Nonsmooth Fractional Programming with Generalized Ratio Invexity

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Abstract: In this paper, we consider nonsmooth fractional programming problems with generalized ratio invexity. We present necessary and sufficient optimality theorems and establish duality theorems for nonsmooth fractional programming under suitable \( \rho \)-invexity assumptions.

1 Introduction

We consider the following nonsmooth fractional programming problem:

\[
\text{(NFP)} \quad \text{Minimize} \quad \frac{f(x)}{g(x)}
\]

subject to \( x \in X = \{ x \in \mathbb{R}^n | h_j(x) \leq 0, \quad j = 1, \ldots, m \} \),

where \( f, g : \mathbb{R}^n \rightarrow \mathbb{R} \) and \( h_j : \mathbb{R} \rightarrow \mathbb{R}, \quad j = l, \ldots, m \), are locally Lipschitz functions. We assume in the sequel that \( f(x) \geq 0 \) and \( g(x) > 0 \) on \( \mathbb{R}^n \).


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Recently, Liang et al. [7] introduced the concept of $(F, \alpha, \rho, d)$-convexity and presented optimality and duality results for a class of nonlinear fractional programming problems under generalized convexity and the properties of sublinear functional. In this paper, we present a result about the fractional objective function based on $\rho$-invexity assumptions. By using $\rho$-invexity of fractional function, we obtain necessary and sufficient optimality conditions and duality theorems for nonsmooth fractional programming problems.

2 Definitions and Generalized Invexity of Fractional Function

The real-valued function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be locally Lipschitz if for any $z \in \mathbb{R}^n$ there exists a positive constant $K$ and a neighborhood $N$ of $z$ such that, for each $x, y \in N$,

$$|f(x) - f(y)| \leq K\|x - y\|,$$

where $\| \cdot \|$ denotes any norm in $\mathbb{R}^n$. The Clarke generalized directional derivative of a locally Lipschitz function $f$ at $x$ in the direction $d \in \mathbb{R}^n$, denote by $f^0(x; d)$, is defined as follows:

$$f^0(x; d) = \lim_{y \rightarrow x} \sup_{t \downarrow 0} t^{-1}(f(y + td) - f(y)),$$

where $y$ is a vector in $\mathbb{R}^n$.

The Clarke generalized subgradient of $f$ at $x$ is denoted by

$$\partial f(x) = \{\xi : f^0(x; d) \geq \xi d, \ \forall d \in \mathbb{R}^n\}.$$

**Definition 2.1** $f$ is said to be regular at $x$ if for all $d \in \mathbb{R}^n$ the one-sided directional derivative $f'(x; d)$ exists and $f'(x; d) = f^0(x; d)$.

**Definition 2.2** A locally Lipschitz function $f : X_0 \rightarrow \mathbb{R}$ is said to be $\rho$-invex at $x_0 \in X_0$ with respect to functions $\eta$ and $\theta : X_0 \times X_0 \rightarrow \mathbb{R}^n$ if there exists $\rho \in \mathbb{R}$ such that for any $x \in X_0$, and any $\xi \in \partial f(x_0)$,

$$f(x) - f(x_0) \geq \xi \eta(x, x_0) + \rho \|\theta(x, x_0)\|^2,$$
where \( \theta(x, x_0) \neq 0 \) if \( x \neq x_0 \).

When \( \rho = 0 \), the definition of \( \rho \)-invexity reduces to the notion of invexity in the sense of Hanson [2].

**Remark.** When \( f \) is of class \( C^1 \) in Definition 2.2, then the above inequality reduces to

\[
 f(x) - f(x_0) \geq f'_x \eta + \rho \|\theta(x, x_0)\|^2
\]

where \( f'_x \) is the Fréchet derivative of \( f \) at \( x_0 \).

