Some Remarks on Automata without Letichevsky Criteria

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Abstract: In this paper we show some properties of finite automata having no Letichevsky criteria

Keywords: Finite automata, Letichevsky criterion.

1. Introduction

We start with some standard concepts and notations. The elements of an alphabet \( X \) are called letters (\( X \) is supposed to be finite and nonempty). A word over an alphabet \( X \) is a finite string consisting of letters of \( X \). The string consisting of zero letters is called the empty word, written by \( \lambda \). The length of a word \( w \), in symbols \(|w|\), means the number of letters in \( w \) when each letter is counted as many times it occurs. By definition, \(|\lambda| = 0\). At the same time, for any set \( H \), \(|H|\) denotes the cardinality of \( H \). If \( u \) and \( v \) are words over an alphabet \( X \), then their catenation \( uv \) is also a word over \( X \). Catenation is an associative operation and the empty word \( \lambda \) is the identity with respect to catenation: \( w\lambda = \lambda w = w \) for any word \( w \). For a word \( w \) and positive integer \( n \), the notation \( w^n \) means the word obtained by catenating \( n \) copies of the word \( w \). \( w^0 \) equals the empty word \( \lambda \). \( w^n \) is called the \( m \)-th power of \( w \) for any non-negative integer \( m \).

Let \( X^* \) be the set of all words over \( X \), moreover, let \( X^* = X^* \setminus \{\lambda\} \). \( X^* \) and \( X^+ \) are the free monoid and the free semigroup, respectively, generated by \( X \) under catenation.

A (finite) directed graph (or, in short, a digraph) \( D = (V, E) \) (of order \(|V| > 0| \) is a pair consisting of sets of vertices \( V \) and edges \( E \subseteq V \times V \). A walk in \( D = (V, E) \) is a sequence of vertices \( v_1, \ldots, v_n, n > 1 \) such that \((v_i, v_{i+1}) \in E, i = 1, \ldots, n - 1 \). A walk is closed if \( v_1 = v_n \). By a (directed) path from a vertex \( a \) to a vertex \( b \neq a \) we shall mean a sequence \( v_1 \ldots v_n, n > 1 \) of pairwise distinct vertices such that \( a = v_1, b = v_n \).
and \((v_i, v_{i+1}) \in E\) for every \(i = 1, \ldots, n - 1\). The positive integer \(n - 1\) is called the length of the path. Thus a path is a walk with all \(n\) vertices distinct. A closed walk with all vertices distinct except \(v_1 = v_n\) is a cycle of length \(n - 1\).

By an automaton we mean a finite automaton without outputs. Given an automaton \(\mathcal{A} = (A, X, \delta)\) with set of states \(A\), set of input letters \(X\), and transition \(\delta : A \times X \rightarrow A\), it is understood that \(\delta\) is extended to \(\delta^* : A \times X^* \rightarrow A\) with \(\delta^*(a, \lambda) = a\), \(\delta^*(a, xq) = \delta^*(\delta(a, x), q)\). In the sequel, we will consider the transition of an automaton in this extended form and thus we will denote it by the same Greek letter \(\delta\). Let \(\mathcal{A} = (A, X, \delta)\) be an automaton. It is said that a state \(a \in A\) generates a state \(b \in A\) if \(\delta(a, p) = b\) holds for some \(p \in X^*\). For every state \(a \in A\) define the state subautomaton \(\mathcal{B} = (B, X, \delta')\) generated by \(a\) such that \(B = \{b \mid b = \delta(a, p), p \in X^*\}\), moreover, \(\delta'(b, x) = \delta(b, x)\) for every pair \(b \in B, x \in X\). \(\mathcal{A}\) is called strongly connected if for every pair \(a, b \in A\) there exists \(p \in X^*\) such that \(\delta(a, p) = b\).

