数理解析研究所講究録 1366 巻 2004 年 44-51

Some Remarks on Automata without Letichevsky Criteria[1]

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Abstract: In this paper we show some properties of finite automata having no Letichevsky criteria

Keywords: Finite automata, Letichevsky criterion.

1. Introduction

We start with some standard concepts and notations. The elements of an *alphabet* X are called *letters* (X is supposed to be finite and nonempty). A *word* over an alphabet X is a finite string consisting of letters of X. The string consisting of zero letters is called the *empty word,* written by λ. The *length* of a word w, in symbols $|w|$, means the number of letters in w when each letter is counted as many times it occurs. By definition, $|\lambda|=0$. At the same time, for any set H, $|H|$ denotes the cardinality of H. If u and v are words over an alphabet X, then their *catenation* uv is also a word over X. Catenation is an associative operation and the empty word λ is the identity with respect to catenation: $w\lambda=\lambda w=w$ for any word w. For a word w and positive integer n, the notation w^{n} means the word obtained by catenating n copies of the word w. w^{0} equals the empty word λ. w^{m} is called the *m-th power* of w for any non-negative integer m.

Let X^{*} be the set of all words over X, moreover, let $X^{+}=X^{*}\setminus\{\lambda\}$. X^{*} and X^{+} are the *free monoid* and the *free semigroup,* respectively, generated by X under catenation.

A (finite) *directed graph* (or, in short, a *digraph*) $\mathcal{D}=(V,E)$ (of order $|V|>0$) is a pair consisting of sets of *vertices* V and *edges* $E\subseteq V\times V$. A *walk* in $\mathcal{D}=(V,E)$ is a sequence of vertices $v_{1},\ldots,v_{n},n>1$ such that $(v_{i},v_{i+1})\in E,i=1,\ldots,n-1$. A walk is *closed* if $v_{1}=v_{n}$. By a *(directed) path* from a vertex a to a vertex $b\neq a$ we shall mean a sequence $v_{1}\ldots v_{n},n>1$ of pairwise distinct vertices such that $a=v_{1},b=v_{n}$

[1] This work was supported by the grant from Dirección General de Universidades, Secretaría de Estado de Educación y Universidades, Ministerio de Educación, Cultura y Deporte (SAB2001-0081), España, the project "Automata and Formal Languages" of the Hungarian Academy of Sciences and the Japan Society for the Promotion of Science, Grant-in-Aid for Science Research 1340016, Japan Society for the Promotion of Science, and the travel grant of the Hungarian Ministry of Education (Mecenatúra, No. MEC-01409/2002).

and $(v_i, v_{i+1}) \in E$ for every $i = 1, \ldots, n-1$. The positive integer $n-1$ is called the *length* of the path. Thus a path is a walk with all n vertices distinct. A closed walk with all vertices distinct except $v_1 = v_n$ is a *cycle* of length $n-1$.

By an *automaton* we mean a finite automaton without outputs. Given an automaton $\mathcal{A} = (A, X, \delta)$ with *set of states* A, *set of input letters* X, and *transition* $\delta : A \times X \to A$, it is understood that δ is extended to $\delta^* : A \times X^* \to A$ with $\delta^*(a, \lambda) = a$, $\delta^*(a, xq) = \delta^*(\delta(a, x), q)$. In the sequel, we will consider the transition of an automaton in this extended form and thus we will denote it by the same Greek letter δ. Let $\mathcal{A} = (A, X, \delta)$ be an automaton. It is said that a state $a \in A$ *generates* a state $b \in A$ if $\delta(a, p) = b$ holds for some $p \in X^*$. For every state $a \in A$ define the *state subautomaton* $\mathcal{B} = (B, X, \delta')$ *generated by* a such that $B = \{b \mid b = \delta(a, p), p \in X^*\}$, moreover, $\delta'(b, x) = \delta(b, x)$ for every pair $b \in B, x \in X$. $\mathcal{A}$ is called *strongly connected* if for every pair $a, b \in A$ there exists $p \in X^*$ such that $\delta(a, p) = b$.

