<table>
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<th>Title</th>
<th>Nonsmooth Fractional Programming with Generalized Ratio Invexity (Nonlinear Analysis and Convex Analysis)</th>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2004), 1365: 116-127</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/25344">http://hdl.handle.net/2433/25344</a></td>
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<td>Right</td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
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<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Nonsmooth Fractional Programming with Generalized Ratio Invexity

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Abstract: In this paper, we consider nonsmooth fractional programming problems with generalized ratio invexity. We present necessary and sufficient optimality theorems and establish duality theorems for nonsmooth fractional programming under suitable $\rho$-invexity assumptions.

1 Introduction

We consider the following nonsmooth fractional programming problem:

\[(NFP) \text{ Minimize } \frac{f(x)}{g(x)} \text{ subject to } x \in X = \{x \in \mathbb{R}^n | h_j(x) \leq 0, j = 1, \cdots, m\},\]

where $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$, and $h_j : \mathbb{R} \rightarrow \mathbb{R}$, $j = l, \cdots, m$, are locally Lipschitz functions. We assume in the sequel that $f(x) \geq 0$ and $g(x) > 0$ on $\mathbb{R}^n$.


* This research of author was supported by the grant No. R01-2003-000-10825-0 from the Basic Research Program of KOSEF
Recently, Liang et al. [7] introduced the concept of \((F, \alpha, \rho, d)\)-convexity and presented optimality and duality results for a class of nonlinear fractional programming problems under generalized convexity and the properties of sublinear functional. In this paper, we present a result about the fractional objective function based on \(\rho\)-invexity assumptions. By using \(\rho\)-invexity of fractional function, we obtain necessary and sufficient optimality conditions and duality theorems for nonsmooth fractional programming problems.

2 Definitions and Generalized Invexity of Fractional Function

The real-valued function \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) is said to be locally Lipschitz if for any \(z \in \mathbb{R}^n\) there exists a positive constant \(K\) and a neighborhood \(N\) of \(z\) such that, for each \(x, y \in N\),

\[
|f(x) - f(y)| \leq K\|x - y\|
\]

where \(\|\cdot\|\) denotes any norm in \(\mathbb{R}^n\). The Clarke generalized directional derivative of a locally Lipschitz function \(f\) at \(x\) in the direction \(d \in \mathbb{R}^n\), denote by \(f^0(x; d)\), is defined as follows:

\[
f^0(x; d) = \limsup_{y \to x} t^{-1}(f(y + td) - f(y)),
\]

where \(y\) is a vector in \(\mathbb{R}^n\).

The Clarke generalized subgradient of \(f\) at \(x\) is denoted by

\[
\partial f(x) = \{\xi : f^0(x; d) \geq \xi d, \ \forall d \in \mathbb{R}^n\}.
\]

**Definition 2.1** \(f\) is said to be regular at \(x\) if for all \(d \in \mathbb{R}^n\) the one-sided directional derivative \(f'(x; d)\) exists and \(f'(x; d) = f^0(x; d)\).

**Definition 2.2** A locally Lipschitz function \(f : X_0 \rightarrow \mathbb{R}\) is said to be \(\rho\)-invex at \(x_0 \in X_0\) with respect to functions \(\eta\) and \(\theta : X_0 \times X_0 \rightarrow \mathbb{R}^n\) if there exists \(\rho \in \mathbb{R}\) such that for any \(x \in X_0\), and any \(\xi \in \partial f(x_0)\),

\[
f(x) - f(x_0) \geq \xi \eta(x, x_0) + \rho \|\theta(x, x_0)\|^2,
\]
where $\theta(x, x_0) \neq 0$ if $x \neq x_0$.

When $\rho = 0$, the definition of $\rho$-invexity reduces to the notion of invexity in the sense of Hanson [2].

**Remark.** When $f$ is of class $C^1$ in Definition 2.2, then the above inequality reduces to

$$f(x) - f(x_0) \geq f'_x \eta + \rho ||\theta(x, x_0)||^2$$

where $f'_x$ is the Frechet derivative of $f$ at $x$.