We say that \(\mathcal{A}\) satisfies Letichevsky’s criterion if there are a state \(a \in A\), input letters \(x, y \in X\), input words \(p, q \in X^*\) such that \(\delta(a, x) \neq \delta(a, y)\) and \(\delta(a, xp) = \delta(a, yq) = a\). It is said that \(\mathcal{A}\) satisfies the semi-Letichevsky criterion if it does not satisfy Letichevsky’s criterion but there are a state \(a \in A\), input letters \(x, y \in X\), an input word \(p \in X^*\) such that \(\delta(a, x) \neq \delta(a, y)\), \(\delta(a, xp) = a\) and for every \(q \in X^*\), \(\delta(a, yq) \neq a\). If \(\mathcal{A}\) do not satisfy either Letichevsky’s criterion or the semi-Letichevsky criterion then we say that \(\mathcal{A}\) does not satisfy any Letichevsky criteria or is without any Letichevsky criteria.

The Letichevsky criterion has a central role in the investigations of products of automata (see [1],[2],[3],[4]). Automata having semi-Letichevsky criterion and automata without any Letichevsky criteria are also important in the classical result of Z. Ésik and Gy. Horváth (see [2],[3]). In this paper we investigate automata without any Letichevsky criteria.

2. Results

First we observe

**Proposition 1** Given an automaton \(\mathcal{A} = (A, X, \delta)\), a state \(a_0 \in A\), four input words \(u, v, p, q \in X^*\) with \(|up|, |vq| > 0\) under which \(\delta(a_0, u) \neq \delta(a_0, v)\), and \(\delta(a_0, up) = \delta(a_0, vq) = a_0\). Then \(\mathcal{A}\) satisfies Letichevsky’s criterion.

**Proof:** First we suppose \(|u|, |v| > 0\). Then there exist input words \(w, w', w_1, w_2 \in X^*\) and input letters \(x, y \in X\) such that \(u = wxw_1, v = w'yw_2\) and \(\delta(a_0, wx) \neq \delta(a_0, wy) = \delta(a_0, w'y)\). Therefore, we can reach Letichevsky’s criterion substituting \(a_0, u, v, p, q\) for \(\delta(a_0, u), x, y, w_1, w_2\).

Now we assume, say, \(|v| = 0\). Then, by our assumptions, \(|q| > 0\) with \(\delta(a_0, q) = a_0\). On the other hand, \(\delta(a_0, u) \neq \delta(a_0, v) = a_0\) implies \(|u| > 0\). In addition, then we have \((a_0 = \delta(a_0, v) =) \delta(a_0, q) \neq \delta(a_0, u)\). Therefore, there are input words \(w, w', w_1, w_2 \in X^*\) and input letters \(x, y \in X\) such that \(u = wxw_1, q = w'yw_2\) and \(\delta(a_0, wx) \neq \delta(a_0, wy) = \delta(a_0, wy)\).
\[ \delta(a_0, wy) = \delta(a_0, w'y). \] We obtain again Letichevsky's criterion substituting \( a_0, u, v, p, q \) for \( \delta(a_0, w), x, y, w_1pw, w_2w. \) \( \square \)

Now we study automata having no Letichevsky's criteria. The following statement is obvious.

**Proposition 2** \( \mathcal{A} = (A, X, \delta) \) is a automaton without any Letichevsky criteria if and only if for every state \( a_0 \in A \), input letters \( x, y \in X \) and an input word \( p \in X^* \) having \( \delta(a_0, xp) = a_0 \), it holds that \( \delta(a_0, x) = \delta(a_0, y). \) \( \square \)

Obviously, if \( \mathcal{A} = (A, X, \delta) \) has the above properties then there exists a nonnegative integer \( n \) such that for every \( p \in X^* \) with \(|p| \geq n\), each \( \delta(a, p) \) generates an autonomous state-subautomaton of \( \mathcal{A} \). Denote by \( n_{\mathcal{A}}(\leq n) \) the minimal nonnegative integer having this property.

