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An Extension of Vector Variational Inequalities With Operator Solutions

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Abstract. In a recent paper, Domokos and Kolumbán introduced variational inequalities with operator solutions to provide a suitable unified approach to several kinds of variational inequality and vector variational inequality in Banach spaces. Inspired by their work, we further develop the new scheme of vector variational inequalities with operator solutions from the single-valued case into the multi-valued one. We prove the existence of solutions of generalized vector variational inequalities with operator solutions. Some applications to generalized vector variational inequalities in a normed space are also provided.

Key Words. Vector variational inequality, C-pseudomonotone operator, generalized hemicontinuity, Fan-Browder fixed point theorem

1. Introduction

Since Giannessi [5] introduced the vector variational inequality, (shortly, VVI) in a finite dimensional Euclidean space, many authors have intensively studied (VVI) and its various extensions [1, 6, 7, 10] (see also the references therein) in abstract spaces.

Several authors have investigated relationships between (VVI) and vector optimization problems, vector complementarity problem [2, 8].

In a recent paper, Domokos and Kolumbán [3] gave an interesting interpretation of variational inequalities (VI) and (VVI) in Banach space settings in terms of variational inequalities with operator solutions (in short, OVVI). They first obtained an existence theorem of the solutions of (OVVI) using Fan's KKM Lemma [4], and then presented a general version of Yu and Yao [10, Theorem 3.3] in a Banach space as a main application and gave some other applications such as the solvability of variational inequality defined on Hausdorff topological vector space, and that of variational inequality on $L^\infty(\Omega)$. However, they dealt with only the single-valued operator.

Inspired by their work, in this report, we further develop the new scheme of (OVVI) from the single-valued case into the multi-valued one, and search some applications, from a theoretical point of view, to exploit the framework of (OVVI). To be more specific, we establish a multi-valued version of (OVVI) called the generalized vector variational inequality with operator solutions (in short, GOVVI). As an application of (GOVVI), we provide a noncompact generalization of Konnov and Yao [6, Theorem 3.1] concerning a generalized (VVI) in a normed space (not necessarily Banach space). In addition, we deal with an existence theorem on (VVI) concerned with upper semicontinuity of multifunction instead of pseudo-monotonicity. As basic tools to obtain main results, we use a Fan-Browder type fixed point theorem due to Park [9, Theorem 5].

2. Preliminaries

Let E, F be Hausdorff topological vector spaces, and let X be a nonempty convex subset of E. A nonempty subset P of E is called *a convex cone* if

$$\lambda P \subset P, \quad \text{for all} \quad \lambda > 0 \quad \text{and} \quad P + P = P.$$

Let $C_1 : X \rightrightarrows F$ be a multifunction such that for each $x \in X$, $C_1(x)$ is a convex cone in F with $\operatorname{int} C_1(x) \neq \emptyset$ and $C_1(x) \neq F$. Let $L(E, F)$ be the space of all continuous linear operators from E to F and $T_1 : X \rightrightarrows L(E, F)$ a multifunction.

Then T_1 is said to be

(1) C_1-*pseudomonotone* if for any $x, y \in X$ and for any $s \in T_1(x)$, we have

$$\langle s, y - x\rangle \notin -\mathrm{int}C_1(x) \ \text{ implies } \ \langle t, y - x\rangle \notin -\mathrm{int}C_1(x) \ \text{ for all } \ t \in T_1(y); \text{ and}$$

(2) *generalized hemicontinuous* if for any $x, y \in X$ and $\alpha \in [0, 1]$, the multifunction

$$\alpha \mapsto \langle T_1(x + \alpha(y - x)), y - x\rangle$$

is upper semicontinuous at 0^+, where

$$\langle T_1(x + \alpha(y - x)), y - x\rangle = \{\langle s, y - x\rangle \mid s \in T_1(x + \alpha(y - x))\}.$$

Now we pay our attention to generalized variational inequalities with operator solutions (in short, GOVVI). From now on, unless otherwise specified, we work under the following settings:

Let X' be a nonempty convex subset of $L(E, F)$ and $T : X' \rightrightarrows E$ be a multifunction. Let $C : X' \rightrightarrows F$ be a multifunction such that for each $f \in X'$, $C(f)$ is a convex cone in F with $0 \notin C(f)$. Then (GOVVI) is defined as follows:

$$\text{Find } f_0 \in X' \text{ such that } \forall f \in X', \exists x \in T(f_0) \text{ with } \langle f - f_0, x\rangle \notin C(f_0). \qquad (GOVVI)$$

When T is single-valued, (GOVVI) reduces to (OVVI) due to Domokos and Kolumbán [3]. As pointed out in [3], the notation (GOVVI) is motivated by the fact that the solutions are sought in the space of continuous linear operators.