**Theorem 2.1** If $f$ and $-g$ are $\rho$-invex with respect to $\eta$ and $\theta$, and $f$ and $-g$ are regular at $x_0$, then the fractional objective function $f(x)/g(x)$ is $\rho$-invex with respect to $\bar{\eta}$ and $\bar{\theta}$, where

$$\bar{\eta}(x, x_0) = (g(x_0)/g(x))\eta(x, x_0), \quad \bar{\theta}(x, x_0) = (1/g(x))^{1/2}\theta(x, x_0).$$

**Proof.** Let $x, x_0 \in X_0$. By the $\rho$-invexity of $f$ and $-g$, we have

$$f(x)/g(x) - f(x_0)/g(x_0) = (f(x) - f(x_0))/g(x) - f(x_0)(g(x) - g(x_0))/g(x)g(x_0) \geq (1/g(x))\xi\eta(x, x_0) + \rho \|(1/g(x))^{1/2}\theta(x, x_0)\|^2$$

$$+ (f(x_0)/(g(x)g(x_0))(-\zeta\eta(x, x_0) + \rho ||\theta(x, x_0)||^2),$$

for any $x \in X_0$, any $\xi \in \partial f(x_0)$ and any $\zeta \in \partial g(x_0)$. Since $f(x) \geq 0$ and $g(x) > 0$,

$$f(x)/g(x) - f(x_0)/g(x_0) \geq (g(x_0)/g(x))((\xi/g(x_0))\eta(x, x_0) + (-f(x_0)\zeta/(g^2(x_0)))\eta(x, x_0))$$

$$+ \rho \|(1/g(x))^{1/2}(1 + (f(x_0)/g(x_0)))^{1/2}\theta(x, x_0)\|^2.$$ 

Since $f$ and $-g$ are regular at $x_0$, we obtain, for any $\delta \in \partial(f(x_0)/g(x_0))$,

$$f(x)/g(x) - f(x_0)/g(x_0) \geq (g(x_0)/g(x))\delta\eta(x, x_0) + \rho \|(1/g(x))^{1/2}(1 + (f(x_0)/g(x_0)))^{1/2}\theta(x, x_0)\|^2.$$

Considering that

$$1 + f(x_0)/g(x_0) \geq 1,$$
we have
\[
\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)} \geq (g(x_0)/g(x)) \delta \eta(x, x_0) + \rho \| (1/g(x))^{1/2} \theta(x, x_0) \|^2.
\]
Therefore, the function \( f(x)/g(x) \) is \( \rho \)-invex, where
\[
\eta(x, x_0) = (g(x_0)/g(x)) \eta(x, x_0),
\]
\[
\theta(x, x_0) = (1/g(x))^{1/2} \theta(x, x_0).
\]

3 Optimality Conditions

The Cottle constraint qualification

The Cottle constraint qualification is satisfied at \( x_0 \) if either \( h_j(x_0) < 0 \) for all \( j = 1, \cdots, m \) or \( 0 \notin \text{conv}\{\partial h_j(x_0) : h_j(x_0) = 0\} \), where \( \text{conv} S \) denotes the convex hull of a set \( S \).

By Theorem 6.1.1 in [1], we can present the following Fritz John necessary conditions.

Theorem 3.1 (Fritz John Necessary Conditions). If \( x_0 \in X \) is an optimal solution of (NFP), then there exist \( \lambda \) and \( r_j, j = 1, 2, \cdots, m \), such that

\[
0 \in \lambda \partial \left( \frac{f(x_0)}{g(x_0)} \right) + \sum_{j=1}^{m} r_j \partial h_j(x_0),
\]
\[
\sum_{j=1}^{m} r_j h_j(x_0) = 0,
\]
\[(\lambda, r_1, \cdots, r_m) \geq 0 \text{ and } (\lambda, r_1, \cdots, r_m) \neq 0.\]

