal form with $F(x_1) = 0$, i.e., (5.2.2) with $A(x_1, x_2) = x_2$, if and only if $\sum_{n=0} a_n/n! = 0$. Moreover, when $\sum_{n=0} a_n/n! \neq 0$, the normalization is divergent as stated in the above paragraph. To discuss a more general case, we need to extend Écalle's theory.

We next focus on a relationship between Theorem 3.4.2 and Theorem 4.3.3 (i). By Theorem 3.4.2, under assumption (A5), a sufficient condition for nonintegrability (4.1.1) is $M_+ \neq M_-^{-1}$ and $M_+ \neq M_-$. By Theorem 4.3.3 (i), a condition for existence of transverse heteroclinic orbits is $M_+ \neq M_-^{-1}$ when $\omega_+ = \omega_-$. So we conjecture that $M_+ = M_-^{-1}$ if (4.1.1) is integrable. In fact, this is true for our example (4.4.1). The condition of Theorem 3.4.2 was obtained from classification of algebraic subgroups of $\mathrm{SL}(2;\mathbb{C})$, while the condition of Theorem 4.3.3 was obtained from a geometric approach. Combining the Morales-Ramis theory with a geometric method, we may be able to improve Theorem 3.4.2 as the condition $M_+ \neq M_-$ is not included for nonintegrable systems.

In Chapter 4, we study some relationships between nonintegrability and chaos for Hamiltonian systems. It is a remaining problem to extend the result to non-Hamiltonian systems such as reversible systems. Here dynamical systems are called reversible if they are of $2n$ dimensions with $n \in \mathbb{N}$ and have linear involutions $R\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $f(Rx) + Rf(x) = 0$ and $\dim\{x \in \mathbb{R}^{2n} \mid Rx = x\} = n$. The situation treated in Chapter 4 is almost the same for reversible systems: There exist families of periodic orbits near saddle-centers by the Lyapunov theorem [25] and we can apply the result of Chapter 3. To solve the remaining problem, we need to extend the Melnikov method [92] to detect transverse heteroclinic orbits in reversible systems.

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Appendix A

Generalization of the Morales-Ramis Theory

In this appendix, we review the extension of the Morales-Ramis theory [65] due to Ayoul and Zung [6] for integrability of general differential equations in the meaning of Bogoyavlenskij. Throughout this section, we consider general complex differential equations of the form

$$\dot{x} = g(x), \quad x \in D, \tag{A.0.1}$$

where D is a domain in $\mathbb{C}^n$ and $g \colon D \to \mathbb{C}^n$ is holomorphic.

A.1 Variational equation

We begin with definitions of VEs and NVEs for (A.0.1). Let Γ be an abstract Riemann surface such that there is an immersion $i \colon \Gamma \to D$. We frequently identify $i(\Gamma)$ with Γ. Following [2, 65], we define VEs and NVEs of (A.0.1) along Γ as follows.

Let E be a meromorphic section of $T\mathbb{C}^n|_\Gamma$. The VE of (A.0.1) along Γ is given by the pullback by i of

$$L_g X = 0, \quad X \in E, \tag{A.1.1}$$

and it is a linear differential equation whose coefficient matrix has only entries in $\mathbb{K}$, where L_g represents the Lie derivative with respect to g and $\mathbb{K}$ is a differential field of meromorphic functions on Γ with derivation L_g. Since any meromorphic vector bundle over a Riemann surface is trivial (see e.g., Appendix A of [62]), we have a basis $\{e_1, \dots, e_n\}$ of E and write

$$X = \sum_{j=1}^{n} \xi_j e_j, \quad L_g e_j = \sum_{k=1}^{n} b_{jk} e_k,$$

where ξ_j, $b_{jk} \in \mathbb{K}$, $j, k = 1, \dots, n$, are uniquely determined. Hence, we obtain

$$L_g X = \sum_{j=1}^{n} (L_g \xi_j + \sum_{k=1} b_{kj} \xi_j) e_j,$$

so that the VE of (A.0.1) is written as

$$L_g \xi + B^{\mathrm{T}} \xi = 0, \tag{A.1.2}$$

where $B = (b_{jk})$, $\xi = (\xi_1, \dots, \xi_n)^{\mathrm{T}}$ and the superscript "T" represents a transpose operator.