In regard to monotonicity and continuity of T, two analogous definitions to those of T_1 in the above are necessary; $T : X' \rightrightarrows E$ is said to be

(1)$'$ *C-pseudomonotone* if for any $f, g \in X'$ and for any $s \in T(f)$, we have

$$\langle g - f, s\rangle \notin C(f) \text{ implies } \langle g - f, t\rangle \notin C(f) \text{ for all } t \in T(g); \text{ and}$$

(2)$'$ *generalized hemicontinuous* if for any $f, g \in X'$ and $\alpha \in [0, 1]$, the multifunction

$$\alpha \mapsto \langle g - f, T(f + \alpha(g - f))\rangle$$

is upper semicontinuous at 0^+, where

$$\langle g - f, T(f + \alpha(g - f))\rangle = \{\langle g - f, s\rangle \mid s \in T(f + \alpha(g - f))\}.$$

In order to prove our main result, we need the following fixed point theorem which is a particular form of Park [9, Theorem 5].

Lemma 2.1. Let X be a nonempty convex subset of a real (not necessarily) Hausdorff topological vector space E, K a nonempty compact subset of X. Let A, $B : X \rightrightarrows X$ be two multifunctions. Suppose that

(i) for each $x \in X$, $Ax \subset Bx$;

(ii) for each $x \in X$, Bx is convex;

(iii) for each $x \in K$, Ax is nonempty ;

(iv) for each $y \in X$, $A^{-1}y = \{x \in X \mid y \in Ax\}$ is open in X; and

(v) for each finite subset N of X, there exists a nonempty compact convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, $Ax \cap L_N \neq \emptyset$.

Then B has a fixed point x_0; that is, $x_0 \in Bx_0$.

3. Generalized vector variational inequality with operator soultions

We begin with the following lemma to get the main result.

Lemma 3.1. Let $T : X' \rightrightarrows E$ be a C-pseudomonotone and generalized hemicontinuous multifunction with $T(f) \neq \emptyset$ for all $f \in X'$. Let $W : X' \rightrightarrows F$ be defined by $W(f) = F \setminus C(f)$ such that the graph $Gr(W)$ of W is closed in $X' \times F$ where $L(E, F)$ is endowed with the topology of pointwise convergence. Then the following two problems are equivalent:

(i) Find $f \in X'$ such that $\forall g \in X'$, $\exists x \in T(f)$ with $\langle g - f, x\rangle \notin C(f)$.

(ii) Find $f \in X'$ such that $\forall g \in X'$, $\forall x \in T(g)$, $\langle g - f, x\rangle \notin C(f)$.

Using Lemma 3.1, we prove the following which is a multi-valued version of (OVVI) in [3] :

Theorem 3.1. Let $T : X' \rightrightarrows E$ be a C-pseudomonotone and generalized hemicontinuous multifunction with $T(f) \neq \emptyset$ for all $f \in X'$. Let $W : X' \rightrightarrows F$ be defined by $W(f) = F \setminus C(f)$ such that the graph $Gr(W)$ of W is closed in $X' \times F$ where $L(E, F)$ is endowed with the topology of pointwise convergence. Let K' be a nonempty compact subset of X'. Assume that for each finite subset N' of X', there exists a nonempty compact convex subset $L_{N'}$ of X' containing N' such that for each $f \in L_{N'} \setminus K'$, there exists $g \in L_{N'}$ satisfying

$$\langle g - f, x\rangle \in C(f) \quad \text{for some} \quad x \in T(g).$$

Then (GOVVI) is solvable.

Proof. First note that $L(E,F)$ equipped with the topology of pointwise convergence is a Hausdorff t.v.s. We define two multifunctions $A,\ B : X' \rightrightarrows X'$ to be

$$A(f) := \{g \in X' \mid \exists x \in T(g) \text{ such that } \langle g - f, x\rangle \in C(f)\},$$

$$B(f) := \{g \in X' \mid \forall x \in T(f),\ \langle g - f, x\rangle \in C(f)\}.$$

The proof is organized in the following parts.

(i) Since T is C-pseudomonotone, we have $A(f) \subset B(f)$ for all $f \in X'$.

(ii) For each $f \in X'$, $B(f)$ is convex. Indeed, let g_1 and g_2 be in $B(f)$. For all $t \in [0,1]$ and $x \in T(f)$, we have

$$\langle tg_1 + (1-t)g_2 - f, x\rangle = t\langle g_1 - f, x\rangle + (1-t)\langle g_2 - f, x\rangle \in C(f),$$

which implies that $tg_1 + (1-t)g_2 \in B(f)$. Hence $B(f)$ is convex.

(iii) Clearly B has no fixed point because $0 \notin C(f)$ for all $f \in X'$.

(iv) For each $g \in X'$, $A^{-1}(g)$ is open in X'. In fact, let $\{f_\lambda\}$ be a net in $(A^{-1}(g))^c$ convergent to $f \in X'$. Then $g \notin A(f_\lambda)$ and hence for each $x \in T(g)$,

$$\langle g - f_\lambda, x\rangle \notin C(f_\lambda).$$

Thus $\langle g - f_\lambda, x\rangle \in W(f_\lambda)$. Since $(f_\lambda, \langle g - f_\lambda, x\rangle) \in Gr(W)$ and $L(E,F)$ is endowed with the topology of pointwise convergence, by virtue of the closedness of $Gr(W)$, we have $(f, \langle g - f, x\rangle) \in Gr(W)$, that is, $\langle g - f, x\rangle \notin C(f)$ for every $x \in T(g)$. Hence $g \notin A(f)$, so $f \in (A^{-1}(g))^c$. This shows that $(A^{-1}(g))^c$ is closed, therefore $A^{-1}(g)$ is open in X'.