Assuming the Cottle constraint qualification, we obtain the Karush-Kuhn-Tucker necessary conditions.
Theorem 3.2 (Karush-Kuhn-Tucker Necessary Conditions). Assume that \( x_0 \in X \) is an optimal solution for \((NFP)\) at which the Cottle constraint qualification is satisfied. Then there exist \( \mu_j \geq 0, j = 1, 2, \ldots, m \), such that

\[
0 \in \partial \left( \frac{f(x_0)}{g(x_0)} \right) + \sum_{j=1}^{m} \mu_j \partial h_j(x_0),
\]

\[
\sum_{j=1}^{m} \mu_j h_j(x_0) = 0,
\]

\[
(\mu_1, \cdots, \mu_m) \geq 0.
\]

Theorem 3.3 (Karush-Kuhn-Tucker Sufficient Conditions). Let \((x_0, \mu)\) satisfy the Karush-Kuhn-Tucker conditions as follows:

\[
0 \in \partial \left( \frac{f(x_0)}{g(x_0)} \right) + \sum_{j=1}^{m} \mu_j \partial h_j(x_0),
\]

\[
\sum_{j=1}^{m} \mu_j h_j(x_0) = 0,
\]

\[
(\mu_1, \cdots, \mu_m) \geq 0.
\]

Assume that \( f \) and \(-g\) are \( \rho \)-invex at \( x_0 \) with respect to \( \eta \) and \( \theta \), and \( f \) and \(-g\) are regular at \( x_0 \), and \( h_j \) is \( \rho_j' \)-invex at \( x_0 \) with respect to \( \overline{\eta} \) and \( \overline{\theta} \) with \( \rho + \sum_{j=1}^{m} \mu_j \rho_j' \geq 0 \).

Then \( x_0 \) is an optimal solution of \((NFP)\).

Proof. Let \( x_0, x \in X \) and \((x_0, \mu)\) satisfy the Karush-Kuhn-Tucker conditions. Then there exist \( \delta \in \partial(f(x_0)/g(x_0)) \) and \( \gamma_j \in \partial h_j(x_0) \) such that \( \delta + \sum_{j=1}^{m} \mu_j \gamma_j = 0 \) and \( \sum_{j=1}^{m} \mu_j h_j(x_0) = 0 \).

Since \( f \) and \(-g\) are \( \rho \)-invex at \( x_0 \) with respect to \( \eta \) and \( \theta \) and regular at \( x_0 \),
then by Theorem 2.1 we have
\[
f(x)/g(x) - f(x_0)/g(x_0) \\
\geq (-g(x)/g(x_0)) \sum_{j=1}^{m} \mu_j \gamma_j \eta(x, x_0) + \rho \| (1/g(x))^{1/2} \theta(x, x_0) \|^2 \\
- \sum_{j=1}^{m} \mu_j h_j(x_0) + \sum_{j=1}^{m} \mu_j h_j(x).
\]

Since \( h_j \) is \( \rho'_j \)-invex at \( x_0 \) with respect to \( \overline{\eta} \) and \( \overline{\theta} \), we obtain
\[
f(x)/g(x) - f(x_0)/g(x_0) \\
\geq (\rho + \sum_{j=1}^{m} \mu_j \rho'_j) \| \overline{\theta}(x, x_0) \|^2 \\
\geq 0.
\]
Therefore, \( x_0 \) is an optimal solution of (NFP).

4 Duality Theorems

We consider the following Mond-Weir dual problem to (NFP):

\[
\text{(NFD)}_M \quad \text{Maximize} \quad \frac{f(u)}{g(u)}
\]

subject to \( 0 \in \partial \left( f(u)/g(u) \right) + \sum_{j=1}^{m} \mu_j \partial h_j(u), \)
\[
\sum_{j=1}^{m} \mu_j h_j(u) \geq 0,
\]
\[
(\mu_1, \cdots, \mu_m) \geq 0.
\]

Theorem 4.1 (Weak Duality). Let \( x \) be feasible for (NFP) and \( (u, \mu) \) feasible for \( \text{(NFD)}_M \). Assume that \( f \) and \(-g\) are \( \rho \)-invex with respect to \( \eta \) and \( \theta \), and \( f \) and \(-g\) are regular functions, and \( h_j \) is \( \rho'_j \)-invex with respect to \( \overline{\eta} \) and \( \overline{\theta} \) with \( \rho + \sum_{j=1}^{m} \mu_j \rho'_j \geq 0. \)
Then

\[
\frac{f(x)}{g(x)} \geq \frac{f(u)}{g(u)}.
\]

