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THREE-STEP ITERATIVE SEQUENCES WITH ERRORS FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN CONVEX METRIC SPACES

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ABSTRACT. In this paper, we will give some necessary and sufficient conditions for three-step iterative sequences with errors to converge to a fixed point for asymptotically quasi-nonexpansive mappings in convex metric spaces. The results of this paper are generalizations and improvements of the corresponding results of Chang, Kim *et al.*, Liu and Xu-Noor.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, we assume that E is a metric space, $F(T)$ and $D(T)$ are the set of all fixed points and domain of T respectively and $\mathbb{N}$ is the set of all positive integers.

Definition 1.1. Let $T : D(T) \subset E \to E$ be a mapping.

(1) The mapping T is said to be *nonexpansive* if

$$d(Tx, Ty) \leq d(x, y), \qquad \forall\, x, y \in D(T).$$

(2) The mapping T is said to be *quasi-nonexpansive* if

$$d(Tx, p) \leq d(x, p), \qquad \forall\, x \in D(T),\ \forall\, p \in F(T).$$

(3) The mapping T is said to be *asymptotically nonexpansive* if there exists a sequence $k_n \in [0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that

$$d(T^n x, T^n y) \leq (1 + k_n) d(x, y), \qquad \forall\, x, y \in D(T),\ \forall\, n \in \mathbb{N}.$$

(4) The mapping T is said to be *asymptotically quasi-nonexpansive* if there exists a sequence $k_n \in [0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that

$$d(T^n x, p) \leq (1 + k_n) d(x, p), \qquad \forall\, x \in D(T),\ \forall\, p \in F(T),\ \forall\, n \in \mathbb{N}.$$

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Remark 1.1. From the Definition 1.1, it follows that if $F(T)$ is nonempty, then a nonexpansive mapping is quasi-nonexpansive, and an asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive. But the converse does not hold.

The iterative approximation problems of fixed points for asymptotically nonexpansive mappings or asymptotically quasi-nonexpansive mappings in Hilbert spaces or Banach spaces have been studied extensively by many authors. In 1973, Petryshyn-Williamson [11] obtained a necessary and sufficient condition for Picard iterative sequences and Mann iterative sequences to converge to a fixed point for quasi-nonexpansive mappings and later, the result of [11] was extended by Ghosh-Debnath [5] to Ishikawa iterative sequences. And recently, Chang [1-3] has proved some another kinds of necessary and sufficient conditions for Ishikawa iterative sequences with errors to converge to a fixed point for asymptotically nonexpansive mappings and Xu-Noor [14] has established a convergence theorem of three-step iterative sequences with errors for asymptotically nonexpansive mappings in uniformly convex Banach spaces. In particular, Liu [7] obtained a necessary and sufficient condition for Ishikawa iterative sequences of asymptotically quasi-nonexpansive mappings in Banach spaces to converge to a fixed point and he [8] has also extended his result [7] to Ishikawa iterative sequences with errors. Furthermore, Kim *et al.* [6] extended to modified three-step iterative sequences with mixed errors.

Rhoades [12] and Naimpally-Singh [10] suggest the following open question.

Open Question. Can the Ishikawa iterative procedure be extended to nonlinear quasi-contractive mapping in a metric space?

Recently, Chang-Kim [4] proved convergence theorems of the Ishikawa type iterative sequences with errors for generalized quasi-contractive mappings in convex metric spaces.

The purpose of this paper is to study some necessary and sufficient conditions for three-step iterative sequences with errors to converge to fixed points for asymptotically quasi-nonexpansive mappings in convex metric spaces. The results of this paper are generalizations and improvements of the corresponding results in Chang [1-3], Ghosh-Debnath [5], Kim *et al.* [6], Liu [7-9] and Xu-Noor [14].

For the sake of convenience, we first recall some definitions and notations.

Definition 1.2. Let (E,d) be a metric space and $I=[0,1]$. A mapping $W: E^3 \times I^3 \to E$ is said to be *a convex structure* on E if it satisfies the following conditions : for all $u,x,y,z \in E$ and for all $\alpha,\beta,\gamma \in I$ with $\alpha+\beta+\gamma=1$,

(1) $W(x,y,z;\alpha,0,0)=x$,

(2) $d(u,W(x,y,z;\alpha,\beta,\gamma)) \le \alpha d(u,x)+\beta d(u,y)+\gamma d(u,z)$.

If (E,d) is a metric space with a convex structure W, then (E,d) is called *a convex metric space* and denotes it by (E,d,W).