We say that $\mathcal{A}$ satisfies *Letichevsky's criterion* if there are a state $a \in A$, input letters $x, y \in X$, input words $p, q \in X^*$ such that $\delta(a, x) \neq \delta(a, y)$ and $\delta(a, xp) = \delta(a, yq) = a$. It is said that $\mathcal{A}$ *satisfies the semi-Letichevsky criterion* if it does not satisfy Letichevsky's criterion but there are a state $a \in A$, input letters $x, y \in X$, an input word $p \in X^*$ such that $\delta(a, x) \neq \delta(a, y), \delta(a, xp) = a$ and for every $q \in X^*$, $\delta(a, yq) \neq a$. If $\mathcal{A}$ do not satisfy either Letichevsky's criterion or the semi-Letichevsky criterion then we say that $\mathcal{A}$ *does not satisfy any Letichevsky criteria* or *is without any Letichevsky criteria.*

The Letichevsky criterion has a central role in the investigations of products of automata (see [1],[2],[3],[4]). Automata having semi-Letichevsky criterion and automata without any Letichevsky criteria are also important in the classical result of Z. Ésik and Gy. Horváth (see [2],[3]). In this paper we investigate automata without any Letichevsky criteria.

2. Results

First we observe

Proposition 1 *Given an automaton $\mathcal{A} = (A, X, \delta)$, a state $a_0 \in A$, four input words $u, v, p, q \in X^*$ with $|up|, |vq| > 0$ under which $\delta(a_0, u) \neq \delta(a_0, v)$, and $\delta(a_0, up) = \delta(a_0, vq) = a_0$. Then $\mathcal{A}$ satisfies Letichevsky's criterion.*

Proof: First we suppose $|u|, |v| > 0$. Then there exist input words $w, w', w_1, w_2 \in X^*$ and input letters $x, y \in X$ such that $u = wxw_1, v = w'yw_2$ and $\delta(a_0, wx) \neq \delta(a_0, wy) = \delta(a_0, w'y)$. Therefore, we can reach Letichevsky's criterion substituting a_0, u, v, p, q for $\delta(a_0, w), x, y, w_1pw, w_2qw$.

Now we assume, say, $|v| = 0$. Then, by our assumptions, $|q| > 0$ with $\delta(a_0, q) = a_0$. On the other hand, $\delta(a_0, u) \neq \delta(a_0, v) = a_0$ implies $|u| > 0$. In addition, then we have $(a_0 = \delta(a_0, v) =)\delta(a_0, q) \neq \delta(a_0, u)$. Therefore, there are input words $w, w', w_1, w_2 \in X^*$ and input letters $x, y \in X$ such that $u = wxw_1, q = w'yw_2$ and $\delta(a_0, wx) \neq$

$\delta(a_0, wy) = \delta(a_0, w'y)$. We obtain again Letichevsky's criterion substituting a_0, u, v, p, q for $\delta(a_0, w), x, y, w_1pw, w_2w$. □

Now we study automata having no Letichevsky's criteria. The following statement is obvious.

Proposition 2 *$\mathcal{A} = (A, X, \delta)$ is a automaton without any Letichevsky criteria if and only if for every state $a_0 \in A$, input letters $x, y \in X$ and an input word $p \in X^*$ having $\delta(a_0, xp) = a_0$, it holds that $\delta(a_0, x) = \delta(a_0, y)$.* □

Obviously, if $\mathcal{A} = (A, X, \delta)$ has the above properties then there exists a nonnegative integer n such that for every $p \in X^*$ with $|p| \geq n$, each $\delta(a, p)$ generates an autonomous state-subautomaton of $\mathcal{A}$. Denote by $n_{\mathcal{A}} (\leq n)$ the minimal nonnegative integer having this property.

Proposition 3 $n_{\mathcal{A}} \leq \max(|A| - 2, 0)$.