**Proposition 3** \( n_{\mathcal{A}} \leq \max(|A| - 2, 0) \).

*Proof:* Take out of consideration the trivial cases. Thus we may assume \(|A| > 2\). Consider \( a \in A, x_1, \ldots, x_{m+2} \in X \) having \( \delta(a, x_1 \cdots x_m x_{m+1}) \neq \delta(a, x_1 \cdots x_m x_{m+2}). \) If \( a, \delta(a, x_1), \delta(a, x_1 x_2), \ldots, \delta(a, x_1 \cdots x_m) \) are not distinct states then \( \mathcal{A} \) satisfies either Letichevsky's criterion or the semi-Letichevsky criterion, a contradiction. Hence, \( m \leq |A| - 3 \). Thus \( n_{\mathcal{A}} \leq |A| - 2. \) \( \square \)

We also note the next direct consequence of Proposition 2.

**Proposition 4** If \( \mathcal{A} \) is a strongly connected automaton without any Letichevsky criteria then \( \mathcal{A} \) is autonomous. \( \square \)

By this observation, we get immediately the following

**Proposition 5** Suppose that \( \mathcal{A} = (A, X, \delta) \) is a strongly connected automaton without any Letichevsky criteria. There exists a \( k > 0 \) such that for every \( a, b \in A \), \( a = b \) if and only if there exists a pair \( p, q \in X^* \) with \(|p| \equiv |q| \pmod{k} \) and \( \delta(a, p) = \delta(b, q). \) \( \square \)

**Lemma 6** Given an automaton \( \mathcal{A} = (A, X, \delta) \) be without any Letichevsky criteria, \( a \in A \) is a state of a strongly connected state-subautomaton of \( \mathcal{A} \) if and only if there exists a nonempty word \( p \in X^* \) with \( \delta(a, p) = a. \)

*Proof:* Let \( a \in A \) be a state of a strongly connected state-subautomaton of \( \mathcal{A} \). By definition, for every nonempty word \( q \in X^* \), there exists a word \( r \in X^* \) with \( \delta(a, qr) = a. \) Conversely, suppose that \( \delta(a, p) = a \) for some \( a \in A \) and \( p \in X^*, p \neq \lambda \). Then for every prefix \( p' \) of \( p \) and input letters \( x, y \in X \), \( \delta(a, p'x) = \delta(a, p'y) \). Therefore, for every \( q \in X^* \), \( \delta(a, q) = \delta(a, r) \), where \( r \) is a prefix of \( p \) with \(|q| \equiv |r| \pmod{|p|} \). But then \( a \) generates a strongly connected state-subautomaton of \( \mathcal{A}. \) \( \square \)

We shall use the following consequence of the above statement.
Proposition 7 Let $A = (A, X, \delta)$ be an automaton without any Letichevsky criteria. Moreover, suppose that $a \in A$ and nonempty $p \in X^*$ then $\delta(a, q) \neq b, q \in X^*$. Conversely, if $\delta(a, r) = c$ for some $c \in A$ and nonempty $r \in X^*$ then $\delta(c, q) \neq a, q \in X^*$.

Lemma 8 Let $A = (A, X, \delta)$ be a automaton without any Letichevsky's criteria. If there are $a \in A, q, q' \in X^*, |q| = |q'| \geq |A| - 1, \delta(a, q) \neq \delta(a, q')$ then for every pair of words $r, r' \in X^*, |r| = |r'|$ we have $\delta(a, qr) \neq \delta(a, q'r')$.

Proof: Suppose that our statement does not hold, i.e., there are $a \in A, q, q'r, r' \in X^*, |q| = |q'| \geq |A| - 1, |r| = |r'|$ having $\delta(a, q) \neq \delta(a, q')$ and $\delta(a, qr) = \delta(a, q'r')$. Then, of course, $|r| = |r'| > 0$. We distinquish the following three cases.

