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<td>数理解析研究所講究録 2004, 1365: 176-184</td>
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STRONG CONVERGENCE OF ISHIKAWA ITERATIONS FOR
ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract—Let $C$ be a nonempty bounded closed convex subset of a uniformly convex Banach space. We prove that if $T : C \to C$ is both compact iterates and asymptotically nonexpansive, the Ishikawa iteration process with errors defined by $x_1 \in C$, $x_{n+1} = \alpha_n x_n + \beta_n T^m y_n + \gamma_n u_n$, and $y_n = \alpha'_n x_n + \beta'_n T^m x_n + \gamma'_n v_n$ converges strongly to some fixed point of $T$. This generalizes the recent theorems due to Rhoades [5], Schu [6] and Schu [7].

Keywords—strong convergence, fixed point, Mann and Ishikawa iteration process, asymptotically nonexpansive mapping.

1. Introduction

Let $C$ be a nonempty bounded closed convex subset of a Banach space $E$ and let $T$ be a mapping of $C$ into itself. Then $T$ is said to be \textit{asymptotically nonexpansive} [1] if there exists a sequence $\{k_n\}$, $k_n \geq 1$, with $\lim_{n \to \infty} k_n = 1$, such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$ and $n \geq 1$. In particular, if $k_n = 1$ for all $n \geq 1$, $T$ is said to be \textit{nonexpansive}. $T$ is said to be \textit{uniformly $L$-Lipschitzian} if there exists a constant $L > 0$, such that

$$\|T^n x - T^n y\| \leq L \|x - y\|$$

for all $x, y \in C$ and $n \geq 1$. $T$ is said to be compact if it maps bounded sets into relatively compact ones. We denote by $F(T)$ the set of all fixed points of $T$, i.e., $F(T) = \{x \in C : Tx = x\}$. We also denote by $N$ the set of all positive integers. A Banach space $E$ is called \textit{uniformly convex} if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1 - \delta)$ holds. When $\{x_n\}$ is a sequence in $E$, then $x_n \to x$ will denote strong convergence of the sequence $\{x_n\}$ to $x$. For a mappings $T$ of $C$ into itself, Rhoades [5] considered the following modified Ishikawa iteration process (cf. Ishikawa [3]) in $C$ defined by

$$x_1 \in C,$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^m y_n,$$

$$y_n = (1 - \beta_n)x_n + \beta_n T^m x_n,$$
where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are two real sequences in \([0, 1]\). If \( \beta_n = 0 \) for all \( n \geq 1 \), then the iteration process (1) becomes the following modified Mann iteration process (cf. Mann [4], Schu [6]):

\[
\begin{align*}
\omega_1 & \in C, \\
x_{n+1} & = (1 - \alpha_n)x_n + \alpha_n T^m x_n,
\end{align*}
\]

(2)

where \( \{\alpha_n\} \) is a real sequence in \([0, 1]\).

Recently, Schu [7] proved that if \( E \) is a uniformly convex Banach space, \( C \) is a nonempty bounded closed and convex subset of \( E \), and \( T : C \to C \) is an asymptotically nonexpansive mapping with \( \{k_n\} \) satisfying \( k_n \geq 1 \), \( \sum_{n=1}^{\infty} (k_n - 1) < \infty \), and \( T^m \) is compact for some \( m \in \mathbb{N} \), then for any \( x_1 \in C \), the sequence \( \{x_n\} \) defined by (2), where \( \{\alpha_n\} \) is chosen so that \( 0 < a \leq \alpha_n \leq b < 1 \), for all \( n \geq 1 \) and some \( a, b \in (0, 1) \), converges strongly to some fixed point of \( T \). This extended a result of Schu [6] to uniformly convex Banach spaces. On the other hand, Rhoades [5] proved that if \( E \) is a uniformly convex Banach space, \( C \) is a nonempty bounded closed convex subset of \( E \), and \( T : C \to C \) is a completely continuous asymptotically nonexpansive mapping with \( \{k_n\} \) satisfying \( k_n \geq 1 \), \( \sum_{n=1}^{\infty} (k_n - 1) < \infty \), \( r = \max\{2, p\} \), then for any \( x_1 \in C \), the sequence \( \{x_n\} \) defined by (1), where \( \{\alpha_n\} \), \( \{\beta_n\} \) satisfy \( a \leq (1 - \alpha_n), (1 - \beta_n) \leq 1 - a \) for all \( n \geq 1 \) and some \( a > 0 \), converges strongly to some fixed point of \( T \). We consider a more general iterative process of the type (cf. Xu [10]) emphasizing the randomness of errors as follows:

\[
\begin{align*}
x_1 & \in C, \\
x_{n+1} & = \alpha_n x_n + \beta_n T^m y_n + \gamma_n u_n, \\
y_n & = \alpha'_n x_n + \beta'_n T^m x_n + \gamma'_n v_n,
\end{align*}
\]

(3)

where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\} \) are real sequences in \([0, 1]\) and \( \{u_n\}, \{v_n\} \) are two sequences in \( C \) such that

(i) \( \alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1 \) for all \( n \geq 1 \),

(ii) \( \sum_{n=1}^{\infty} \gamma_n < \infty \) and \( \sum_{n=1}^{\infty} \gamma'_n < \infty \).

If \( \gamma_n = \gamma'_n = 0 \) for all \( n \geq 1 \), then the iteration process (3) reduces to the Ishikawa iteration process [3], while setting \( \beta'_n = 0 \) and \( \gamma'_n = 0 \) for all \( n \geq 1 \), (3) reduces to the Mann iteration process with errors, which is a generalized case of the Mann iteration process [4].

In this paper, we prove strong convergence theorems of the Ishikawa (and Mann) iteration process with errors defined by (3) for a compact iterates and asymptotically nonexpansive mapping in a uniformly convex Banach space, which generalize the recent theorems due to Rhoades [5], Schu [6] and Schu [7].

2. Strong convergence theorems

We first begin with the following:
Lemma 1 [9]. Let \( \{a_n\} \) and \( \{b_n\} \) be two sequences of nonnegative real numbers such that 
\[
\sum_{n=1}^{\infty} b_n < \infty \text{ and } \forall n \geq 1, a_{n+1} \leq a_n + b_n
\]
for all \( n \geq 1 \). Then \( \lim_{n \to \infty} a_n \) exists.

Lemma 2 [2]. Let \( E \) be a uniformly convex Banach space. Let \( x, y \in E \). If \( \|x\| \leq 1 \), \( \|y\| \leq 1 \), and \( \|x-y\| \geq \epsilon > 0 \), then \( \|\lambda x + (1-\lambda)y\| \leq 1 - 2\lambda(1-\lambda)\delta(\epsilon) \) for \( \lambda \) with \( 0 \leq \lambda \leq 1 \).

Lemma 3 (cf. [6]). Let \( E \) be a normed space and let \( C \) be a nonempty bounded convex subset of \( E \). Let \( T : C \to C \) be a uniformly \( L \)-Lipschitzian mapping. Define the sequence \( \{x_n\} \) defined by (3). Set \( w_n = \|T^n x_n - x_n\| \), for all \( n \geq 1 \). Then
\[
\|x_n - Tx_n\| \leq w_n + L(2 + 2L + L^2)w_{n-1} + L^2(1 + L)M^*\gamma_{n-1} + L(1 + L)M^*\gamma_{n-1},
\]
for all \( n \geq 1 \), where \( M^* := \sup_{n \geq 1} \|x_n - x_n\| \vee \sup_{n \geq 1} \|x_n - x_n\| < \infty \).