Proof: Take out of consideration the trivial cases. Thus we may assume $|A| > 2$. Consider $a \in A, x_1, \ldots, x_{m+2} \in X$ having $\delta(a, x_1 \cdots x_m x_{m+1}) \neq \delta(a, x_1 \cdots x_m x_{m+2})$. If $a, \delta(a, x_1), \delta(a, x_1x_2), \ldots, \delta(a, x_1 \cdots x_m), \delta(a, x_1 \cdots x_m x_{m+1}), \delta(a, x_1 \cdots x_m x_{m+2})$ are not distinct states then $\mathcal{A}$ satisfies either Letichevsky's criterion or the semi-Letichevsky criterion, a contradiction. Hence, $m \leq |A| - 3$. Thus $n_{\mathcal{A}} \leq |A| - 2$. □

We also note the next direct consequence of Proposition 2.

Proposition 4 *If $\mathcal{A}$ is a strongly connected automaton without any Letichevsky criteria then $\mathcal{A}$ is autonomous.* □

By this observation, we get immediately the following

Proposition 5 *Suppose that $\mathcal{A} = (A, X, \delta)$ is a strongly connected automaton without any Letichevsky criteria. There exists a $k > 0$ such that for every $a, b \in A$, $a = b$ if and only if there exists a pair $p, q \in X^*$ with $|p| \equiv |q| (\bmod k)$ and $\delta(a, p) = \delta(b, q)$.*□

Lemma 6 *Given an automaton $\mathcal{A} = (A, X, \delta)$ be without any Letichevsky criteria, $a \in A$ is a state of a strongly connected state-subautomaton of $\mathcal{A}$ if and only if there exists a nonempty word $p \in X^*$ with $\delta(a, p) = a$.*

Proof: Let $a \in A$ be a state of a strongly connected state-subautomaton of $\mathcal{A}$. By definition, for every nonempty word $q \in X^*$, there exists a word $r \in X^*$ with $\delta(a, qr) = a$. Conversely, suppose that $\delta(a, p) = a$ for some $a \in A$ and $p \in X^*, p \neq \lambda$. Then for every prefix p' of p and input letters $x, y \in X$, $\delta(a, p'x) = \delta(a, p'y)$. Therefore, for every $q \in X^*$, $\delta(a, q) = \delta(a, r)$, where r is a prefix of p with $|q| \equiv |r| (\bmod |p|)$. But then a generates a strongly connected state-subautomaton of $\mathcal{A}$. □

We shall use the following consequence of the above statement.

Proposition 7 *Let $\mathcal{A} = (A, X, \delta)$ be an automaton without any Letichevsky criteria. Moreover, suppose that $a \in A$ is not a state of any strongly connected state-subautomaton of $\mathcal{A}$. If $\delta(b,p) = a$ for some $b \in A$ and nonempty $p \in X^*$ then $\delta(a,q) \neq b, q \in X^*$. Conversely, if $\delta(a,r) = c$ for some $c \in A$ and nonempty $r \in X^*$ then $\delta(c,q) \neq a, q \in X^*$.*

Lemma 8 *Let $\mathcal{A} = (A, X, \delta)$ be a automaton without any Letichevsky's criteria. If there are $a \in A, q, q' \in X^*, |q| = |q'| \geq |A| - 1, \delta(a,q) \neq \delta(a,q')$ then for every pair of words $r, r' \in X^*, |r| = |r'|$ we have $\delta(a,qr) \neq \delta(a,q'r')$.*

Proof: Suppose that our statement does not hold, i.e., there are $a \in A, q, q'r, r' \in X^*, |q| = |q'| \geq |A| - 1, |r| = |r'|$ having $\delta(a,q) \neq \delta(a,q')$ and $\delta(a,qr) = \delta(a,q'r')$. Then, of course, $|r| = |r'| > 0$. We distuinguish the following three cases.