Case 1. There are $q_1, r_1, q_2, r_2, q_1', r_1', q_2', r_2'$ with $q = q_1 r_1 = q_2 r_2, q' = q_1 r_1' = q_2 r_2'$, $|q| < |q_1|, |q'| < |q_2|$, such that $\delta(a, q_1) = \delta(a, q_2), \delta(a, q_1') = \delta(a, q_2')$. But then, by Proposition 2, $\delta(a, q_1 w) = \delta(a, q_1 w')$ and $\delta(a, q_2 w) = \delta(a, q_2 w')$ for every $w, w' \in X^*, |w| = |w'|$. Thus, because of $\delta(a, q_1) = \delta(a, q_2)$ and $\delta(a, q_1') = \delta(a, q_2')$, we obtain that, for every $w, w' \in X^*$ there are $z, z' \in X^*$ with $\delta(a, q_1 wz) = \delta(a, q_1')$ and $\delta(a, q_2 w'z') = \delta(a, q_2')$. Thus $q_1 r_1 = q, q_1' r_1' = q'$ imply that $\delta(a, q_1 wz) = \delta(a, q_1')$ and $\delta(a, q_2 w'z') = \delta(a, q_2')$ hold for some $z, z' \in X^*$. This means that $\delta(a, q_1 r_1) = \delta(a, q_2)$ and $\delta(a, q_2' r_1') = \delta(a, q_2')$. Put $b = \delta(a, q_1 r_1)(= \delta(a, q_2' r_1')), c = \delta(a, q), c' = \delta(a, q')$. Then $\delta(b, q_1 r_1) = c \neq c' = \delta(b, q_2' r_1')$ and $\delta(c, r) = \delta(c', r') = b$. But then $|r| = |r'| > 0$ implies $|z_1 r_1 r_1|, |z_2 r_1' r_1'| > 0$. Therefore, by Proposition 1, $A$ satisfies Letichevsky's criterion, a contradiction.

Case 2. There are $q_1, r_1, q_2, r_2$ with $q = q_1 r_1 = q_2 r_2, |q_1| < |q_2|$, such that $\delta(a, q_1) = \delta(a, q_2)$, but $\delta(a, q_1') \neq \delta(a, q_2')$ holds for every distinct prefixes $q_1, q_2$ of $q$. Then, because of $|q| = |q'| \geq |A| - 1$, we necessarily have $|q| = |q'| = |A| - 1$, moreover, we also have that for every $d \in A$ there exists a prefix $q_1'$ of $q$ with $\delta(a, q_1') = d$. (Indeed, we assumed $\delta(a, q_1') \neq \delta(a, q_2)$ for every distinct prefixes $q_1, q_2$ of $q'$, where $|q'| = |A| - 1$.)

And then for every $d \in A$ there exists an $r_1' \in X^*$ having $\delta(d, r_1') = \delta(a, q')$. On the other hand, we may assume $\delta(a, q_1 r_1 r_1') = \delta(a, q)$ as in the previous case.

Now we suppose again $\delta(a, q_1') = \delta(a, q_2')$ as before. Substituting $d$ for $\delta(a, q_1 r_1)$, there exists an $r_1' \in X^*$ holding $\delta(a, q_1' r_1 r_1') = \delta(a, q_1')$. Put $b = \delta(a, q_1 r_1), c = \delta(a, q), c' = \delta(a, q')$. But then $|r| = |r'| > 0$ implies $|z_1 r_1 r_1|, |z_2 r_1' r_1'| > 0$. Therefore, by Proposition 1 we obtain again that $A$ satisfies Letichevsky's criterion contrary of our assumptions.

Case 3. Let $\delta(a, q_1) \neq \delta(a, q_2)$ and $\delta(a, q_1') \neq \delta(a, q_2')$ for every distinct prefixes $q_1, q_2$ of $q$ and $q_1', q_2'$ of $q'$, respectively. Then for every $d \in A$ there are $r_1, r_1' \in X^*$ having $\delta(d, r_1) = \delta(a, q)$ and $\delta(d, r_1') = \delta(a, q')$. Therefore, assuming $\delta(a, q_1 r_1) = \delta(a, q_1' r_1')$ for some $r, r' \in X^*$, and substituting $d$ for $\delta(a, q_1 r_1) = \delta(a, q_1' r_1')$, we obtain $\delta(a, q_1 r_1) = \delta(a, q), \delta(a, q_1' r_1') = \delta(a, q')$ (with $\delta(a, q) = \delta(a, q_1 r_1)$). Put $c = \delta(a, q), c' = \delta(a, q')$.