(v) By the given hypothesis, we know that for each finite subset N' of X', there exists a nonempty compact convex subset $L_{N'}$ of X' containing N' such that for each $f \in L_{N'} \setminus K'$, there exists $g \in L_{N'}$ satisfying $g \in A(f)$, hence $L_{N'} \cap A(f) \neq \emptyset$.

(vi) From (i)-(v), we see, by Lemma 2.1, there must be an $f_0 \in K'$ such that $A(f_0) = \emptyset$, namely,

$$\langle g - f_0, x \rangle \notin C(f_0) \text{ for any } g \in X',\ x \in T(g).$$

It follows from Lemma 3.1 that f_0 is a solution of (GOVVI). This completes the proof.

As an application of Theorem 3.1 in multi-valued settings, we can obtain the existence of a solution of a generalized (VVI) in a normed space :

Theorem 3.2. Let Y and Z be two normed spaces. Let X be a nonempty convex subset of Y and $C_1 : X \rightrightarrows Z$ be a multifunction such that for each $x \in X$, $C_1(x)$ is a convex cone in Z with $\mathrm{int}C_1(x) \neq \emptyset$ and $C_1(x) \neq Z$. Let $T_1 : X \rightrightarrows L(Y, Z)$ be a C_1-pseudomonotone and generalized hemicontinuous multifunction with nonempty values. Let $W_1 : X \rightrightarrows Z$ be defined by $W_1(x) = Z \setminus -\mathrm{int}C_1(x)$ such that the graph $Gr(W_1)$ of W_1 is weakly closed in $X \times Z$. Assume that K is a nonempty weakly compact subset of X and for each finite subset N of X, there exists a nonempty weakly compact convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, there exists $y \in L_N$ satisfying

$$\langle s, y - x \rangle \in -\mathrm{int}C_1(x) \quad \text{for some} \quad s \in T_1(y).$$

Then there exists $x_0 \in X$ such that

$$\forall x \in X,\ \exists t \in T_1(x_0) \text{ with } \langle t, x - x_0 \rangle \notin -\mathrm{int}C_1(x_0).$$

Remark 3.1. Theorem 3.2 is a noncompact generalization of Konnov and Yao [6, Theorem 3.1] in normed spaces (not necessarily Banach spaces) without assuming the convex cone $C_1(x)$ being closed.

Now we are interested in (GOVVI) concerned with the upper semicontinuity of T instead of pseudomonotonicity and hemicontinuity. To this end, we replace the topology of pointwise convergence by that of bounded convergence on $L(E, F)$.

Theorem 3.3. Suppose that $L(E, F)$ is endowed with the topology of bounded convergence. Let $T : X' \rightrightarrows E$ be an upper semicontinuous multifunction such that $T(f)$ is a nonempty compact subset of E for all $f \in X'$, and the range $T(X')$ is contained in a compact subset of E. Let $W : X' \rightrightarrows F$ be defined by $W(f) = F \setminus C(f)$ such that the graph $Gr(W)$ of W is closed in $X' \times F$. Let K' be a nonempty compact subset of X'. Assume that for each finite subset N' of X', there exists a nonempty compact convex subset $L_{N'}$ of X' containing N' such that for each $f \in L_{N'} \setminus K'$, there exists $g \in L_{N'}$ satisfying

$$\langle g - f, x \rangle \in C(f) \quad \text{for all} \quad x \in T(f).$$

Then (GOVVI) is solvable.

As a direct consequence of Theorem 3.3, we obtain the following.

Theorem 3.4. Let Y and Z be two normed spaces. Let X be a nonempty convex subset of Y and $C_1 : X \rightrightarrows Z$ be a multifunction such that for each $x \in X$, $C_1(x)$ is a convex cone in Z with $\text{int}C_1(x) \neq \emptyset$ and $C_1(x) \neq Z$. Let $T_1 : X \rightrightarrows L(Y, Z)$ be an upper semicontinuous multifunction with nonempty compact values and the range $T_1(X)$ be contained in a compact subset of $L(Y, Z)$ where $L(Y, Z)$ is the normed space of the continuous linear operators between Y and Z with the usual norm. Let $W_1 : X \rightrightarrows Z$ be defined by $W_1(x) = Z \setminus -\text{int}C_1(x)$ such that the graph $Gr(W_1)$ of W_1 is closed in $X \times Z$. Assume that K is a nonempty compact subset of X and for each finite subset

N of X, there exists a nonempty compact convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, there exists $y \in L_N$ satisfying

$$\langle s, y - x \rangle \in -\mathrm{int}C_1(x) \quad \text{for all} \quad s \in T_1(x).$$

Then there exists $x_0 \in X$ such that

$$\forall x \in X, \ \exists t \in T_1(x_0) \text{ with } \langle t, x - x_0 \rangle \notin -\mathrm{int}C_1(x_0).$$

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