Remark 1.2. Every linear normed space is a convex metric space, where a convex structure $W(x,y,z;\alpha,\beta,\gamma) = \alpha x + \beta y + \gamma z$, for all $x,y,z \in E$ and $\alpha,\beta,\gamma \in I$ with $\alpha+\beta+\gamma = 1$. But there exist some convex metric spaces which can not be embedded into any linear normed spaces (see, Takahashi [13]).

Definition 1.3. (1) Let (E,d,W) be a convex metric space, $T: E \to E$ be a mapping and let $x_1 \in E$ be a given point. Then the sequence $\{x_n\}$ defined by

$$\begin{cases} x_{n+1} = W(x_n, T^n y_n, u_n; a_n, b_n, c_n), \\ y_n = W(x_n, T^n z_n, v_n; \bar{a_n}, \bar{b_n}, \bar{c_n}), \\ z_n = W(x_n, T^n x_n, w_n; \hat{a_n}, \hat{b_n}, \hat{c_n}), \quad \forall\, n \in \mathbb{N}, \end{cases} \tag{1.1}$$

is called *the three-step iterative sequence with errors* for the mapping T, where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\bar{a_n}\}$, $\{\bar{b_n}\}$, $\{\bar{c_n}\}$, $\{\hat{a_n}\}$, $\{\hat{b_n}\}$ and $\{\hat{c_n}\}$ are nine sequences in $[0,1]$ satisfying the following conditions:

$$a_n + b_n + c_n = \bar{a_n} + \bar{b_n} + \bar{c_n} = \hat{a_n} + \hat{b_n} + \hat{c_n} = 1, \quad \forall\, n \in \mathbb{N},$$

and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are three bounded sequences in E.

(2) In (1.1), if $\hat{b_n} \equiv 0$ and $\hat{c_n} \equiv 0$, for all $n = 1,2,\cdots$, then $z_n = x_n$. Then the sequence $\{x_n\}$ defined by

$$\begin{cases} x_{n+1} = W(x_n, T^n y_n, u_n; a_n, b_n, c_n), \\ y_n = W(x_n, T^n x_n, v_n; \bar{a_n}, \bar{b_n}, \bar{c_n}), \quad \forall\, n \in \mathbb{N}, \end{cases} \tag{1.2}$$

is called *the Ishikawa type (or two-step) iterative sequence with errors* for the mapping T, where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\bar{a_n}\}$, $\{\bar{b_n}\}$ and $\{\bar{c_n}\}$ are six sequences in $[0,1]$ satisfying the following conditions :

$$a_n + b_n + c_n = \bar{a_n} + \bar{b_n} + \bar{c_n} = 1, \quad \forall\, n \in \mathbb{N},$$

and $\{u_n\}$, $\{v_n\}$ are two bounded sequences in E.

2. Main results

In order to obtain the main theorems, we will first prove the following lemma.

Lemma 2.1. *Let (E,d,W) be a convex metric space, $T: E \to E$ be an asymptotically quasi-nonexpansive mapping satisfying $\sum_{n=1}^{\infty} k_n < \infty$ where $\{k_n\}$ is the sequence appeared in Definition 1.1, and $F(T)$ be a nonempty set. For a given $x_1 \in E$, let $\{x_n\}$ be the three-step iterative sequence with errors defined by (1.1). Then*

(a) $d(x_{n+1}, p) \le (1+k_n)^3 d(x_n, p) + h_n, \ \forall\, p \in F(T),\ n \in \mathbb{N}$,
where $h_n = b_n(1+k_n)\theta_n + c_n d(u_n, p)$, $\theta_n = \bar{b_n}\hat{c_n}(1+k_n)d(w_n,p) + \bar{c_n} d(v_n,p)$ *and* $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ *are three bounded sequences in* E.

(b) *there exists a constant* $M > 0$ *such that*
$d(x_m, p) \le M d(x_n, p) + M \sum_{j=n}^{m-1} h_j, \quad \forall\, p \in F(T),\ m > n.$