\footnote{This holds automatically if $|q| = |q'| \geq |A|$.}
Then \( \delta(d, r_1) = c, \delta(d, r'_1) = c', \delta(c, r) = \delta(c', r') = d \) such that, by \(|r| = |r'| > 0, |r_1r|, |r'_1r'| > 0\). By Proposition 1, this implies that \( \mathcal{A} \) satisfies Letichevsky’s criterion, a contradiction again.

**Theorem 9** Let \( \mathcal{A} = (\mathcal{A}, \mathcal{X}, \delta) \) be a automaton without any Letichevsky’s criteria. For every state \( a \in \mathcal{A} \) we have one of the following two possibilities:

(i) there exist \( q, q' \in \mathcal{X}^* \) such that \(|q| = |q'| \geq |\mathcal{A}| - 1 \) such that \( \delta(a, qr) \neq \delta(a, q'r') \) for every \( r, r' \in \mathcal{X}^*, |r| = |r'| \).

(ii) \( \delta(a, q) = \delta(a, q') \) for every \( q, q' \in \mathcal{X}^* \) such that \(|q| = |q'| \geq |\mathcal{A}| - 1 \).

**Proof:** Suppose that (i) does not hold. Then for every \( q, q' \in \mathcal{X}^* \) such that \(|q| = |q'| \geq |\mathcal{A}| - 1 \) there exist \( r, r' \in \mathcal{X}^* \) having \( \delta(a, qr) = \delta(a, q'r') \). Using Lemma 8, \( \delta(a, qr) = \delta(a, q'r') \), \(|r| = |r'| \) and \(|q| = |q'| \geq |\mathcal{A}| - 1 \) implies \( \delta(a, q) = \delta(a, q') \). Thus (ii) holds whenever (i) does not hold. \( \square \)

The following statement is obvious.

**Lemma 10** Given a digraph \( \mathcal{D} = (V, E) \), let \( v \in V, p_1, p_2, p_3, p_4 \in V^* \) such that \( p_1p_2p_3vp_4v \) and \( p_1p_2p_3vp_4v \) are walks and \( vp_4v \) is a cycle. \( |p_2| \equiv |p_2'| \) (mod \( |p_4v| \)) if and only if there are positive integers \( k, \ell \) having \( p_1p_2p_3v(p_4v)^k = p_1p_2p_3v(p_4v)^\ell \). \( \square \)

We finish the paper studying both types of states given in Theorem 9.

**Proposition 11** Let \( \mathcal{A} = (\mathcal{A}, \mathcal{X}, \delta) \) be an automaton without any Letichevsky’s criteria. Consider a state \( a \in \mathcal{A} \) and suppose that there are \( q, q' \in \mathcal{X}^* \) such that \(|q| = |q'| \geq |\mathcal{A}| - 1 \), \( \delta(a, q) \neq \delta(a, q') \). Then there are \( q, q' \) having this property for which \( q = uv \) and \( q' = uv' \) for some \( u, v, v' \in \mathcal{X}^* \) such that for every prefixes \( r \) of \( v \) and \( r' \) of \( v' \) with \(|r| = |r'| > 0 \) we have \( \delta(a, ur) \neq \delta(a, ur') \), and simultaneously, for every \( w, z_1, z_2, w', z'_1, z'_2 |w| \geq 0 \), \( w|w'| > 0 \) with \( v = wz_1z_2, v' = w'z'_1z'_2 \) we obtain \( z_1 = z'_1 \) whenever \( \delta(a, uw) = \delta(a, uw') \), and \( |z_1| = |z'_1| \).

**Proof:** Consider \( a \in \mathcal{A} \) and suppose that our conditions hold, i.e., there are \( q, q' \in \mathcal{X}^* \) having \(|q| = |q'| \geq |\mathcal{A}| - 1 \), \( \delta(a, q) \neq \delta(a, q') \). Then Proposition 3 implies that \( \delta(a, q) \) and \( \delta(a, q') \) generate autonomous state subautomata of \( \mathcal{A} \). We will distinguish the following cases (omitting some of the analogous cases):

**Case 1.** There are \( u, u', v, v' \in \mathcal{X}^* \) such that \( q = uv, q' = u'v' \) and for every nonempty prefixes \( r \) of \( v \) and \( r' \) of \( v' \), \( \delta(a, ur) \neq \delta(a, u'r') \), \( \delta(a, ur) \neq \delta(a, ur') \), and \( \delta(a, ur) \neq \delta(a, ur'). \footnote{u = u' = \lambda \ is \ possible.} \) Let, say, \(|u| \geq |u'| \) and let \( v'' \) be a prefix of \( v' \) \( |v''| = |v| \). Change \( q' \) for \( uv'' \) and then we will have our requirements.