Case 1. There are $q_1, r_1, q_2, r_2, q_1', r_1', q_2', r_2'$ with $q = q_1r_1 = q_2r_2, q' = q_1'r_1' = q_2'r_2', |q_1| < |q_2|, |q_1'| < |q_2'|$ such that $\delta(a,q_1) = \delta(a,q_2), \delta(a,q_1') = \delta(a,q_2')$. [2] But then, by Proposition 2, $\delta(a,q_1w) = \delta(a,q_1w')$ and $\delta(a,q_1'w) = \delta(a,q_1'w')$ for every $w, w' \in X^*, |w| = |w'|$. Thus, because of $\delta(a,q_1) = \delta(a,q_2)$ and $\delta(a,q_1') = \delta(a,q_2')$, we obtain that, for every $w, w' \in X^*$ there are $z, z' \in X^*$ with $\delta(a,q_1wz) = \delta(a,q_1)$ and $\delta(a,q_1'w'z') = \delta(a,q_1')$. Thus $q_1r_1 = q, q_1'r_1' = q'$ imply that $\delta(a,qrz) = \delta(a,q_1)$ and $\delta(a,q'r'z') = \delta(a,q_1')$ hold for some $z, z' \in X^*$. This means that $\delta(a,qrzr_1) = \delta(a,q)$ and $\delta(a,q'r'z'r_1') = \delta(a,q')$. Put $b = \delta(a,qr)(= \delta(a,q'r')), c = \delta(a,q), c' = \delta(a,q')$. Then $\delta(b,zr_1) = c \neq c' = \delta(b,z'r_1')$ and $\delta(c,r) = \delta(c',r') = b$. But then $|r| = |r'| > 0$ implies $|zr_1r|, |z'r_1'r'| > 0$. Therefore, by Proposition 1, $\mathcal{A}$ satisfies Letichevsky's criterion, a contradiction.

Case 2. There are q_1, r_1, q_2, r_2 with $q = q_1r_1 = q_2r_2, |q_1| < |q_2|$, such that $\delta(a,q_1) = \delta(a,q_2)$, but $\delta(a,q_1') \neq \delta(a,q_2')$ holds for every distinct prefixes q_1', q_2' of q'. Then, because of $|q| = |q'| \geq |A| - 1$, we necessarily have $|q| = |q'| = |A| - 1$, moreover, we also have that for every $d \in A$ there exists a prefix q_1' of q' with $\delta(a,q_1') = d$. (Indeed, we assumed $\delta(a,q_1') \neq \delta(a,q_2')$ for every distinct prefixes q_1', q_2' of q', where $|q'| = |A| - 1$.)

And then for every $d \in A$ there exists an $r_1' \in X^*$ having $\delta(d,r_1') = \delta(a,q')$. On the other hand, we may assume $\delta(a,qrzr_1) = \delta(a,q)$ as in the previous case.

Now we suppose again $\delta(a,qr) = \delta(a,q'r')$ as before. Substituting d for $\delta(a,qrzr_1)$, there exists an $r_1' \in X^*$ holding $\delta(a,qrzr_1r_1') = \delta(a,q_1')$. Put $b = \delta(a,qr), c = \delta(a,q), c' = \delta(a,q')$. But then $|r| = |r'| > 0$ implies $|zr_1r|, |zr_1r_1'r'| > 0$. Therefore, by Proposition 1 we obtain again that $\mathcal{A}$ satisfies Letichevsky's criterion contrary of our assumptions.

Case 3. Let $\delta(a,q_1) \neq \delta(a,q_2)$ and $\delta(a,q_1') \neq \delta(a,q_2')$ for every distinct prefixes q_1, q_2 of q and q_1', q_2' of q', respectively. Then for every $d \in A$ there are $r_1, r_1' \in X^*$ having $\delta(d,r_1) = \delta(a,q)$ and $\delta(d,r_1') = \delta(a,q')$. Therefore, assuming $\delta(a,qr) = \delta(a,q'r')$ for some $r, r' \in X^*$, and substituting d for $\delta(a,qr) = \delta(a,q'r')$, we obtain $\delta(a,qrr_1) = \delta(a,q), \delta(a,qrr_1') = \delta(a,q')$ (with $\delta(a,qr) = \delta(a,q'r')$). Put $c = \delta(a,q), c' = \delta(a,q')$.

[2] This holds automatically if $|q| = |q'| \geq |A|$.