Proof. (a) Since T is asymptotically quasi-nonexpansive, for each $p \in F(T)$,

$$\begin{aligned} d(x_{n+1}, p) &= d(W(x_n, T^n y_n, u_n; a_n, b_n, c_n), p) \\ &\leq a_n d(x_n, p) + b_n d(T^n y_n, p) + c_n d(u_n, p) \\ &\leq a_n d(x_n, p) + b_n (1 + k_n) d(y_n, p) + c_n d(u_n, p), \end{aligned} \tag{2.1}$$

$$\begin{aligned} d(y_n, p) &= d(W(x_n, T^n z_n, v_n; \bar{a_n}, \bar{b_n}, \bar{c_n}), p) \\ &\leq \bar{a_n} d(x_n, p) + \bar{b_n} d(T^n z_n, p) + \bar{c_n} d(v_n, p) \\ &\leq \bar{a_n} d(x_n, p) + \bar{b_n} (1 + k_n) d(z_n, p) + \bar{c_n} d(v_n, p) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} d(z_n, p) &= d(W(x_n, T^n x_n, w_n; \hat{a_n}, \hat{b_n}, \hat{c_n}), p) \\ &\leq \hat{a_n} d(x_n, p) + \hat{b_n} d(T^n x_n, p) + \hat{c_n} d(w_n, p) \\ &\leq \hat{a_n} d(x_n, p) + \hat{b_n} (1 + k_n) d(x_n, p) + \hat{c_n} d(w_n, p). \end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2), we have

$$\begin{aligned} d(y_n, p) &\leq \bar{a_n} d(x_n, p) \\ &\quad + \bar{b_n}(1 + k_n)\{\hat{a_n} d(x_n, p) + \hat{b_n}(1 + k_n) d(x_n, p) \\ &\quad + \hat{c_n} d(w_n, p)\} + \bar{c_n} d(v_n, p) \\ &= \bar{a_n} d(x_n, p) + \bar{b_n}\hat{a_n}(1 + k_n) d(x_n, p) \\ &\quad + \bar{b_n}\hat{b_n}(1 + k_n)^2 d(x_n, p) + \bar{b_n}\hat{c_n}(1 + k_n) d(w_n, p) \\ &\quad + \bar{c_n} d(v_n, p) \\ &\leq \bar{a_n} d(x_n, p) + \bar{b_n}\hat{a_n}(1 + k_n)^2 d(x_n, p) \\ &\quad + \bar{b_n}\hat{b_n}(1 + k_n)^2 d(x_n, p) + \theta_n \\ &\leq \bar{a_n} d(x_n, p) + \bar{b_n}(1 + k_n)^2 d(x_n, p) + \theta_n, \end{aligned} \tag{2.4}$$

where $\theta_n = \bar{b_n}\hat{c_n}(1 + k_n) d(w_n, p) + \bar{c_n} d(v_n, p)$. And again, substituting (2.4) into (2.1), it follows that

$$
\begin{aligned}
d(x_{n+1}, p) &\le a_n d(x_n, p) \\
&\quad + b_n(1+k_n)\{\bar{a_n} d(x_n, p) + \bar{b_n}(1+k_n)^2 d(x_n, p) + \theta_n\} \\
&\quad + c_n d(u_n, p) \\
&= a_n d(x_n, p) + b_n \bar{a_n}(1+k_n) d(x_n, p) \\
&\quad + b_n \bar{b_n}(1+k_n)^3 d(x_n, p) + b_n(1+k_n)\theta_n + c_n d(u_n, p) \\
&= a_n d(x_n, p) + (1 - a_n - c_n)\bar{a_n}(1+k_n) d(x_n, p) \\
&\quad + (1 - a_n - c_n)\bar{b_n}(1+k_n)^3 d(x_n, p) + h_n \\
&\le a_n d(x_n, p) + (1 - a_n)\bar{a_n}(1+k_n)^3 d(x_n, p) \\
&\quad + (1 - a_n)\bar{b_n}(1+k_n)^3 d(x_n, p) + h_n \\
&\le a_n(1+k_n)^3 d(x_n, p) \\
&\quad + (1 - a_n)(1+k_n)^3(\bar{a_n} + \bar{b_n}) d(x_n, p) + h_n \\
&\le a_n(1+k_n)^3 d(x_n, p) + (1 - a_n)(1+k_n)^3 d(x_n, p) + h_n \\
&= (1+k_n)^3 d(x_n, p) + h_n,
\end{aligned}
$$

where $h_n = b_n(1+k_n)\theta_n + c_n d(u_n, p)$. This completes the proof of (a).