**Case 2.** There exist a prefix \( u \) of \( q \) having \( \delta(a, u) = \delta(a, q) \). Let \( t_2 \in \mathcal{X}^* \) be a nonempty word with minimal length having \( \delta(a, q't_2t_2) = \delta(a, q't_1) \) for some word
$t_1 \in X^*$ and assume that $t_2$ is minimal in the sense that for every nonempty $p \in X^*$, $\delta(a, q't_1p) = \delta(a, q't_1)$ implies $|t_2| \leq |p|$.\footnote{The finiteness of the state set of $A$ implies the existence of $t_1$ and $t_2$.} Then, using that $\delta(a, q')$ generates an autonomous state subautomaton of $A$, we have $q = uv$, where $v$ is a nonempty prefix of $t_1t_2^k$ for a suitable $k \geq 0$.

Prove that in this case $u \equiv |q'| \text{(mod } |t_2|)$ is impossible. Assume the contrary. Recall again that $\delta(a, q')$ generates an autonomous state subautomaton of $A$. But then, applying Lemma 10, there are words $r, r' \in X^*, |r| = |r'|$ having $\delta(a, qr) = \delta(a, q'r')$. By Lemma 8, then $|q| = |q'| < |A| - 1$ contrary of our assumptions. Thus we have the following cases.

Case 2.1. Suppose $u \not\equiv |q'| \text{(mod } |t_2|)$ such that for every prefixes $u_1$ of $u$ and $u_1'$ of $q'$ with $u_1u_1' \neq \lambda$, $\delta(a, u_1) = \delta(a, u_1')$ implies $u_1 = u$ and $u_1' = q'$. Then we obtain our requirements again (having $q = uv$, where $v$ is a nonempty prefix of $t_1t_2^k$ for a suitable $k \geq 0$).

Case 2.2. Assume $u \not\equiv |q'| \text{(mod } |t_2|)$, and simultaneously, let for some prefixes $u_1$ of $u$ and $u_1'$ of $q'$, $\delta(a, u_1) = \delta(a, u_1')$ such that $u = u_1v_1$, $q' = u_1v_1'$, furthermore, $\lambda \in \{u_1, v_1\}$ implies $\lambda \not\in \{u_1', v_1'\}$ and $\lambda \in \{u_1', v_1'\}$ implies $\lambda \not\in \{u_1, v_1\}$. If $v_1 = \lambda$ and $v_1' \neq \lambda$ then $\delta(a, u_1) = \delta(a, u_1v_1') \neq \delta(a, u_1v_1v)(= \delta(a, uv))$ such that $v$ is a nonempty suffix of $q$. But then $A$ has either Letichevs'ky's criterion or the semi-Letichevs'ky criterion, a contradiction. Similarly, it also lead to a contradiction is we assume $v_1 \neq \lambda$ and $v_1' = \lambda$. Thus $\lambda \not\in \{v_1, v_1'\}$ can be assumed and we may also assume $\lambda \not\in \{u_1, u_1'\}$ analogously.

By $u \not\equiv |q'| \text{(mod } |t_2|)$, either $|u_1| \not\equiv |u_1'| \text{(mod } |t_2|)$, or $|v_1| \not\equiv |v_1'| \text{(mod } |t_2|)$.

Case 2.2.1. Suppose $|u_1| \not\equiv |u_1'| \text{(mod } |t_2|)$ and let, say, $|v_1| \geq |v_1'|$. Take a prefix $v'$ of $t_1t_2^k$ for a suitable $k \geq 0$ with $|u_1v_1v'| = |q|$ and let us consider $u_1v_1v'$ instead of $q'$.