We take $\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}$ as the basis $\{e_1, \dots, e_n\}$ of E, where $x = (x_1, \dots, x_n)^{\mathrm{T}} \in \mathbb{C}^n$. Since

$$L_g \frac{\partial}{\partial x_j} = \left[\sum_{k=1}^{n} g_k \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j}\right] = -\sum_{k=1}^{n} \frac{\partial g_k}{\partial x_j} \frac{\partial}{\partial x_k},$$

we have $b_{jk} = -\frac{\partial g_k}{\partial x_j}$, so that the VE (A.1.2) becomes

$$L_g \xi = \hat{A}(t)\xi, \quad t \in \Gamma \tag{A.1.3}$$

where $\hat{A}(t) := -B^{\mathrm{T}} = Dg(i(t))$ and $[\,\cdot, \cdot\,]$ represents the Lie bracket, as in Definition 3.1.1. Note that $\hat{A}(t)$ is holomorphic in t. Let $t = t_0$ be a point on Γ. If $g(i(t_0)) \neq 0$, then we can choose a local coordinate s near t_0 on Γ such that $s(t_0) = 0$ and $L_g = \frac{d}{ds}$, and rewrite (A.1.3) as

$$\frac{d\xi}{ds} = \hat{A}(s)\xi, \tag{A.1.4}$$

which has no singularity near $s = 0$. Assume that $g(i(t_0)) = 0$. Then there exist a local coordinate s near t_0 on Γ such that $s(t_0) = 0$ and $L_g = h(s)\frac{d}{ds}$, where $h(s)$ is a holomorphic function on Γ with $h(0) = 0$. Hence, we rewrite (A.1.3) as

$$h(s)\frac{d\xi}{ds} = \hat{A}(s)\xi, \quad \text{i.e.,} \quad \frac{d\xi}{ds} = \frac{1}{h(s)}\hat{A}(s)\xi, \tag{A.1.5}$$

which may have a singularity at $s = 0$.

Suppose that equation (A.0.1) has an $(n-m)$-dimensional locally invariant manifold $\mathscr{M}$ containing $i(\Gamma)$. Let $\bar{E}$ be a meromorphic section of $(T\mathbb{C}^n/T\mathscr{M})|_\Gamma$. Since $\mathscr{M}$ is invariant under the flow of (A.0.1), we have $L_g\colon \bar{E} \to \bar{E}$. The NVE of (A.0.1) to $\mathscr{M}$ along Γ is given by the pullback by i of

$$L_g \bar{X} = 0, \quad \bar{X} \in \bar{E}. \tag{A.1.6}$$

Letting $\{\bar{e}_1, \dots, \bar{e}_m\}$ be a basis of $\bar{E}$, we write

$$\bar{X} = \sum_{j=1}^{n} \eta_j \bar{e}_j, \quad L_g \bar{e}_j = \sum_{k=1}^{n} \bar{b}_{jk} \bar{e}_k,$$

so that equation (A.1.6) becomes

$$L_g \eta + \bar{B}^{\mathrm{T}} \eta = 0, \tag{A.1.7}$$

where $\bar{B} = (\bar{b}_{jk})$ and $\eta = (\eta_1, \dots, \eta_m)^{\mathrm{T}}$. Since $\mathscr{M}$ is of $n-m$ dimensions, there exists local coordinates $(y_1, \dots, y_n)$ in $\mathbb{C}^n$ such that $\mathscr{M}$ is represented by $y_k = 0$, $k = n-m+1, \dots, n$. We take $\left\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}\right\}$ as the basis $\{e_1, \dots, e_n\}$ of E and identify $\bar{E}$ with the vector bundle spanned by $\left\{\dfrac{\partial}{\partial y_{n-m+1}},\dots,\dfrac{\partial}{\partial y_n}\right\}$, so that $\bar{e}_1 = \dfrac{\partial}{\partial y_{n-m+1}},\dots,\bar{e}_m = \dfrac{\partial}{\partial y_n}$. Using the local invariance of $\mathscr{M}$, we express (A.1.3)