(b) If $a \ge 0$, then $1 + a \le e^a$ and $(1+a)^3 \le e^{3a}$. Therefore, from (a) we can obtain that

$$
\begin{aligned}
d(x_m, p) &\le (1 + k_{m-1})^3 d(x_{m-1}, p) + h_{m-1} \\
&\le e^{3k_{m-1}} d(x_{m-1}, p) + h_{m-1} \\
&\le e^{3k_{m-1}}[(1 + k_{m-2})^3 d(x_{m-2}, p) + h_{m-2}] + h_{m-1} \\
&\le e^{3(k_{m-1}+k_{m-2})} d(x_{m-2}, p) + e^{3k_{m-1}} h_{m-2} + h_{m-1} \\
&\le e^{3(k_{m-1}+k_{m-2})} d(x_{m-2}, p) + e^{3k_{m-1}} (h_{m-1} + h_{m-2}) \\
&\le \cdots \\
&\le e^{3\sum_{j=n}^{m-1} k_j} d(x_n, p) + e^{3\sum_{j=n}^{m-1} k_j} \sum_{j=n}^{m-1} h_j \\
&\le M d(x_n, p) + M \sum_{j=n}^{m-1} h_j,
\end{aligned}
$$

where $M = e^{3\sum_{j=n}^{\infty} k_j}$. This completes the proof of (b). □

We also need the following lemma for the proof of our main results.

Lemma 2.2 [8]. *Let the number of sequences* $\{a_n\}$, $\{b_n\}$ *and* $\{\lambda_n\}$ *satisfy that* $a_n \ge 0$, $b_n \ge 0$, $\lambda_n \ge 0$, $a_{n+1} \le (1 + \lambda_n)a_n + b_n$, $\forall\, n \in \mathbb{N}$, $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$. *Then*

(a) $\lim_{n\to\infty} a_n$ *exist.*

(b) *if* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Now, we are in a position to prove the main theorems. $D_d(y, S)$ denotes the distance from y to set S, that is, $D_d(y, S) = \inf\{d(y, s) : s \in S\}$.

Theorem 2.1. *Let (E, d, W) be a complete convex metric space, $T : E \to E$ be an asymptotically quasi-nonexpansive mapping and $F(T)$ be a nonempty set. For a given $x_1 \in E$, let $\{x_n\}$ be the three-step iterative sequence with errors defined by* (1.1) *and $\{k_n\}, \{c_n\}, \{\bar{c_n}\}, \{\hat{c_n}\}$ be four sequences satisfying the following conditions :*

(i) $\sum_{n=1}^{\infty} k_n < \infty$,

(ii) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} \bar{c_n} < \infty$, $\sum_{n=1}^{\infty} \hat{c_n} < \infty$,

where $\{k_n\}$ is a sequence appeared in Definition 1.1 *and $\{c_n\}$, $\{\bar{c_n}\}$, $\{\hat{c_n}\}$ are three sequences appeared in* (1.1). *Then the iterative sequence $\{x_n\}$ converges to a fixed point of T if and only if*

$$\liminf_{n\to\infty} D_d(x_n, F(T)) = 0.$$

Proof. The necessity is obvious. Now, we prove the sufficiency. Suppose that the condition $\liminf_{n\to\infty} D_d(x_n, F(T)) = 0$ is satisfied. Then from Lemma 2.1 (a), we have

$$d(x_{n+1}, p) \leq (1 + k_n)^3 d(x_n, p) + h_n, \quad \forall\, p \in F(T), \ \forall\, n \in \mathbb{N}, \tag{2.5}$$

where $h_n = b_n(1 + k_n)\theta_n + c_n d(u_n, p)$ and $\theta_n = \bar{b_n}\hat{c_n}(1 + k_n)d(w_n, p) + \bar{c_n}d(v_n, p)$. Since $0 \leq b_n, \bar{b_n} \leq 1$, $\sum_{n=1}^{\infty} k_n < \infty$, $\sum_{n=1}^{\infty} \hat{c_n} < \infty$, $\sum_{n=1}^{\infty} \bar{c_n} < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$ and $\{w_n\}$, $\{v_n\}$, $\{u_n\}$ are three bounded sequences, we have $\sum_{n=1}^{\infty} \theta_n < \infty$ and so $\sum_{n=1}^{\infty} h_n < \infty$. From (2.5), we can obtain that

$$D_d(x_{n+1}, F(T)) \leq (1 + k_n)^3 D_d(x_n, F(T)) + h_n.$$

Since $\liminf_{n\to\infty} D_d(x_n, F(T)) = 0$, by Lemma 2.2, we have

$$\lim_{n\to\infty} D_d(x_n, F(T)) = 0.$$

Now, we will prove that $\{x_n\}$ is a Cauchy sequence. Let $\epsilon > 0$. By Lemma 2.1 (b), there exists a constant $M > 0$ such that

$$d(x_m, p) \leq M d(x_n, p) + M \sum_{j=n}^{m-1} h_j, \quad \forall\, p \in F(T), \ m > n. \tag{2.6}$$