Case 2.2.2. Suppose $|u_1| \equiv |u_1'| \text{(mod } |t_2|)$. Let, say, $|u_1| \geq |v_1'|$. Take a prefix $v'$ of $t_1t_2^k$ for a suitable $k \geq 0$ with $|u_1v_1v'| = |q|$ and change $u_1v_1v'$ for $q'$.

In both of the above Case 2.2.1 and Case 2.2.2, we have words\footnote{in Case 2a, of course, $w = \lambda$.} $w, w_1, w_2, w_1', w_2' \in X^*, \lambda \not\in \{w_1, w_1'\}$, $w_1 \not\equiv |w_1'| \text{(mod } |t_2|)$, $w_2'$ is a prefix of $w_2$ (or, in the opposite case, $w_2$ is a prefix of $w_2'$), $q = wu_1w_2$, $q' = wu_1w_2'$, such that $\delta(a, wu_1) = \delta(a, wu_1')$. Then let $w, w_1, w_2, w_1', w_2' \in X^*$ be arbitrary having these properties for which min$(|w_1|, |w_2|)$ is minimal.

If for every nonempty proper prefixes $z_1$ of $w_1$ and $z_1'$ of $w_1'$ we have $\delta(a, w) \not\in \{\delta(a, wz_1'), \delta(a, wz_1')\}$ and $\delta(a, wz_1) \neq \delta(a, wz_1')$ then we are ready having our properties for $q = wu_1w_2$, $q' = wu_1w_2'$.

Now we assume $|w_1| \not\equiv |w_1'| \text{(mod } |t_2|)$ such that for some prefixes $z_1$ of $w_1$ and $z_1'$ of $w_1'$, $\delta(a, z_1) = \delta(a, z_1')$ such that $w_1 = z_1z_2$, $w_1' = z_1'z_2'$, furthermore, $\lambda \in \{z_1, z_2\}$ implies $\lambda \not\in \{z', z_2'\}$ and $\lambda \in \{z_1, z_2\}$ implies $\lambda \not\in \{z_1, z_2\}$. We can prove $\lambda \not\in \{z_1, z_1', z_2, z_2'\}$ similarly as before. Then either $|z_1| \not\equiv |z_1'| \text{(mod } |t_2|)$ or $|z_2| \not\equiv |z_2'| \text{(mod } |t_2|)$. It remains to prove that these cases are impossible.
If $|z_1| \neq |z_1| \pmod{|t_2|}$ and, say, $|z_2| \geq |z_2|$ then considering the prefix $w'_2$ of $w_2$ having $|z'_1w'_2| = |z_1w_2|$, we can take $w, z_1, z_2w_2, z_1', z_2w'_2$ as $w, w_1, w_2, w'_1, w'_2$ contrary of the minimality of $\min(|w_1|, |w_2|)$.

If $|z_1| \equiv |z_1| \pmod{|t_2|}$ with $|z_2| \neq |z_2| \pmod{|t_2|}$ and, say, $|z_1| \geq |z_1|$ then considering the prefix $w''_2$ of $w'_2$ having $|z'_2w''_2| = |z_2w_2|$, we can take $w, z_2, z'_2, w_2, w''_2$ as $w, w_1, w'_1, w_2, w'_2$ contradicting the minimality of $\min(|w_1|, |w_2|)$.

The proof is complete.

\[ \square \]

Proposition 12 Let $A = (A, X, \delta)$ be an automaton without any Letichevsky's criteria. Consider $a, a_0 \in A, p \in X^*$ with $\delta(a_0, p) = a$ and suppose that $\delta(a, r) = \delta(a, r')$ holds for every $r, r' \in X^*, |pr| = |pr'| \geq |A| - 1$. Assume that $\delta(a, q) \neq \delta(a, q')$ holds for some $q, q' \in X^*, |pq| = |pq'|(< |A| - 1)$ and let $q, q'$ be words of maximal length having this property. Then there are $q, q'$ with this property having

(i) $q = uv$ and $q' = uv'$ for some $u, v, v' \in X^*$ such that for every prefixes $r$ of $v$ and $r'$ of $v'$ with $|r| = |r'| > 0$ we have $\delta(a, ur) \neq \delta(a, ur')$, and simultaneously, for every $w, z_1, z_2, u, v', z'_1, z'_2$ with $v = wz_1z_2, v' = w'z'_1z'_2$ we obtain $z_1 = z'_1$ whenever $\delta(a, u) = \delta(a, u', v')$, and $|z_1| = |z'_1|;

(ii) for every distinct prefixes $p_1, p_2$ of $pq, \delta(a_0, p_1) \neq \delta(a_0, p_2)$.