Then $\delta(d,r_1) = c, \delta(d,r_1') = c', \delta(c,r) = \delta(c',r') = d$ such that, by $|r| = |r'| > 0$, $|r_1 r|, |r_1' r'| > 0$. By Proposition 1, this implies that $\mathcal{A}$ satisfies Letichevsky's criterion, a contradiction again. □

Theorem 9 *Let $\mathcal{A} = (A, X, \delta)$ be a automaton without any Letichevsky's criteria. For every state $a \in A$ we have one of the following two possibilities:*

(i) there exist $q, q' \in X^*, |q| = |q'| \geq |A| - 1$ such that $\delta(a, qr) \neq \delta(a, q'r')$ for every $r, r' \in X^*, |r| = |r'|$,

(ii) $\delta(a,q) = \delta(a,q')$ for every $q, q' \in X^*, |q| = |q'| \geq |A| - 1$.

Proof: Suppose that (i) does not hold. Then for every $q, q' \in X^*, |q| = |q'| \geq |A| - 1$ there exist $r, r' \in X^*, |r| = |r'|$ having $\delta(a, qr) = \delta(a, q'r')$. Using Lemma 8, $\delta(a, qr) = \delta(a, q'r'), |r| = |r'|$ and $|q| = |q'| \geq |A| - 1$ implies $\delta(a,q) = \delta(a,q')$. Thus (ii) holds whenever (i) does not hold. □

The following statement is obvious.

Lemma 10 *Given a digraph $\mathcal{D} = (V, E)$, let $v \in V, p_1, p_2, p_2', p_3, p_4 \in V^*$ such that $p_1 p_2 p_3 v p_4 v$ and $p_1 p_2' p_3 v p_4 v$ are walks and $v p_4 v$ is a cycle. $|p_2| \equiv |p_2'| (\mathrm{mod}\, |p_4 v|)$ if and only if there are positive integers k, ℓ having $|p_1 p_2 p_3 v (p_4 v)^k| = |p_1 p_2' p_3 v (p_4 v)^\ell|$.* □

We finish the paper studying both types of states given in Theorem 9.

Proposition 11 *Let $\mathcal{A} = (A, X, \delta)$ be an automaton without any Letichevsky's criteria. Consider a state $a \in A$ and suppose that there are $q, q' \in X^*, |q| = |q'| \geq |A| - 1$, $\delta(a,q) \neq \delta(a,q')$. Then there are q, q' having this property for which $q = uv$ and $q' = uv'$ for some $u, v, v' \in X^*$ such that for every prefixes r of v and r' of v' with $|r| = |r'| > 0$ we have $\delta(a, ur) \neq \delta(a, ur')$, and simultaneously, for every $w, z_1, z_2, w', z_1', z_2', |w|, |w'| > 0$ with $v = w z_1 z_2, v' = w' z_1' z_2'$ we obtain $z_1 = z_1'$ whenever $\delta(a, uw) = \delta(a, uw')$, and $|z_1| = |z_1'|$.*

Proof: Consider $a \in A$ and suppose that our conditions hold, i.e., there are $q, q' \in X^*$ having $|q| = |q'| \geq |A| - 1, \delta(a,q) \neq \delta(a,q')$. Then Proposition 3 implies that $\delta(a,q)$ and $\delta(a,q')$ generate autonomous state subautomata of $\mathcal{A}$. We will distinguish the following cases (omitting some of the analogous cases):

Case 1. There are $u, u', v, v', \in X^*$ such that $q = uv, q' = u'v', \delta(a,u) = \delta(a,u')$ and for every nonempty prefixes r of v and r' of v', $\delta(a,u) \neq \delta(a, u'r'), \delta(a,u') \neq \delta(a, ur)$, and $\delta(a, ur) \neq \delta(a, u'r')$.[3] Let, say, $|u| \geq |u'|$ and let v'' be a prefix of v' with $|v''| = |v|$. Change q' for uv'' and then we will have our requirements.

Case 2. There exist a prefix u of q having $\delta(a,u) = \delta(a,q')$. Let $t_2 \in X^*$ be a nonempty word with minimal length having $\delta(a, q't_1t_2) = \delta(a, q't_1)$ for some word

[3] $u = u' = \lambda$ is possible.