Since $\lim_{n\to\infty} D_d(x_n, F(T)) = 0$ and $\sum_{n=1}^{\infty} h_n < \infty$, there exists a constant N_1 such that for all $n \geq N_1$,

$$D_d(x_n, F(T)) < \frac{\epsilon}{4M} \quad \text{and} \quad \sum_{j=N_1}^{\infty} h_j < \frac{\epsilon}{6M}.$$

We note that there exists $p_1 \in F(T)$ such that $d(x_{N_1}, p_1) < \frac{\epsilon}{3M}$. It follows from (2.6) that for all $m > n \geq N_1$,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, p_1) + d(x_n, p_1) \\ &\leq Md(x_{N_1}, p_1) + M \sum_{j=N_1}^{m-1} h_j + Md(x_{N_1}, p_1) + M \sum_{j=N_1}^{n-1} h_j \\ &< M\frac{\epsilon}{3M} + M\frac{\epsilon}{6M} + M\frac{\epsilon}{3M} + M\frac{\epsilon}{6M} \\ &= \epsilon. \end{aligned} \tag{2.7}$$

Since ϵ is an arbitrary positive number, (2.7) implies that $\{x_n\}$ is a Cauchy sequence. From the completeness of this space, $\lim_{n\to\infty} x_n$ exists. Let $\lim_{n\to\infty} x_n = p$. It will be proven that p is a fixed point. Let $\bar{\epsilon} > 0$. Since $\lim_{n\to\infty} x_n = p$, there exists a natural number N_2 such that for all $n \geq N_2$,

$$d(x_n, p) < \frac{\bar{\epsilon}}{2(2 + k_1)}. \tag{2.8}$$

$\lim_{n\to\infty} D_d(x_n, F(T)) = 0$ implies that there exists a natural number $N_3 \geq N_2$ such that for all $n \geq N_3$,

$$D_d(x_n, F(T)) < \frac{\bar{\epsilon}}{3(4 + 3k_1)}.$$

Therefore, there exists a $\bar{p} \in F(T)$ such that

$$d(x_{N_3}, \bar{p}) < \frac{\bar{\epsilon}}{2(4 + 3k_1)}. \tag{2.9}$$

From (2.8) and (2.9), we have

$$\begin{aligned} d(Tp, p) &\leq d(Tp, \bar{p}) + d(\bar{p}, Tx_{N_3}) + d(Tx_{N_3}, \bar{p}) + d(\bar{p}, x_{N_3}) + d(x_{N_3}, p) \\ &= d(Tp, \bar{p}) + 2d(Tx_{N_3}, \bar{p}) + d(x_{N_3}, \bar{p}) + d(x_{N_3}, p) \\ &\leq (1 + k_1)d(p, \bar{p}) + 2(1 + k_1)d(x_{N_3}, \bar{p}) + d(x_{N_3}, \bar{p}) \\ &\quad + d(x_{N_3}, p) \\ &\leq (1 + k_1)\{d(p, x_{N_3}) + d(x_{N_3}, \bar{p})\} + 2(1 + k_1)d(x_{N_3}, \bar{p}) \\ &\quad + d(x_{N_3}, \bar{p}) + d(x_{N_3}, p) \\ &= (2 + k_1)d(x_{N_3}, p) + (4 + 3k_1)d(x_{N_3}, \bar{p}) \\ &< (2 + k_1)\frac{\bar{\epsilon}}{2(2 + k_1)} + (4 + 3k_1)\frac{\bar{\epsilon}}{2(4 + 3k_1)} \\ &= \bar{\epsilon}. \end{aligned}$$

Since $\bar{\epsilon}$ is an arbitrary positive number, this implies that $Tp = p$, that is, p is a fixed point. This completes the proof of Theorem 2.1. □

In (1.1), if $\hat{b_n} \equiv 0$ and $\hat{c_n} \equiv 0$, for all $n = 1, 2, \cdots$, then $z_n = x_n$. Therefore, the following corollary can be obtained from Theorem 2.1 immediately.