Proof: Consider $a \in A$ and suppose that our conditions hold.

First we suppose that, whenever $uu' \neq \lambda$, $\delta(a, u) = \delta(a, u')$ implies $u = q$ and $u' = q'$ for every prefixes $u$ of $q$ and $u'$ of $q'$. It is clear then that we are ready.

Assume the opposite case and let $q = uv, q' = u'v'$ with $\lambda \notin \{uu', vv\}$ such that $\delta(a, u) = \delta(a, u')$.

Let $\min(|u|, |u'|)$ be maximal with the above property and prove that in this case $u = u'$ can be assumed. Indeed, if it true if $|u| = |u'|$ because we can consider, say, $uu'$ instead of $u'v'$.

Finally, prove that, say, $|u| > |u'|$ is impossible. Indeed, otherwise we could change $q'$ for $uu'$, where $u''$ is a prefix of $v'$ with $|u''| = |v'|$. This contradicts the maximality of $\min(|u|, |u'|)$.

Now we prove (ii) omitting some analogous cases. If there are no distinct prefixes $p_1, p_2 \in X^*$ of $pq$ with $\delta(a_0, p_1) = \delta(a_0, p_2)$ for $pq$ and $pq$. Therefore, in this case, we are ready. Otherwise, we may suppose $\delta(a_0, p_1') = \delta(a_0, p_2')$ for some distinct prefixes $p_1', p_2' \in X^*$ of $pq$. Let, say, $p_1' = p_2'r'$ for some nonempty $r' \in X$. By Lemma 2 and $\delta(a_0, pq) \neq \delta(a_0, pq')$, this implies that $\delta(a_0, p_2')$ generates an autonomous state-subautomaton $B$ of $A$. Moreover, $\delta(a_0, p_1') = \delta(a_0, p_2'r') = \delta(a_0, p_2'), r' \neq \lambda$ implies that this autonomous state-subautomaton is strongly connected. On the other hand, by the maximality of $|q| = |q'|$, $\delta(a_0, pqx) = \delta(a_0, pq'x')$ holds for every $x, x' \in X$. Thus, $\delta(a_0, pqx) = \delta(a_0, pqx')$ is also a state of the state-subautomaton $B$ of $A$. Recall that by the maximality of $q$ and $q'$, we have $\delta(a_0, pqx) = \delta(a_0, p'q'x'), x, x' \in X$. Then $\delta(a_0, pq) \neq \delta(a_0, pq')$ and $\delta(a_0, pqx) = \delta(a_0, pq'x')$ imply that $\delta(a_0, pq)$ is not a state of $B$. Therefore, for every prefix $p_1$ of $pq$, $\delta(a_0, p_1)$ is not a state of $B$. 
Suppose that, contrary of our assumptions, $\delta(a_0, p_1) = \delta(a_0, p_2)$ holds for distinct prefixes $p_1$ and $p_2$ of $pq$ and put, say, $p_1 = p_2 r_1$ (where $r_1 \neq \lambda$ is assumed). In other words, $\delta(a_0, p_2 r_1) = \delta(a_0, p_2)$ holds such that $\delta(a_0, p_2)$ is not a state of $B$. But $\delta(a_0, pq x) = \delta(a_0, pq' x')$, $x, x' \in X$ implies that there exists an $r_2 \in X^*$ such that $\delta(a_0, p_2 r_2)$ is a state of $B$. Clearly, then $A$ satisfies either Letichevsky's criterion or the semi-Letichevsky criterion, a contradiction. This completes the proof. \hfill \square

References