**Theorem 2.1** If \( f \) and \(-g\) are \( \rho \)-invex with respect to \( \eta \) and \( \theta \), and \( f \) and \(-g\) are regular at \( x_0 \), then the fractional objective function \( f(x)/g(x) \) is \( \rho \)-invex with respect to \( \overline{\eta} \) and \( \overline{\theta} \), where

\[
 \eta(x, x_0) = (g(x_0)/g(x))\eta(x,x_0), \quad \overline{\theta}(x, x_0) = (1/g(x))^{1/2}\theta(x,x_0).
\]

**Proof.** Let \( x, x_0 \in X_0 \). By the \( \rho \)-invexity of \( f \) and \(-g\), we have

\[
 f(x)/g(x) - f(x_0)/g(x_0) = \frac{(f(x) - f(x_0))}{g(x)} - \frac{f(x_0)(g(x) - g(x_0))}{g(x)g(x_0)} \geq (1/g(x))\xi \eta(x, x_0) + \rho \|(1/g(x))^{1/2}\theta(x, x_0)\|^2
\]

\[
 + \frac{(f(x_0))/(g(x)g(x_0))(-\zeta \eta(x, x_0) + \rho \|(\frac{1}{g(x)})^{1/2}(1 + (f(x_0)/g(x_0)))^{1/2}\theta(x, x_0)\|^2)}{g(x_0)/g(x)} \delta \eta(x, x_0) + \rho \|(\frac{1}{g(x)})^{1/2}(1 + (f(x_0)/g(x_0)))^{1/2}\theta(x, x_0)\|^2.
\]

Since \( f \) and \(-g\) are regular at \( x_0 \), we obtain, for any \( \delta \in \partial(f(x_0)/g(x_0)) \),

\[
 f(x)/g(x) - f(x_0)/g(x_0) \geq (g(x_0)/g(x))\delta \eta(x, x_0) + \rho \|(1/g(x))^{1/2}(1 + (f(x_0)/g(x_0)))^{1/2}\theta(x, x_0)\|^2.
\]

Considering that

\[
 1 + f(x_0)/g(x_0) \geq 1,
\]
we have

\[
f(x)/g(x) - f(x_0)/g(x_0) \\
\geqq (g(x_0)/g(x))\delta\eta(x, x_0) + \rho\|(1/g(x))^{1/2}\theta(x, x_0)\|^2.
\]

Therefore, the function \( f(x)/g(x) \) is \( \rho \)-invex, where

\[
\bar{\eta}(x, x_0) = (g(x_0)/g(x))\eta(x, x_0), \\
\bar{\theta}(x, x_0) = (1/g(x))^{1/2}\theta(x, x_0).
\]

## 3 Optimality Conditions

### The Cottle constraint qualification

The Cottle constraint qualification is satisfied at \( x_0 \) if either \( h_j(x_0) < 0 \) for all \( j = 1, \cdots, m \) or \( 0 \notin \text{conv}\{\partial h_j(x_0) : h_j(x_0) = 0\} \), where \( \text{conv}S \) denotes the convex hull of a set \( S \).

By Theorem 6.1.1 in [1], we can present the following Fritz John necessary conditions.

**Theorem 3.1 (Fritz John Necessary Conditions).** If \( x_0 \in X \) is an optimal solution of (NFP), then there exist \( \lambda \) and \( r_j, j = 1, 2, \cdots, m \), such that

\[
0 \in \lambda\partial\left(\frac{f(x_0)}{g(x_0)}\right) + \sum_{j=1}^{m} r_j\partial h_j(x_0),
\]

\[
\sum_{j=1}^{m} r_j h_j(x_0) = 0,
\]

\[
(\lambda, r_1, \cdots, r_m) \geqq 0 \text{ and } (\lambda, r_1, \cdots, r_m) \neq 0.
\]

Assuming the Cottle constraint qualification, we obtain the Karush-Kuhn-Tucker necessary conditions.
Theorem 3.2 (Karush-Kuhn-Tucker Necessary Conditions). Assume that \( x_0 \in X \) is an optimal solution for (NFP) at which the Cottle constraint qualification is satisfied. Then there exist \( \mu_j \geq 0, j = 1, 2, \cdots, m \), such that
\[
0 \in \partial \left( \frac{f(x_0)}{g(x_0)} \right) + \sum_{j=1}^{m} \mu_j \partial h_j(x_0),
\]
\[
\sum_{j=1}^{m} \mu_j h_j(x_0) = 0,
\]
\[
(\mu_1, \cdots, \mu_m) \geq 0.
\]