Proof. Since \( f \) and \(-g\) are \( \rho \)-invex with respect to \( \eta \) and \( \theta \), and regular, and \((u, \mu)\) is feasible for \((\text{NFD})_M\), then by Theorem 2.1 we have

\[
\frac{f(x)}{g(x)} - \frac{f(u)}{g(u)} \geq (-g(u)/g(x)) \sum_{j=1}^{m} \mu_j \gamma_j \eta(x, u) + \rho ||(1/g(x))^{1/2}\theta(x, u)||^2,
\]

for some \( \gamma_j \in \partial h_j(u) \). Since \( h_j \) is \( \rho_j \)-invex with respect to \( \overline{\eta} \) and \( \overline{\theta} \), we obtain

\[
\frac{f(x)}{g(x)} - \frac{f(u)}{g(u)} \geq (\rho + \sum_{j=1}^{m} \mu_j \rho_j') ||\overline{\theta}(x, u)||^2 \geq 0.
\]

**Theorem 4.2 (Strong Duality).** Let \( \bar{x} \) be an optimal solution for \((\text{NFP})\) at which the Cottle constraint qualification is satisfied. Then there exists \( \bar{\mu} \) such that \((\bar{x}, \bar{\mu})\) is feasible for \((\text{NFD})_M\). Moreover, if \( f, g \) and \( h \) satisfy the conditions of Theorem 4.1, then \((\bar{x}, \bar{\mu})\) is an optimal solution of \((\text{NFD})_M\) and the optimal values of \((\text{NFP})\) and \((\text{NFD})_M\) are equal.

Proof. From the Karush-Kuhn-Tucker necessary conditions, there exists \( \bar{\mu}_j \geq 0, j = 1, 2, \ldots, m \) such that

\[
0 \in \partial \left( \frac{f(\bar{x})}{g(\bar{x})} \right) + \sum_{j=1}^{m} \bar{\mu}_j \partial h_j(\bar{x}),
\]

\[
\sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}) = 0.
\]

Thus \((\bar{x}, \bar{\mu})\) is feasible for \((\text{NFD})_M\). So, by Theorem 4.1, \((\bar{x}, \bar{\mu})\) is an optimal solution of \((\text{NFD})_M\).
Theorem 4.3 (Strict Converse Duality). Let \( \bar{x} \) be feasible for \((NFP)\) and \((\bar{u}, \bar{\mu})\) be feasible for \((NFD)_M\) such that \( f(\bar{x})/g(\bar{x}) \leq f(\bar{u})/g(\bar{u}) \). Assume that \( f \) and \(-g\) are \( \rho\)-invex at \( \bar{u} \) with respect to \( \eta \) and \( \theta \), and \( f \) and \(-g\) are regular at \( \bar{u} \), and \( h_j \) is \( \rho'_j\)-invex with respect to \( \bar{\eta} \) and \( \bar{\theta} \) with \( \rho + \sum_{j=1}^{m} \bar{\mu}_j \rho'_j \geq 0 \).

Then
\[
\bar{x} = \bar{u}.
\]

Proof. Since \( f \) and \(-g\) are \( \rho\)-invex at \( \bar{u} \) with respect to \( \eta \) and \( \theta \), and regular at \( \bar{u} \) and \((\bar{u}, \bar{\mu})\) is feasible for \((NFD)_M\), then by Theorem 2.1 we have
\[
f(\bar{u})/g(\bar{u}) - f(\bar{x})/g(\bar{x})
\leq (g(\bar{u})/g(\bar{x})) \sum_{j=1}^{m} \bar{\mu}_j \gamma_j \eta(\bar{x}, \bar{u}) - \rho \|(1/g(\bar{x}))^{1/2} \theta(\bar{x}, \bar{u})\|^2
+ \sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{u}) - \sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}),
\]
for some \( \gamma_j \in \partial h_j(\bar{u}) \). Since \( h_j \) is \( \rho'_j\)-invex with respect to \( \bar{\eta} \) and \( \bar{\theta} \), we obtain
\[
f(\bar{u})/g(\bar{u}) - f(\bar{x})/g(\bar{x})
\leq -(\rho + \sum_{j=1}^{m} \bar{\mu}_j \rho'_j) \|\bar{\theta}(\bar{x}, \bar{u})\|^2
\leq 0.
\]
Thus \( \bar{x} = \bar{u} \).