$t_1 \in X^*$ and assume that t_2 is minimal in the sense that for every nonempty $p \in X^*$, $\delta(a, q't_1p) = \delta(a, q't_1)$ implies $|t_2| \leq |p|$.[4] Then, using that $\delta(a, q')$ generates an autonomous state subautomaton of $\mathcal{A}$, we have $q = uv$, where v is a nonempty prefix of $t_1t_2^k$ for a suitable $k \geq 0$.

Prove that in this case $u \equiv |q'|(\mathrm{mod}\,|t_2|)$ is impossible. Assume the contrary. Recall again that $\delta(a, q')$ generates an autonomous state subautomaton of $\mathcal{A}$. But then, applying Lemma 10, there are words $r, r' \in X^*, |r| = |r'|$ having $\delta(a, qr) = \delta(a, q'r')$. By Lemma 8, then $|q| = |q'| < |A| - 1$ contrary of our assumptions. Thus we have the following cases.

Case 2.1. Suppose $u \not\equiv |q'|(\mathrm{mod}\,|t_2|)$ such that for every prefixes u_1 of u and u_1' of q' with $u_1u_1' \neq \lambda$, $\delta(a, u_1) = \delta(a, u_1')$ implies $u_1 = u$ and $u_1' = q'$. Then we obtain our requirements again (having $q = uv$, where v is a nonempty prefix of $t_1t_2^k$ for a suitable $k \geq 0$).

Case 2.2. Assume $u \not\equiv |q'|(\mathrm{mod}\,|t_2|)$, and simultaneously, let for some prefixes u_1 of u and u_1' of q', $\delta(a, u_1) = \delta(a, u_1')$ such that $u = u_1v_1$, $q' = u_1'v_1'$, furthermore, $\lambda \in \{u_1, v_1\}$ implies $\lambda \notin \{u_1', v_1'\}$ and $\lambda \in \{u_1', v_1'\}$ implies $\lambda \notin \{u_1, v_1\}$. If $v_1 = \lambda$ and $v_1' \neq \lambda$ then $\delta(a, u_1') = \delta(a, u_1'v_1') \neq \delta(a, u_1'v_1'v)(= \delta(a, uv))$ such that v is a nonempty suffix of q. But then $\mathcal{A}$ has either Letichevsky's criterion or the semi-Letichevsky criterion, a contradiction. Similarly, it also lead to a contradiction is we assume $v_1 \neq \lambda$ and $v_1' = \lambda$. Thus $\lambda \notin \{v_1, v_1'\}$ can be assumed and we may also assume $\lambda \notin \{u_1, u_1'\}$ analogously.

By $u \not\equiv |q'|(\mathrm{mod}\,|t_2|)$, either $|u_1| \not\equiv |u_1'|(\mathrm{mod}\,|t_2|)$, or $|v_1| \not\equiv |v_1'|(\mathrm{mod}\,|t_2|)$.

Case 2.2.1. Suppose $|u_1| \not\equiv |u_1'|(\mathrm{mod}\,|t_2|)$ and let, say, $|v_1| \geq |v_1'|$. Take a prefix v' of $t_1t_2^k$ for a suitable $k \geq 0$ with $|u_1'v_1v'| = |q|$ and let us consider $u_1'v_1v'$ instead of q'.

Case 2.2.2. Suppose $|u_1| \equiv |u_1'|(\mathrm{mod}\,|t_2|)$. Then $|v_1| \not\equiv |v_1'|(\mathrm{mod}\,|t_2|)$. Let, say, $|u_1| \geq |u_1'|$. Take a prefix v' of $t_1t_2^k$ for a suitable $k \geq 0$ with $|u_1v_1'v'| = |q|$ and change $u_1v_1'v'$ for q'.

In both of the above Case 2.2.1 and Case 2.2.2, we have words[5] $w, w_1, w_2, w_1', w_2' \in X^*, \lambda \notin \{w_1, w_1'\}$, $w_1 \not\equiv |w_1'|(\mathrm{mod}\,|t_2|)$, w_2' is a prefix of w_2 (or, in the opposite case, w_2 is a prefix of w_2'), $q = ww_1w_2, q' = ww_1'w_2'$, such that $\delta(a, ww_1) = \delta(a, ww_1')$. Then let $w, w_1, w_2, w_1', w_2' \in X^*$ be arbitrary having these properties for which $\min(|w_1|, |w_2|)$ is minimal.