Proof. Since
\[
\|y_n - x_n\| = \|\alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n - x_n\|
\leq \beta'_n \|T^n x_n - x_n\| + \gamma'_n \|v_n - x_n\|
\leq w_n + \gamma'_n M^*,
\]
\[
\|T^ny_n - x_n\| \leq \|T^ny_n - T^nx_n\| + \|T^n x_n - x_n\|
\leq L\|y_n - x_n\| + w_n
\leq L\{w_n + \gamma'_n M^*\} + w_n
= (1 + L)w_n + LM^*\gamma'_n
\]
and thus
\[
\|x_n - x_{n-1}\| = \|\alpha_{n-1} x_{n-1} + \beta_{n-1} T^{n-1} y_{n-1} + \gamma_{n-1} v_{n-1} - x_{n-1}\|
\leq \beta_{n-1} \|T^{n-1} y_{n-1} - x_{n-1}\| + \gamma_{n-1} \|v_{n-1} - x_{n-1}\|
\leq (1 + L)w_{n-1} + LM^*\gamma'_{n-1} + M^*\gamma_{n-1},
\]
\[
\|T^{n-1} x_n - x_n\| \leq \|T^{n-1} x_n - T^{n-1} x_{n-1}\| + \|T^{n-1} x_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\|
\leq w_{n-1} + (1 + L)\|x_n - x_{n-1}\|
\leq w_{n-1} + (1 + L)\{(1 + L)w_{n-1} + LM^*\gamma'_{n-1} + M^*\gamma_{n-1}\}.
\]
Hence we obtain
\[
\|x_n - Tx_n\| \leq \|x_n - T^nx_n\| + \|T^nx_n - Tx_n\|
\leq w_n + L\|T^{n-1} x_n - x_n\|
\leq w_n + L\{w_{n-1} + (1 + L)\{(1 + L)w_{n-1} + LM^*\gamma'_{n-1} + M^*\gamma_{n-1}\}\}
= w_n + L(2 + 2L + L^2)w_{n-1} + L^2(1 + L)M^*\gamma'_{n-1} + L(1 + L)M^*\gamma_{n-1}.
\]
\(\square\)

Using Lemma 1, we have the following:
Lemma 4. Let $C$ be a nonempty bounded closed convex subset of a uniformly convex Banach space $E$ and let $T : C \to C$ be an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1$, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Suppose that the sequence $\{x_n\}$ defined by (3). Then

$$\lim_{n \to \infty} \|x_n - z\|$$

exists, for any $z \in F(T)$.

Proof. The existence of a fixed point of $T$ follows from Goebel-Kirk [1]. For a fixed $z \in F(T)$, since $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ are bounded, let

$$M := \sup_{n \geq 1} \|x_n - z\| \vee \sup_{n \geq 1} \|u_n - z\| \vee \sup_{n \geq 1} \|v_n - z\| < \infty.$$

Put $c_n = k_n - 1$. Since

$$\|T^n y_n - z\| \leq k_n \|y_n - z\|$$

$$= (1 + c_n)\{\alpha_n' x_n + \beta_n' T^n x_n + \gamma_n' u_n - z\|$$

$$\leq (1 + c_n)\{\alpha_n' \|x_n - z\| + \beta_n' \|T^n x_n - z\| + \gamma_n' \|u_n - z\|\}$$

$$\leq (1 + c_n)\{\alpha_n' \|x_n - z\| + \beta_n' \|x_n - z\| + c_n \|x_n - z\| + \gamma_n' \|u_n - z\|\}$$

$$\leq \alpha_n' \|x_n - z\| + \beta_n' \|x_n - z\| + c_n \|x_n - z\| + \gamma_n' \|u_n - z\|$$

$$\leq (1 - \gamma_n') \|x_n - z\| + 4M c_n + M \gamma_n'$$

we have

$$\|x_{n+1} - z\| = \|\alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n - z\|$$

$$\leq \alpha_n \|x_n - z\| + \beta_n \|T^n y_n - z\| + \gamma_n \|u_n - z\|$$

$$\leq \alpha_n \|x_n - z\| + \beta_n \{(1 - \gamma_n') \|x_n - z\| + 4M c_n + M \gamma_n'\} + \gamma_n M$$

$$= (1 - (\gamma_n + \beta_n \gamma_n')) \|x_n - z\| + 4M \beta_n c_n + M (\gamma_n + \beta_n \gamma_n)$$

$$\leq \|x_n - z\| + 4M c_n + M (\gamma_n + \gamma_n').$$

By Lemma 1, we readily see that $\lim_{n \to \infty} \|x_n - z\|$ exists. □

By using Lemma 1–Lemma 4, we have the following:

Theorem 1. Let $C$ be a nonempty bounded closed convex subset of a uniformly convex Banach space $E$ and let $T : C \to C$ be an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1$, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Suppose $x_1 \in C$, and the sequence $\{x_n\}$ defined by (3) satisfies $0 < a \leq \alpha_n \leq b < 1$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $0 \leq \beta_n' \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0, 1)$ or $0 < a \leq \beta_n \leq 1$, $0 < a \leq \alpha_n' \leq b < 1$, $\sum_{n=1}^{\infty} \beta_n' = \infty$ for all $n \geq 1$ and some $a, b \in (0, 1)$. Then $\liminf_{n \to \infty} \|x_n - Tx_n\| = 0$. 