Theorem 3.3 (Karush-Kuhn-Tucker Sufficient Conditions). Let \((x_0, \mu)\) satisfy the Karush-Kuhn-Tucker conditions as follows:
\[
0 \in \partial \left( \frac{f(x_0)}{g(x_0)} \right) + \sum_{j=1}^{m} \mu_j \partial h_j(x_0),
\]
\[
\sum_{j=1}^{m} \mu_j h_j(x_0) = 0,
\]
\[
(\mu_1, \cdots, \mu_m) \geq 0.
\]

Assume that \( f \) and \(-g\) are \( \rho \)-invex at \( x_0 \) with respect to \( \eta \) and \( \theta \), and \( f \) and \(-g\) are regular at \( x_0 \), and \( h_j \) is \( \rho_j' \)-invex at \( x_0 \) with respect to \( \overline{\eta} \) and \( \overline{\theta} \) with \( \rho + \sum_{j=1}^{m} \mu_j \rho_j' \geq 0 \).

Then \( x_0 \) is an optimal solution of (NFP).

Proof. Let \( x_0, x \in X \) and \((x_0, \mu)\) satisfy the Karush-Kuhn-Tucker conditions. Then there exist \( \delta \in \partial(f(x_0)/g(x_0)) \) and \( \gamma_j \in \partial h_j(x_0) \) such that \( \delta + \sum_{j=1}^{m} \mu_j \gamma_j = 0 \) and \( \sum_{j=1}^{m} \mu_j h_j(x_0) = 0 \).

Since \( f \) and \(-g\) are \( \rho \)-invex at \( x_0 \) with respect to \( \eta \) and \( \theta \) and regular at \( x_0 \),
then by Theorem 2.1 we have
\[
\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)} \geq (-\frac{g(x)}{g(x_0)}) \sum_{j=1}^{m} \mu_j \gamma_j \eta(x, x_0) + \rho \| (1/g(x))^{1/2} \theta(x, x_0) \|^2 
- \sum_{j=1}^{m} \mu_j h_j(x_0) + \sum_{j=1}^{m} \mu_j h_j(x).
\]

Since $h_j$ is $\rho'_j$-invex at $x_0$ with respect to $\overline{\eta}$ and $\overline{\theta}$, we obtain
\[
\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)} \geq (\rho + \sum_{j=1}^{m} \mu_j \rho'_j) \| \overline{\theta}(x, x_0) \|^2 
- \sum_{j=1}^{m} \mu_j h_j(x_0) \geq 0.
\]

Therefore, $x_0$ is an optimal solution of (NFP).

4 Duality Theorems

We consider the following Mond-Weir dual problem to (NFP):

\begin{align*}
\text{(NFD)}_M & \quad \text{Maximize} \quad \frac{f(u)}{g(u)} \\
& \quad \text{subject to} \quad 0 \in \partial \left(\frac{f(u)}{g(u)}\right) + \sum_{j=1}^{m} \mu_j \partial h_j(u), \\
& \quad \sum_{j=1}^{m} \mu_j h_j(u) \geq 0, \\
& \quad (\mu_1, \cdots, \mu_m) \geq 0.
\end{align*}

**Theorem 4.1 (Weak Duality).** Let $x$ be feasible for (NFP) and $(u, \mu)$ feasible for (NFD)$_M$. Assume that $f$ and $-g$ are $\rho$-invex with respect to $\eta$ and $\theta$, and $f$ and $-g$ are regular functions, and $h_j$ is $\rho'_j$-invex with respect to $\overline{\eta}$ and $\overline{\theta}$ with $\rho + \sum_{j=1}^{m} \mu_j \rho'_j \geq 0$. 
Then
\[
\frac{f(x)}{g(x)} \geq \frac{f(u)}{g(u)}.
\]