If for every nonempty proper prefixes z_1 of w_1 and z_1' of w_1' we have $\delta(a, w) \notin \{\delta(a, wz_1'), \delta(a, wz_1')\}$ and $\delta(a, wz_1) \neq \delta(a, wz_1')$ then we are ready having our properties for $q = ww_1w_2, q' = ww_1'w_2'$.

Now we assume $|w_1| \not\equiv |w_1'|(\mathrm{mod}\,|t_2|)$ such that for some prefixes z_1 of w_1 and z_1' of w_1', $\delta(a, z_1) = \delta(a, z_1')$ such that $w_1 = z_1z_2$, $w_1' = z_1'z_2'$, furthermore, $\lambda \in \{z_1, z_2\}$ implies $\lambda \notin \{z_1', z_2'\}$ and $\lambda \in \{z_1', z_2'\}$ implies $\lambda \notin \{z_1, z_2\}$. We can prove $\lambda \notin \{z_1, z_1', z_2, z_2'\}$ similarly as before. Then either $|z_1| \not\equiv |z_1'|(\mathrm{mod}\,|t_2|)$ or $|z_2| \not\equiv |z_2'|(\mathrm{mod}\,|t_2|)$. It remains to prove that these cases are impossible.

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[4]The finiteness of the state set of $\mathcal{A}$ implies the existence of t_1 and t_2.

[5]in Case 2a, of course, $w = \lambda$, .

If $|z_1| \not\equiv |z_1'| (\bmod |t_2|)$ and, say, $|z_2| \geq |z_2'|$ then considering the prefix w_2'' of w_2' having $|z_1' w_2''| = |z_1 w_2|$, we can take $w, z_1, z_2 w_2, z_1', z_2 w_2''$ as w, w_1, w_2, w_1', w_2' contrary of the the minimality of $\min(|w_1|, |w_2|)$.

If $|z_1| \equiv |z_1'| (\bmod |t_2|)$ with $|z_2| \not\equiv |z_2'| (\bmod |t_2|)$ and, say, $|z_1| \geq |z_1'|$ then considering the prefix w_2'' of w_2' having $|z_2' w_2''| = |z_2 w_2|$, we can take $wz_1, z_2, z_2', w_2, w_2''$ as w, w_1, w_1', w_2, w_2' contradicting the minimality of $\min(|w_1|, |w_2|)$.

The proof is complete. □

Proposition 12 *Let $\mathcal{A} = (A, X, \delta)$ be an automaton without any Letichevsky's criteria. Consider $a, a_0 \in A, p \in X^*$ with $\delta(a_0, p) = a$ and suppose that $\delta(a, r) = \delta(a, r')$ holds for every $r, r' \in X^*, |pr| = |pr'| \geq |A| - 1$. Assume that $\delta(a, q) \neq \delta(a, q')$ holds for some $q, q' \in X^*, |pq| = |pq'| (< |A| - 1)$ and let q, q' be words of maximal length having this property. Then there are q, q' with this property having*

(i) $q = uv$ and $q' = uv'$ for some $u, v, v' \in X^*$ such that for every prefixes r of v and r' of v' with $|r| = |r'| > 0$ we have $\delta(a, ur) \neq \delta(a, ur')$, and simultaneously, for every $w, z_1, z_2, w', z_1', z_2'$ with $v = wz_1z_2, v' = w'z_1'z_2'$ we obtain $z_1 = z_1'$ whenever $\delta(a, uw) = \delta(a, uw')$, and $|z_1| = |z_1'|$;

(ii) for every distinct prefixes p_1, p_2 of pq, $\delta(a_0, p_1) \neq \delta(a_0, p_2)$.

Proof: Consider $a \in A$ and suppose that our conditions hold.

First we suppose that, whenever $uu' \neq \lambda$, $\delta(a, u) = \delta(a, u')$ implies $u = q$ and $u' = q'$ for every prefixes u of q and u' of q'. It is clear that then we are ready.