Proof. The existence of a fixed point of $T$ follows from Goebel-Kirk [1]. For a fixed $z \in F(T)$, since $\{x_n\}, \{u_n\}$ and $\{v_n\}$ are bounded, let

$$M := \sup_{n \geq 1} \|x_n - z\| \vee \sup_{n \geq 1} \|u_n - z\| \vee \sup_{n \geq 1} \|v_n - z\| < \infty.$$  

By Lemma 4, we see that $\lim_{n \to \infty} \|x_n - z\| = r$ exists. If $r = 0$, then the conclusion is obvious. So, we assume $r > 0$. Note that $d_n := \max\{\gamma_n, \gamma_n/a, \gamma_n/a\} \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} d_n < \infty$. Put $c_n = k_n - 1$. Since $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, we have

$$\lim_{n \to \infty} c_n = 0.$$  

Since $\|T^n y_n - z\| \leq \|x_n - z\| + 4M c_n + M d_n$ and

$$\left\| \frac{\alpha_n x_n}{\alpha_n + \gamma_n} + \frac{\gamma_n u_n}{\alpha_n + \gamma_n} - z \right\| \leq \|x_n - z\| + 4M c_n + M d_n,$$

by using Lemma 2 and Takahashi [8], we obtain

$$\|x_{n+1} - z\| = \|\alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n - z\| = \|\beta_n (T^n y_n - z) + (1 - \beta_n) \left( \frac{\alpha_n x_n}{\alpha_n + \gamma_n} + \frac{\gamma_n u_n}{\alpha_n + \gamma_n} - z \right)\| \leq (\|x_n - z\| + 4M c_n + M d_n) \left[ 1 - 2\beta_n (1 - \beta_n) \right] \times \delta_E \left( \frac{1}{\alpha_n + \gamma_n} \cdot \frac{\|\alpha_n (T^n y_n - x_n) + \gamma_n (T^n y_n - u_n)\|}{\|x_n - z\| + 4M c_n + M d_n} \right).$$

Thus, by using $0 < a \leq \alpha_n \leq b < 1$, we obtain

$$2\beta_n a (\|x_n - z\| + 4M c_n + M d_n) \delta_E \left( \frac{1}{\alpha_n + \gamma_n} \cdot \frac{\|\alpha_n (T^n y_n - x_n) + \gamma_n (T^n y_n - u_n)\|}{\|x_n - z\| + 4M c_n + M d_n} \right) \leq 2\beta_n (1 - \beta_n) (\|x_n - z\| + 4M c_n + M d_n) \delta_E \left( \frac{1}{\alpha_n + \gamma_n} \cdot \frac{\|\alpha_n (T^n y_n - x_n) + \gamma_n (T^n y_n - u_n)\|}{\|x_n - z\| + 4M c_n + M d_n} \right) \leq \|x_n - z\| - \|x_{n+1} - z\| + 4M c_n + M d_n.$$  

Since

$$2a \sum_{n=1}^{\infty} \beta_n (\|x_n - z\| + 4M c_n + M d_n) \delta_E \left( \frac{1}{\alpha_n + \gamma_n} \cdot \frac{\|\alpha_n (T^n y_n - x_n) + \gamma_n (T^n y_n - u_n)\|}{\|x_n - z\| + 4M c_n + M d_n} \right) < \infty,$$