We propose the following Wolfe dual problem to \((NFP)\):

\[
\text{(NFD)}_W \quad \text{Maximize} \quad \frac{f(u)}{g(u)} + \sum_{j=1}^{m} \mu_j h_j(u)
\]
subject to \( \sum_{j=1}^{m} \mu_j \partial h_j(u) \leq 0, \)
\( \mu_1, \cdots, \mu_m \geq 0. \)
Theorem 4.4 (Weak Duality). Let \( x \) be feasible for (NFP) and \((u, \mu)\) feasible for (NFD)\(_W\).
Assume that \( f \) and \(-g\) are \( \rho \)-invex with respect to \( \eta \) and \( \theta \), and \( f \) and \(-g\) are regular functions, and \( h_j \) is \( \rho'_j \)-invex with respect to \( \overline{\eta} \) and \( \overline{\theta} \) with \( \rho + \sum_{j=1}^{m} \mu_j \rho'_j \geq 0 \).
Then
\[
\frac{f(x)}{g(x)} \leq \frac{f(u)}{g(u)} + \sum_{j=1}^{m} \mu_j h_j(u).
\]

Proof. Since \( f \) and \(-g\) are \( \rho \)-invex with respect to \( \eta \) and \( \theta \), regular and \((u, \mu)\) is feasible for (NFD)\(_W\), then by Theorem 2.1 we have
\[
f(x)/g(x) - ((f(u)/g(u)) + \sum_{j=1}^{m} \mu_j h_j(u)) \\
\geq (-g(u)/g(x)) \sum_{j=1}^{m} \mu_j \gamma_j \eta(x, y) + \rho \|1/g(x)^{1/2}\theta(x, u)\|^2 - \sum_{j=1}^{m} \mu_j h_j(u)
\]
for some \( \gamma_j \in \partial h_j(u) \). Since \( h_j \) is \( \rho'_j \)-invex with respect to \( \overline{\eta} \) and \( \overline{\theta} \), we obtain
\[
f(x)/g(x) - ((f(u)/g(u)) + \sum_{j=1}^{m} \mu_j h_j(u)) \\
\geq - \sum_{j=1}^{m} \mu_j h_j(x) + (\rho + \sum_{j=1}^{m} \mu_j \rho'_j) \|\overline{\theta}(x, u)\|^2 \\
\geq 0.
\]

Theorem 4.5 (Strong Duality). Let \( \bar{x} \) be an optimal solution for (NFP) at which the Cottle constraint qualification is satisfied. Then there exists \( \bar{\mu} \) such that \((\bar{x}, \bar{\mu})\) is feasible for (NFD)\(_W\).
Moreover, if \( f, g \) and \( h \) satisfy the conditions of Theorem 4.4, then \((\bar{x}, \bar{\mu})\) is an optimal solution of (NFD)\(_W\) and the optimal values of (NFP) and (NFD)\(_W\) are equal.
Proof. From the Karush-Kuhn-Tucker necessary conditions, there exists $\bar{\mu}_j \geq 0$, $j = 1, 2, \ldots, m$ such that

$$0 \in \partial \left( \frac{f(\bar{x})}{g(\bar{x})} \right) + \sum_{j=1}^{m} \bar{\mu}_j \partial h_j(\bar{x}),$$

$$\sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}) = 0.$$

Thus $(\bar{x}, \bar{\mu})$ is feasible for $(NFD)_W$. So, by Theorem 4.4, $(\bar{x}, \bar{\mu})$ is an optimal solution of $(NFD)_W$.