Proof. Since \( f \) and \(-g\) are \( \rho \)-invex with respect to \( \eta \) and \( \theta \), and regular, and \((u, \mu)\) is feasible for \((\text{NFD})_M\), then by Theorem 2.1 we have
\[
\frac{f(x)}{g(x)} - \frac{f(u)}{g(u)} \geq (-g(u)/g(x)) \sum_{j=1}^{m} \mu_j \gamma^j \eta(x, u) + \rho \|(1/g(x))^{1/2} \theta(x, u)\|^2,
\]
for some \( \gamma_j \in \partial h_j(u) \). Since \( h_j \) is \( \rho^j \)-invex with respect to \( \overline{\eta} \) and \( \overline{\theta} \), we obtain
\[
\frac{f(x)}{g(x)} - \frac{f(u)}{g(u)} \geq (\rho + \sum_{j=1}^{m} \mu_j \rho^j) \|\overline{\theta}(x, u)\|^2
\]
\[\geq 0.\]

Theorem 4.2 (Strong Duality). Let \( \bar{x} \) be an optimal solution for \((\text{NFP})\) at which the Cottle constraint qualification is satisfied. Then there exists \( \bar{\mu} \) such that \((\bar{x}, \bar{\mu})\) is feasible for \((\text{NFD})_M\). Moreover, if \( f, g \) and \( h \) satisfy the conditions of Theorem 4.1, then \((\bar{x}, \bar{\mu})\) is an optimal solution of \((\text{NFD})_M\) and the optimal values of \((\text{NFP})\) and \((\text{NFD})_M\) are equal.

Proof. From the Karush-Kuhn-Tucker necessary conditions, there exists \( \bar{\mu}_j \geq 0, j = 1, 2, \ldots, m \) such that
\[
0 \in \partial \left( \frac{f(\bar{x})}{g(\bar{x})} \right) + \sum_{j=1}^{m} \bar{\mu}_j \partial h_j(\bar{x}),
\]
\[
\sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}) = 0.
\]
Thus \((\bar{x}, \bar{\mu})\) is feasible for \((\text{NFD})_M\). So, by Theorem 4.1, \((\bar{x}, \bar{\mu})\) is an optimal solution of \((\text{NFD})_M\).
Theorem 4.3 (Strict Converse Duality). Let $\overline{x}$ be feasible for (NFP) and $(\overline{u}, \overline{\mu})$ be feasible for $(NFD)_M$ such that $f(\overline{x})/g(\overline{x}) \leq f(\overline{u})/g(\overline{u})$. Assume that $f$ and $-g$ are $\rho$-invex at $\overline{u}$ with respect to $\eta$ and $\theta$, and $f$ and $-g$ are regular at $\overline{u}$, and $h_j$ is $\rho_j'$-invex with respect to $\overline{\eta}$ and $\overline{\theta}$ with $\rho + \sum_{j=1}^{m} \overline{\mu}_j \rho_j' \geq 0$.

Then

$$\overline{x} = \overline{u}.$$

Proof. Since $f$ and $-g$ are $\rho$-invex at $\overline{u}$ with respect to $\eta$ and $\theta$, and regular at $\overline{u}$ and $(\overline{u}, \overline{\mu})$ is feasible for $(NFD)_M$, then by Theorem 2.1 we have

$$f(\overline{u})/g(\overline{u}) - f(\overline{x})/g(\overline{x})$$

$$\leq (g(\overline{u})/g(\overline{x})) \sum_{j=1}^{m} \overline{\mu}_j \gamma_j \eta(\overline{x}, \overline{u}) - \rho ||(1/g(\overline{x}))^{1/2} \theta(\overline{x}, \overline{u})||^2$$

$$+ \sum_{j=1}^{m} \overline{\mu}_j h_j(\overline{u}) - \sum_{j=1}^{m} \overline{\mu}_j h_j(\overline{x}),$$

for some $\gamma_j \in \partial h_j(\overline{u})$. Since $h_j$ is $\rho_j'$-invex with respect to $\overline{\eta}$ and $\overline{\theta}$, we obtain

$$f(\overline{u})/g(\overline{u}) - f(\overline{x})/g(\overline{x})$$

$$\leq -(\rho + \sum_{j=1}^{m} \overline{\mu}_j \rho_j') ||\theta(\overline{x}, \overline{u})||^2$$

$$\leq 0.$$

Thus $\overline{x} = \overline{u}$.