$\sup_{n \geq 1} \|T^n y_n - u_n\| < \infty$, and $\delta_E$ is strictly increasing and continuous, we obtain

$$\lim_{n \to \infty} \inf_{n \geq 1} \|T^n y_n - x_n\| = 0.$$  

(5)
Since
\[
||T^m x_n - x_n|| \leq ||T^m x_n - T^m y_n|| + ||T^m y_n - x_n||
\]
\[
\leq (1 + c_n) ||x_n - y_n|| + ||T^m y_n - x_n||
\]
\[
= (1 + c_n) ||x_n - \alpha_n' x_n - \beta_n' T^n x_n - \gamma_n' v_n|| + ||T^m y_n - x_n||
\]
\[
\leq (1 + c_n) \beta_n ||T^m x_n - x_n|| + (1 + c_n) \gamma_n' ||x_n - v_n|| + ||T^m y_n - x_n||
\]
\[
\leq (1 + c_n) b ||T^m x_n - x_n|| + (1 + c_n) \gamma_n' ||x_n - v_n|| + ||T^m y_n - x_n||
\]
\[
= b ||T^m x_n - x_n|| + c_n b ||T^m x_n - x_n|| + (1 + c_n) \gamma_n' ||x_n - v_n|| + ||T^m y_n - x_n||
\]
\[
\leq b ||T^m x_n - x_n|| + c_n (2 + c_n) b ||x_n - z|| + (1 + c_n) \gamma_n' ||x_n - v_n|| + ||T^m y_n - x_n||,
\]
we obtain
\[
(1 - b) ||T^m x_n - x_n|| \leq c_n (2 + c_n) b ||x_n - z|| + (1 + c_n) \gamma_n' ||x_n - v_n|| + ||T^m y_n - x_n||
\]
\[
\leq c_n (2 + c_n) b M + 2 (1 + c_n) \gamma_n' M + ||T^m y_n - x_n||.
\]
By using (4) and (5), we obtain
\[
\liminf_{n \to \infty} ||T^m x_n - x_n|| = 0.
\]
On the other hand, if 0 < a ≤ \beta_n ≤ 1, 0 < a ≤ \alpha_n' ≤ b < 1, \sum_{n=1}^{\infty} \beta_n' = \infty for all n ≥ 1 and some a, b ∈ (0, 1), then we have
\[
||x_{n+1} - z|| = ||\alpha_n x_n + \beta_n T^m y_n + \gamma_n u_n - z||
\]
\[
\leq \alpha_n ||x_n - z|| + \beta_n ||T^m y_n - z|| + \gamma_n ||u_n - z||
\]
\[
\leq \alpha_n ||x_n - z|| + \beta_n (1 + c_n) ||y_n - z|| + \gamma_n ||u_n - z||
\]
\[
\leq \alpha_n ||x_n - z|| + \beta_n ||y_n - z|| + \beta_n c_n ||y_n - z|| + M \gamma_n
\]
\[
= (1 - \beta_n - \gamma_n) ||x_n - z|| + \beta_n ||y_n - z|| + \beta_n c_n ||y_n - z|| + M \gamma_n
\]
\[
\leq (1 - \beta_n) ||x_n - z|| + \beta_n ||y_n - z|| + \beta_n c_n ||y_n - z|| + M \gamma_n
\]
and hence
\[
\frac{||x_{n+1} - z|| - ||x_n - z||}{\beta_n} - \frac{||y_n - z||}{a} \leq ||y_n - z|| - ||x_n - z|| + c_n ||y_n - z|| + M \gamma_n
\]
\[
\leq ||y_n - z|| - ||x_n - z|| + c_n (||x_n - z|| + M c_n + M \gamma_n) + M d_n.
\]
So, we have
\[
||x_n - z|| - ||y_n - z|| \leq \frac{||x_n - z|| - ||x_{n+1} - z||}{\beta_n} + c_n (||x_n - z|| + M c_n + M \gamma_n) + M d_n
\]
\[
(7) \leq \frac{||x_n - z|| - ||x_{n+1} - z||}{a} + c_n (M (1 + c_n) + M \gamma_n) + M d_n.
\]
Since
\[ \|T^nx_n - z\| \leq (1 + c_n)\|x_n - z\| \]
\[ \leq \|x_n - z\| + Mc_n + Md_n \]
and
\[ \left\| \frac{\alpha_n'x_n + \gamma_n'v_n}{\alpha_n' + \gamma_n'} - z \right\| \leq \|x_n - z\| + Mc_n + Md_n, \]
we obtain
\[ \|y_n - z\| = \left\| \alpha_n'x_n + \beta_n'T^nx_n + \gamma_n'v_n - z \right\| \]
\[ = \left\| \beta_n'(T^nx_n - z) + (1 - \beta_n') \left( \frac{\alpha_n'x_n}{\alpha_n' + \gamma_n'} + \frac{\gamma_n'v_n}{\alpha_n' + \gamma_n'} - z \right) \right\| \]
\[ \leq (\|x_n - z\| + Mc_n + Md_n)[1 - 2\beta_n'(1 - \beta_n') \cdot \frac{||\alpha_n'(T^nx_n-x_n)+\gamma_n'(T^nx_n-v_n)||}{||x_n-z||+Mc_n+Md_n}]. \tag{8} \]