$$L_g \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{n-m} \\ \eta_1 \\ \vdots \\ \eta_m \end{pmatrix} = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ 0 & A_{22}(t) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{n-m} \\ \eta_1 \\ \vdots \\ \eta_m \end{pmatrix},$$

where $\hat{A}(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ 0 & A_{22}(t) \end{pmatrix}$ and $A_{11}(t)$, $A_{12}(t)$ and $A_{22}(t)$ are, respectively, $(n-m)\times(n-m)$, $(n-m)\times m$ and $m\times m$ matrices. Equation (A.1.7) is locally written as

$$L_g\eta = A(t)\eta, \quad A(t) := A_{22}(t), \quad t\in\Gamma,$$

which is also represented by

$$\frac{d\eta}{ds} = A(s)\eta \tag{A.1.8}$$

and

$$h(s)\frac{d\eta}{ds} = A(s)\eta, \quad \text{i.e.,} \quad \frac{d\eta}{ds} = \frac{1}{h(s)}A(s)\eta, \tag{A.1.9}$$

by the same coordinate s near $t=t_0$ on Γ as (A.1.4) and (A.1.5) if $g(i(t_0))\neq 0$ and $g(i(t_0))=0$, respectively. Notice that $A(s)$ is holomorphic in s and the eigenvalues of $A(0)$ are the same as those of $Dg(i(0))$ for which the associated eigenvectors are normal to $\mathscr{M}$.

Let $\hat{\mathrm{G}}$ and G be the differential Galois groups for the VE (A.1.3) and NVE (A.1.7), respectively. As in Section 3.2, we denote their connected identity components by the superscript "0".

Proposition A.1.1. *If $\hat{\mathrm{G}}^0$ is commutative, then so is G^0.*

The proof of this proposition is found in that of Theorem 3.4 of [2].

A.2 Necessary condition for integrability

The Morales-Ramis theory [62, 65] is a powerful technique to prove nonintegrability of Hamiltonian systems based on the differential Galois theory. Ayoul and Zung [6] extended the theory to general dynamical systems. Their fundamental idea is as follows.

We first introduce a new state variable $p\in\mathbb{C}^n$ and define a Hamiltonian function

$$H(x,p) = p^{\mathrm{T}}g(x).$$

The associated Hamiltonian system is given by

$$\dot{x} = g(x), \quad \dot{p} = -Dg(x)^{\mathrm{T}}p. \tag{A.2.1}$$

We can apply the original Morales-Ramis theory to (A.2.1).

Let $\tilde{x}(t)$ be a non-stationary solution to (A.0.1). We take an abstract Riemann surface corresponding to the solution $\tilde{x}(t)$ as Γ (see e.g. [62, 65, 46]). For example, $\tilde{x}(t)$ is doubly periodic, i.e. $\tilde{x}(z+\omega_1) = \tilde{x}(z+\omega_2) = \tilde{x}(z)$, then $\Gamma = \mathbb{C}/\{n_1\omega_1 + n_2\omega_2 | n_1, n_2 \in \mathbb{Z}\}$ is a complex torus and the associated immersion $i\colon \Gamma \to \mathbb{C}^n$ is given by $i(t) = \tilde{x}(t)$. If $\tilde{x}(t)$ is a homoclinic or heteroclinic orbit, then Γ is extended such that $i(\Gamma)$ contain the associated equilibria under some appropriate condition. See Section 4.1 of [62] or Section 4 of [65] (see also Section 4). According to the recipes of Section A.1, we obtain the VE and NVE of (A.0.1) along Γ and its differential Galois group $\hat{\mathrm{G}}$ and G, respectively. In this situation we state the result of Ayoul and Zung [6] for (A.0.1) as follows.

Theorem A.2.1. *If equation* (A.0.1) *is meromorphically integrable on* $i(\Gamma)$*, then* $\hat{\mathrm{G}}^0$ *is commutative.*

A more general situation is treated in [6]. From Proposition A.1.1 we immediately obtain the following result as a corollary of Theorem A.2.1.

Corollary A.2.2. *If equation* (A.0.1) *is meromorphically integrable on* $i(\Gamma)$*, then* G^0 *is also commutative.*

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