Corollary 2.1. *Let $(E,d,W),T$ and $F(T)$ be as in Theorem* 2.1. *For a given $x_1 \in E$, let $\{x_n\}$ be the Ishikawa type iterative sequence with errors defined by* (1.2) *and $\{k_n\},\{c_n\},\{\bar{c_n}\}$ be three sequences satisfying the conditions* (i) *and* (ii) *in Theorem* 2.1. *Then the iterative sequence $\{x_n\}$ converges to a fixed point of T if and only if*

$$\liminf_{n\to\infty} D_d(x_n, F(T)) = 0.$$

By using the same method in Theorem 2.1, we can easily obtain the following theorem.

Theorem 2.2. *Let (E,d,W) be a complete convex metric space, $T: E \to E$ be a quasi-nonexpansive mapping and $F(T)$ be a nonempty set. For a given $x_1 \in E$, let $\{x_n\}$ be the three-step iterative sequence with errors defined by* (1.1) *and $\{k_n\},\{c_n\},\{\bar{c_n}\},\{\hat{c_n}\}$ be four sequences satisfying the conditions* (i) *and* (ii) *in Theorem* 2.1. *Then the iterative sequence $\{x_n\}$ converges to a fixed point of T if and only if*

$$\liminf_{n\to\infty} D_d(x_n, F(T)) = 0.$$

Theorem 2.3. *Let (E,d,W) be a complete convex metric space, $T : E \to E$ be an asymptotically nonexpansive mapping and $F(T)$ be a nonempty set. For a given $x_1 \in E$, let $\{x_n\}$ be the three-step iterative sequence with errors defined by* (1.1) *and $\{k_n\},\{c_n\},\{\bar{c_n}\},\{\hat{c_n}\}$ be four sequences satisfying the conditions* (i) *and* (ii) *in Theorem* 2.1. *Then the iterative sequence $\{x_n\}$ converges to a fixed point of T if and only if*

$$\liminf_{n\to\infty} D_d(x_n, F(T)) = 0.$$

Proof. Since $F(T)$ is a nonempty set, by Definition 1.1, an asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive mapping. Thus, the conclusion can be obtained from Theorem 2.1 immediately. □

From Theorem 2.1, we can also obtain the following result for the Banach space.

Theorem 2.4. *Let E be a real Banach space, $T : E \to E$ be an asymptotically quasi-nonexpansive mapping satisfying the condition* (i) *in Theorem* 2.1 *and $F(T)$ be a nonempty set. Let $\{x_n\}$ be the three-step iterative sequence with errors defined by*

$$\begin{cases} x_1 \in E, \\ x_{n+1} = a_n x_n + b_n T^n y_n + c_n u_n, \\ y_n = \bar{a_n} x_n + \bar{b_n} T^n z_n + \bar{c_n} v_n, \\ z_n = \hat{a_n} x_n + \hat{b_n} T^n x_n + \hat{c_n} w_n, \quad \forall\, n \in \mathbb{N}, \end{cases}$$

where $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ *are three bounded sequences in* E *and* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\bar{a_n}\}$, $\{\bar{b_n}\}$, $\{\bar{c_n}\}$, $\{\hat{a_n}\}$, $\{\hat{b_n}\}$, $\{\hat{c_n}\}$ *are nine sequences in* $[0,1]$ *satisfying* $a_n + b_n + c_n = \bar{a_n}+\bar{b_n}+\bar{c_n} = \hat{a_n}+\hat{b_n}+\hat{c_n} = 1$, $\forall n \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} \bar{c}_n < \infty$, $\sum_{n=1}^{\infty} \hat{c_n} < \infty$. *Then the iterative sequence* $\{x_n\}$ *converges to a fixed point of* T *if and only if*

$$\liminf_{n\to\infty} D(x_n, F(T)) = 0,$$

where $D(y,S) = \inf\{\|y-s\| : s \in S\}$.

proof. Since E is a Banach space, it is a complete convex metric space with a convex structure $W(x,y,z:\alpha,\beta,\gamma) := \alpha x+\beta y+\gamma z$, for all $x,y,z \in E$ and for all $\alpha,\beta,\gamma \in [0,1]$ with $\alpha+\beta+\gamma = 1$. Therefore, the conclusion of Theorem 2.4 can be obtained from Theorem 2.1 immediately. □

Remark 2.1. (1) Theorem 2.1, 2.2 and 2.3 are three new convergence theorems of three-step iterative sequences with errors for nonlinear mappings in convex metric spaces. These three theorems generalize and improve the corresponding results of [7-9] and [1-3, 5, 11, 14].

(2) Theorem 2.4 generalizes and improves the corresponding results of Kim *et al.* [6], Liu [8, 9] and Xu-Noor [14].

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