By using (7), (8) and \(0 < a \leq \alpha_n' \leq b < 1\), we obtain
\[ 2\beta_n'a(\|x_n - z\| + Mc_n + Md_n)\delta_E \left( \frac{1}{\alpha_n' + \gamma_n'} \cdot \left\| \frac{\alpha_n'(T^nx_n - x_n) + \gamma_n'(T^nx_n - v_n)}{\alpha_n' + \gamma_n'} - z \right\| \right) \]
\[ \leq 2\beta_n'(1 - \beta_n')(\|x_n - z\| + Mc_n + Md_n)\delta_E \left( \frac{1}{\alpha_n' + \gamma_n'} \cdot \left\| \frac{\alpha_n'(T^nx_n - x_n) + \gamma_n'(T^nx_n - v_n)}{\alpha_n' + \gamma_n'} \right\| \right) \]
\[ \leq \|x_n - z\| - \|y_n - z\| + Mc_n + Md_n \]
\[ \leq \frac{\|x_n - z\| - \|x_{n+1} - z\|}{a} + c_n\{M(1 + c_n) + M\gamma_n\} + Md_n + Mc_n + Md_n \]
\[ = \frac{\|x_n - z\| - \|x_{n+1} - z\|}{a} + c_n\{M(2 + c_n) + M\gamma_n\} + 2Md_n. \]

Hence
\[ 2a \sum_{n=1}^{\infty} \beta_n'(\|x_n - z\| + Mc_n + Md_n)\delta_E \left( \frac{1}{\alpha_n' + \gamma_n'} \cdot \left\| \frac{\alpha_n'(T^nx_n - x_n) + \gamma_n'(T^nx_n - v_n)}{\alpha_n' + \gamma_n'} \right\| \right) < \infty. \]

We also obtain (6) similarly to the argument above. By using Lemma 3, we obtain \( \liminf_{n \to \infty} \|Tx_n - x_n\| = 0 \). \( \Box \)

Our Theorem 2 improves Theorem 1.5 of Schu [6], Theorem 2.2 of Schu [7] and Theorem 3 of Rhoades [5] to a more general Ishikawa type scheme under much less restrictions on the iterative parameters \( \{\alpha_n\} \) and \( \{\beta_n\} \).

**Theorem 2.** Let \( E \) be a uniformly convex Banach space, and let \( C \) be a nonempty bounded closed convex subset of \( E \), and let \( T : C \to C \) be an asymptotically nonexpansive mapping with \( \{k_n\} \) satisfying \( k_n \geq 1 \), \( \sum_{n=1}^{\infty}(k_n - 1) < \infty \), and let \( T^m \) be compact for some \( m \in \mathbb{N} \). If
$x_1 \in C$, and the sequence $\{x_n\}$ defined by (3) satisfies $0 < a \leq \alpha_n \leq b < 1$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $0 \leq \beta'_n \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0, 1)$ or $0 < a \leq \beta_n \leq 1$, $0 < a \leq \alpha'_n \leq b < 1$, $\sum_{n=1}^{\infty} \beta'_n = \infty$ for all $n \geq 1$ and some $a, b \in (0, 1)$, then $\{x_n\}$ converges strongly to some fixed point of $T$.