We propose the following Wolfe dual problem to (NFP):

$$(NFD)_W \text{ Maximize } f(u) + \sum_{j=1}^{m} \mu_j h_j(u)$$

subject to $0 \in \partial (f(u)/g(u)) + \sum_{j=1}^{m} \mu_j \partial h_j(u)$,

$$(\mu_1, \cdots, \mu_m) \geq 0.$$
Theorem 4.4 (Weak Duality). Let $x$ be feasible for $(NFP)$ and $(u, \mu)$ feasible for $(NFD)_W$. Assume that $f$ and $-g$ are $\rho$-invex with respect to $\eta$ and $\theta$, and $f$ and $-g$ are regular functions, and $h_j$ is $\rho_j'$-invex with respect to $\tilde{\eta}$ and $\tilde{\theta}$ with $\rho + \sum_{j=1}^{m} \mu_j \rho_j' \geq 0$.

Then

$$\frac{f(x)}{g(x)} \geq \frac{f(u)}{g(u)} + \sum_{j=1}^{m} \mu_j h_j(u).$$

Proof. Since $f$ and $-g$ are $\rho$-invex with respect to $\eta$ and $\theta$, regular and $(u, \mu)$ is feasible for $(NFD)_W$, then by Theorem 2.1 we have

$$f(x)/g(x) - ((f(u)/g(u)) + \sum_{j=1}^{m} \mu_j h_j(u))$$

$$\geq (-g(u)/g(x)) \sum_{j=1}^{m} \mu_j \gamma_j \eta(x, y) + \rho ||(1/g(x)^{1/2}\theta(x, u)||^2 - \sum_{j=1}^{m} \mu_j h_j(u)$$

for some $\gamma_j \in \partial h_j(u)$. Since $h_j$ is $\rho_j'$-invex with respect to $\tilde{\eta}$ and $\tilde{\theta}$, we obtain

$$f(x)/g(x) - ((f(u)/g(u)) + \sum_{j=1}^{m} \mu_j h_j(u))$$

$$\geq -\sum_{j=1}^{m} \mu_j h_j(x) + (\rho + \sum_{j=1}^{m} \mu_j \rho_j') ||\tilde{\theta}(x, u)||^2$$

$$\geq 0.$$

Theorem 4.5 (Strong Duality). Let $\bar{x}$ be an optimal solution for $(NFP)$ at which the Cottle constraint qualification is satisfied. Then there exists $\bar{\mu}$ such that $(\bar{x}, \bar{\mu})$ is feasible for $(NFD)_W$. Moreover, if $f$, $g$ and $h$ satisfy the conditions of Theorem 4.4, then $(\bar{x}, \bar{\mu})$ is an optimal solution of $(NFD)_W$ and the optimal values of $(NFP)$ and $(NFD)_W$ are equal.
Proof. From the Karush-Kuhn-Tucker necessary conditions, there exists $\overline{\mu}_j \geq 0, j = 1, 2, \ldots, m$ such that

$$0 \in \partial \left( \frac{f(\bar{x})}{g(\bar{x})} \right) + \sum_{j=1}^{m} \overline{\mu}_j \partial h_j(\bar{x}),$$

$$\sum_{j=1}^{m} \overline{\mu}_j h_j(\bar{x}) = 0.$$ 

Thus $(\bar{x}, \overline{\mu})$ is feasible for $(\text{NFD})_W$. So, by Theorem 4.4, $(\bar{x}, \overline{\mu})$ is an optimal solution of $(\text{NFD})_W$.