Theorem 4.6 (Strict Converse Duality). Let $\hat{x}$ be an optimal solution for $(NFP)$ at which the Cottle constraint qualification is satisfied. Assume that $f$ and $-g$ are $\rho$-invex at $\hat{x}$ with respect to $\eta$ and $\theta$, and $f$ and $-g$ are regular at $\hat{x}$, and $h_j$ is $\rho_j'$-invex with respect to $\overline{\eta}$ and $\overline{\theta}$ with $\rho + \sum_{j=1}^{m} \hat{\mu}_j \rho_j' > 0$. If $(\hat{x}, \hat{\mu})$ is an optimal solution of $(NFD)_W$, then $\hat{x} = \bar{x}$ and the optimal values of $(NFP)$ and $(NFD)_W$ are equal.

Proof. Assume that $\hat{x} \neq \bar{x}$. Since $\bar{x}$ is an optimal solution of $(NFP)$, there exists $\bar{\mu} \geq 0$ such that $(\bar{x}, \bar{\mu})$ is an optimal solution of $(NFD)_W$. Then

$$f(\bar{x})/g(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}) = f(\hat{x})/g(\hat{x}) + \sum_{j=1}^{m} \hat{\mu}_j h_j(\hat{x}) = \max_{(x, \mu) \in Y} \frac{f(x)}{g(x)} + \sum_{j=1}^{m} \mu_j h_j(x)$$

where $Y$ is a feasible set of $(NFD)_W$. Because $(\hat{x}, \hat{\mu}) \in Y$, we have

$$0 \in \partial \left( \frac{f(\hat{x})}{g(\hat{x})} \right) + \sum_{j=1}^{m} \hat{\mu}_j \partial h_j(\hat{x}).$$

Since $f$ and $-g$ are $\rho$-invex at $\hat{x}$ with respect to $\eta$ and $\theta$, and regular at $\hat{x}$, then by Theorem 2.1 we have

$$f(\hat{x})/g(\hat{x}) - f(\bar{x})/g(\bar{x}) \geq (-g(\bar{x})/g(\bar{x})) \sum_{j=1}^{m} \hat{\mu}_j \gamma_j \eta(\bar{x}, \hat{x}) + \rho \| (1/g(\bar{x}))^{1/2} \theta(\bar{x}, \hat{x}) \|^2$$
for some $\gamma_j \in \partial h_j(\hat{x})$. Since $h_j$ is $\rho_j'$-invex with respect to $\bar{\eta}$ and $\bar{\theta}$, we obtain

$$f(\bar{x})/g(\bar{x}) + \sum_{j=1}^{m} \hat{\mu}_j h_j(\bar{x}) - (f(\hat{x})/g(\hat{x}) + \sum_{j=1}^{m} \hat{\mu}_j h_j(\hat{x}))$$

$$\geq (\rho + \sum_{j=1}^{m} \hat{\mu}_j \rho_j') ||\bar{\theta}(\bar{x}, \mathrm{i})||^2 > 0.$$ 

It follows then that

$$f(\bar{x})/g(\bar{x}) + \sum_{j=1}^{m} \hat{\mu}_j h_j(\bar{x})$$

$$> f(\hat{x})/g(\hat{x}) + \sum_{j=1}^{m} \hat{\mu}_j h_j(\hat{x}) = f(\bar{x})/g(\bar{x}) + \sum_{j=1}^{m} \hat{\mu}_j h_j(\bar{x})$$

or that

$$\sum_{j=1}^{m} \hat{\mu}_j h_j(\bar{x}) > \sum_{j=1}^{m} \hat{\mu}_j h_j(\bar{x}).$$

But from Theorem 3.2, we have that $\sum_{j=1}^{m} \hat{\mu}_j h_j(\bar{x}) = 0$, hence $\sum_{j=1}^{m} \hat{\mu}_j h_j(\bar{x}) > 0$ which contradicts the facts that $\hat{\mu}_j \geq 0$ and $h_j(\bar{x}) \leq 0$. Hence $\hat{x} = \bar{x}$.

**References**