Proof. From Theorem 1, there exists a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that

$$\lim_{k \to \infty} \|x_{n_k} - Tx_{n_k}\| = 0.$$  

Since

$$\|T^m x_{n_k} - x_{n_k}\| \leq \|T^m x_{n_k} - T^{m-1} x_{n_k}\| + \|T^{m-1} x_{n_k} - T^{m-2} x_{n_k}\| + \cdots + \|Tx_{n_k} - x_{n_k}\|$$

we obtain

$$\lim_{k \to \infty} \|x_{n_k} - T^m x_{n_k}\| = 0.$$  

Since $T^m$ is compact, there exist a subsequence $\{x_{n_{k_i}}\}$ of the sequence $\{x_{n_k}\}$ and a point $p \in C$ such that $x_{n_{k_i}} \to p$. Thus we obtain $p \in F(T)$ by the continuity of $T$ and (9). Hence we obtain $\lim_{n \to \infty} \|x_n - p\| = 0$ by Lemma 4. □

Our Theorem 3 improves Theorem 1.5 of Schu [6], Theorem 2.2 of Schu [7] and Theorem 3 of Rhoades [5] under much less restrictions on the iterative parameters $\{\alpha_n\}$ and $\{\beta_n\}$.

**Theorem 3.** Let $E$ be a uniformly convex Banach space, and let $C$ be a nonempty bounded closed convex subset of $E$, and let $T : C \to C$ be an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1$, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, and let $T^m$ be compact for some $m \in \mathbb{N}$. If $x_1 \in C$, and the sequence $\{x_n\}$ defined by (1) satisfies $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$, $0 \leq \beta_n \leq b < 1$

for all $n \geq 1$ and some $b \in (0, 1)$ or $0 < a \leq \alpha_n \leq 1$, $\sum_{n=1}^{\infty} \beta_n (1 - \beta_n) = \infty$ for all $n \geq 1$ and some $a \in (0, 1)$, then $\{x_n\}$ converges strongly to some fixed point of $T$.

As a direct consequence, taking $\beta'_n = 0$ and $\gamma'_n = 0$ for $n \in \mathbb{N}$ in Theorem 2, we obtain the following result, which improves Theorem 2.2 of Schu [7] and Theorem 2 of Rhoades [5] under much less restrictions on the iterative parameter $\{\alpha_n\}$.

**Theorem 4.** Let $E$ be a uniformly convex Banach space, and let $C$ be a nonempty bounded closed convex subset of $E$, and let $T : C \to C$ be an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1$, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, and let $T^m$ be compact for some $m \in \mathbb{N}$. Suppose that $x_1 \in C$, and the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + \beta_n T^m x_n + \gamma_n u_n,$$  

for all $n \geq 1$ and some $b \in (0, 1)$ or $0 < a \leq \alpha_n \leq 1$, $\sum_{n=1}^{\infty} \beta_n (1 - \beta_n) = \infty$ for all $n \geq 1$ and some $a \in (0, 1)$, then $\{x_n\}$ converges strongly to some fixed point of $T$. 

Hence we obtain $\lim_{n \to \infty} \|x_n - p\| = 0$ by Lemma 4. □
where \( \{\alpha_n\} , \{\beta_n\} , \{\gamma_n\} \) are sequences in \([0,1]\) satisfying \( 0 < a \leq \alpha_n \leq b < 1 \) for some \( a, b \in (0,1) \), \( \sum_{n=1}^{\infty} \beta_n = \infty \), \( \alpha_n + \beta_n + \gamma_n = 1 \) for all \( n \geq 1 \), \( \sum_{n=1}^{\infty} \gamma_n < \infty \) and \( \{u_n\} \) is a sequence in \( C \). Then \( \{x_n\} \) converges strongly to some fixed point of \( T \).

**Remark.** If \( \{\alpha_n\} \) is bounded away from both 0 and 1, i.e., \( a \leq \alpha_n \leq b \) for all \( n \geq 1 \) and some \( a, b \in (0,1) \), then \( \sum_{n=1}^{\infty} \alpha_n = \infty \) and \( \sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty \) hold. However, the converse is not true.

**References**