Theorem 4.6 (Strict Converse Duality). Let $\hat{x}$ be an optimal solution for $(\text{NFP})$ at which the Cottle constraint qualification is satisfied. Assume that $f$ and $-g$ are $\rho$-invex at $\hat{x}$ with respect to $\eta$ and $\theta$, and $f$ and $-g$ are regular at $\hat{x}$, and $h_j$ is $\rho_j'$-invex with respect to $\overline{\eta}$ and $\overline{\theta}$ with $\rho + \sum_{j=1}^{m} \hat{\mu}_j \rho_j' > 0$. If $(\hat{x}, \hat{\mu})$ is an optimal solution of $(\text{NFD})_W$, then $\hat{x} = \bar{x}$ and the optimal values of $(\text{NFP})$ and $(\text{NFD})_W$ are equal.

Proof. Assume that $\hat{x} \neq \bar{x}$. Since $\bar{x}$ is an optimal solution of $(\text{NFP})$, there exists $\bar{\mu} \geq 0$ such that $(\bar{x}, \bar{\mu})$ is an optimal solution of $(\text{NFD})_W$. Then

$$f(\bar{x})/g(\bar{x}) + \sum_{j=1}^{m} \overline{\mu}_j h_j(\bar{x}) = f(\bar{x})/g(\bar{x}) + \sum_{j=1}^{m} \mu_j h_j(\bar{x}) = \max_{(x, \mu) \in Y} \frac{f(x)}{g(x)} + \sum_{j=1}^{m} \mu_j h_j(x)$$

where $Y$ is a feasible set of $(\text{NFD})_W$. Because $(\hat{x}, \hat{\mu}) \in Y$, we have

$$0 \in \partial \left( \frac{f(\hat{x})}{g(\hat{x})} \right) + \sum_{j=1}^{m} \hat{\mu}_j \partial h_j(\hat{x}).$$

Since $f$ and $-g$ are $\rho$-invex at $\hat{x}$ with respect to $\eta$ and $\theta$, and regular at $\hat{x}$, then by Theorem 2.1 we have

$$f(\hat{x})/g(\hat{x}) - f(\bar{x})/g(\bar{x}) \geq (-g(\hat{x})/g(\hat{x})) \sum_{j=1}^{m} \overline{\mu}_j \gamma_i \eta(x, \hat{x}) + \rho \| (1/g(\hat{x}))^{1/2} \theta(x, \hat{x}) \|^2.$$
for some $\gamma_j \in \partial h_j(\hat{x})$. Since $h_j$ is $\rho_j'$-invex with respect to $\bar{\eta}$ and $\bar{\theta}$, we obtain

$$f(\bar{x})/\bar{g}(\bar{x}) + \sum_{j=1}^{m} \hat{\mu}_j h_j(\bar{x}) - (f(\hat{x})/g(\hat{x}) + \sum_{j=1}^{m} \hat{\mu}_j h_j(\hat{x})) \geq \left( \rho + \sum_{j=1}^{m} \hat{\mu}_j \rho_j' \right) ||\bar{\theta}(\bar{x}, \bar{x})||^2 > 0.$$ 

It follows then that

$$f(\bar{x})/\bar{g}(\bar{x}) + \sum_{j=1}^{m} \hat{\mu}_j h_j(\bar{x}) > f(\hat{x})/g(\hat{x}) + \sum_{j=1}^{m} \hat{\mu}_j h_j(\hat{x}),$$

or that

$$\sum_{j=1}^{m} \hat{\mu}_j h_j(\bar{x}) > \sum_{j=1}^{m} \hat{\mu}_j h_j(\hat{x}).$$

But from Theorem 3.2, we have that $\sum_{j=1}^{m} \hat{\mu}_j h_j(\bar{x}) = 0$, hence $\sum_{j=1}^{m} \hat{\mu}_j h_j(\bar{x}) > 0$ which contradicts the facts that $\hat{\mu}_j \geq 0$ and $h_j(\bar{x}) \leq 0$. Hence $\hat{x} = \bar{x}$.

References