Assume the opposite case and let $q = uv, q' = u'v'$ with $\lambda \notin \{uu', vv'\}$ such that $\delta(a, u) = \delta(a, u')$.

Let $\min(|u|, |u'|)$ be maximal with the above property and prove that in this case $u = u'$ can be assumed. Indeed, if it true if $|u| = |u'|$ because we can consider, say, uv' instead of $u'v'$.

Finally, prove that, say, $|u| > |u'|$ is impossible. Indeed, otherwise we could change q' for uv'', where v'' is a prefix of v' with $|v''| = |v'|$. This contradicts of the maximality of $\min(|u|, |u'|)$.

Now we prove *(ii)* omitting some analogous cases. If there are no distinct prefixes $p_1', p_2' \in X^*$ of pq' with $\delta(a_0, p_1') = \delta(a_0, p_2')$ for pq' and pq. Therefore, in this case, we are ready. Otherwise, we may suppose $\delta(a_0, p_1') = \delta(a_0, p_2')$ for some distinct prefixes $p_1', p_2' \in X^*$ of pq'. Let, say, $p_1' = p_2'r'$ for some nonempty $r' \in X$. By Lemma 2 and $\delta(a_0, pq) \neq \delta(a_0, pq')$, this implies that $\delta(a_0, p_2')$ generates an autonomous state-subautomaton $\mathcal{B}$ of $\mathcal{A}$. Moreover, $\delta(a_0, p_1') = \delta(a_0, p_2'r') = \delta(a_0, p_2'), r' \neq \lambda$ implies that this autonomous state-subautomaton is strongly connected. On the other hand, by the maximality of $|q| (= |q'|)$, $\delta(a_0, pqx) = \delta(a_0, pq'x')$ holds for every $x, x' \in X$. Thus, $\delta(a_0, pqx)$ is also a state of the state-subautomaton $\mathcal{B}$ of $\mathcal{A}$. Recall that by the maximality of q and q', we have $\delta(a_0, pqx) = \delta(a_0, p'q'x'), x, x' \in X$. Then $\delta(a_0, pq) \neq \delta(a_0, pq')$ and $\delta(a_0, pqx) = \delta(a_0, pq'x')$ imply that $\delta(a_0, pq)$ is not a state of $\mathcal{B}$. Therefore, for every prefix p_1 of pq, $\delta(a_0, p_1)$ is not a state of $\mathcal{B}$.

Suppose that, contrary of our assumptions, $\delta(a_0, p_1) = \delta(a_0, p_2)$ holds for distinct prefixes p_1 and p_2 of pq and put, say, $p_1 = p_2 r_1$ (where $r_1 \neq \lambda$ is assumed). In other words, $\delta(a_0, p_2 r_1) = \delta(a_0, p_2)$ holds such that $\delta(a_0, p_2)$ is not a state of $\mathcal{B}$. But $\delta(a_0, pqx) = \delta(a_0, pq'x'), x, x' \in X$ implies that there exists an $r_2 \in X^*$ such that $\delta(a_0, p_2 r_2)$ is a state of $\mathcal{B}$. Clearly, then $\mathcal{A}$ satisfies either Letichevsky's criterion or the semi-Letichevsky criterion, a contradiction. This completes the proof. □

References

[1] Z. Ésik, Homomorphically complete classes of automata with respect to the α_2-product. *Acta Sci. Math.*, **48** (1985), 135–141.

[2] Z. Ésik and Gy. Horváth, The α_2-product is homomorphically general. *Papers on Automata Theory V*, No. DM83-3, Dep. Math., Karl Marx University of Economics, Budapest (1983), 49–62.

[3] F. Gécseg, *Products of Automata.* EATCS Monographs on Theor. Comput. Sci., Vol. **7**, Springer-Verlag, Berlin - Heidelberg - New York - Tokyo, 1986.

[4] A. A. Letichevsky, Conditions of completeness for finite automata (in Russian). *Žurn. Mat. i Mat. Fiz.*, 1 (1961), 702–710.