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Some Remarks on Automata without Letichevsky Criteria

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Abstract: In this paper we show some properties of finite automata having no Letichevsky criteria.

Keywords: Finite automata, Letichevsky criterion.

1. Introduction

We start with some standard concepts and notations. The elements of an alphabet $X$ are called letters ($X$ is supposed to be finite and nonempty). A word over an alphabet $X$ is a finite string consisting of letters of $X$. The string consisting of zero letters is called the empty word, written by $\lambda$. The length of a word $w$, in symbols $|w|$, means the number of letters in $w$ when each letter is counted as many times it occurs. By definition, $|\lambda| = 0$. At the same time, for any set $H$, $|H|$ denotes the cardinality of $H$. If $u$ and $v$ are words over an alphabet $X$, then their catenation $uv$ is also a word over $X$. Catenation is an associative operation and the empty word $\lambda$ is the identity with respect to catenation: $w\lambda = \lambda w = w$ for any word $w$. For a word $w$ and positive integer $n$, the notation $w^n$ means the word obtained by catenating $n$ copies of the word $w$. $w^0$ equals the empty word $\lambda$. $w^n$ is called the $m$-th power of $w$ for any non-negative integer $m$.

Let $X^*$ be the set of all words over $X$, moreover, let $X^+ = X^* \setminus \{\lambda\}$. $X^*$ and $X^+$ are the free monoid and the free semigroup, respectively, generated by $X$ under catenation.

A (finite) directed graph (or, in short, a digraph) $D = (V, E)$ (of order $|V| > 0$) is a pair consisting of sets of vertices $V$ and edges $E \subseteq V \times V$. A walk in $D = (V, E)$ is a sequence of vertices $v_1, \ldots, v_n, n > 1$ such that $(v_i, v_{i+1}) \in E$, $i = 1, \ldots, n - 1$. A walk is closed if $v_1 = v_n$. By a (directed) path from a vertex $a$ to a vertex $b \neq a$ we shall mean a sequence $v_1 \ldots v_n, n > 1$ of pairwise distinct vertices such that $a = v_1, b = v_n$.

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and \((u_i, v_{i+1}) \in E\) for every \(i = 1, \ldots, n - 1\). The positive integer \(n - 1\) is called the length of the path. Thus a path is a walk with all \(n\) vertices distinct. A closed walk with all vertices distinct except \(v_1 = v_n\) is a cycle of length \(n - 1\).

By an automaton we mean a finite automaton without outputs. Given an automaton \(A = (A, X, \delta)\) with set of states \(A\), set of input letters \(X\), and transition \(\delta : A \times X \to A\), it is understood that \(\delta\) is extended to \(\delta^* : A \times X^+ \to A\) with \(\delta^*(a, \lambda) = a\), \(\delta^*(a, xq) = \delta^*(\delta(a, x), q)\). In the sequel, we will consider the transition of an automaton in this extended form and thus we will denote it by the same Greek letter \(\delta\). Let \(A = (A, X, \delta)\) be an automaton. It is said that a state \(a \in A\) generates a state \(b \in A\) if \(\delta(a, p) = b\) holds for some \(p \in X^+\). For every state \(a \in A\) define the state subautomaton \(B = (B, X, \delta')\) generated by \(a\) such that \(B = \{ b \mid b = \delta(a, p), p \in X^+\}\), moreover, \(\delta'(b, x) = \delta(b, x)\) for every pair \(b, x \in X\). \(A\) is called strongly connected if for every pair \(a, b \in A\) there exists \(p \in X^+\) such that \(\delta(a, p) = b\).

We say that \(A\) satisfies Letichevsky’s criterion if there are a state \(a \in A\), input letters \(x, y \in X\), input words \(p, q \in X^+\) such that \(\delta(a, x) \neq \delta(a, y)\) and \(\delta(a, xp) = \delta(a, yq) = a\). It is said that \(A\) satisfies the semi-Letichevsky criterion if it does not satisfy Letichevsky’s criterion but there are a state \(a \in A\), input letters \(x, y \in X\), an input word \(p \in X^+\) such that \(\delta(a, x) \neq \delta(a, y), \delta(a, xp) = a\) and for every \(q \in X^+\), \(\delta(a, yq) \neq a\). If \(A\) do not satisfy either Letichevsky’s criterion or the semi-Letichevsky criterion then we say that \(A\) does not satisfy any Letichevsky criteria or is without any Letichevsky criteria.

The Letichevsky criterion has a central role in the investigations of products of automata (see [1],[2],[3],[4]). Automata having semi-Letichevsky criterion and automata without any Letichevsky criteria are also important in the classical result of Z. Ésik and Gy. Horváth (see [2],[3]). In this paper we investigate automata without any Letichevsky criteria.

## 2. Results

First we observe

**Proposition 1** Given an automaton \(A = (A, X, \delta)\), a state \(a_0 \in A\), four input words \(u, v, p, q \in X^+\) with \(|up|, |vq| > 0\) under which \(\delta(a_0, u) \neq \delta(a_0, v)\) and \(\delta(a_0, up) = \delta(a_0, vq) = a_0\). Then \(A\) satisfies Letichevsky’s criterion.

**Proof:** First we suppose \(|u|, |v| > 0\). Then there exist input words \(w, w', w_1, w_2 \in X^+\) and input letters \(x, y \in X\) such that \(u = wxw_1, v = w'yw_2\) and \(\delta(a_0, wx) \neq \delta(a_0, wy) = \delta(a_0, w'y)\). Therefore, we can reach Letichevsky’s criterion substituting \(a_0, u, v, p, q\) for \(\delta(a_0, u), x, y, w_1pw, w_2qw\).

Now we assume, say, \(|u| = 0\). Then, by our assumptions, \(|q| > 0\) with \(\delta(a_0, q) = a_0\). On the other hand, \(\delta(a_0, u) \neq \delta(a_0, v) = a_0\) implies \(|u| > 0\). In addition, then we have \((a_0 = \delta(a_0, v) =) \delta(a_0, q) \neq \delta(a_0, u)\). Therefore, there are input words \(w, w', w_1, w_2 \in X^+\) and input letters \(x, y \in X\) such that \(u = wxw_1, q = w'yw_2\) and \(\delta(a_0, wx) \neq \delta(a_0, wy) = \delta(a_0, w'y)\).
\[ \delta(a_0, wy) = \delta(a_0, w'y). \] We obtain again Letichevsky's criterion substituting \( a_0, u, v, p, q \) for \( \delta(a_0, w), x, y, w_1pw, w_2w. \)

Now we study automata having no Letichevsky's criteria. The following statement is obvious.

**Proposition 2** \( \mathcal{A} = (A, X, \delta) \) is a automaton without any Letichevsky criteria if and only if for every state \( a_0 \in A \), input letters \( x, y \in X \) and an input word \( p \in X^* \) having \( \delta(a_0, xp) = a_0 \), it holds that \( \delta(a_0, x) = \delta(a_0, y). \)

Obviously, if \( \mathcal{A} = (A, X, \delta) \) has the above properties then there exists a nonnegative integer \( n \) such that for every \( p \in X^* \) with \( |p| \geq n \), each \( \delta(a, p) \) generates an autonomous state-subautomaton of \( \mathcal{A} \). Denote by \( n_{\mathcal{A}}(\leq n) \) the minimal nonnegative integer having this property.

**Proposition 3** \( n_{\mathcal{A}} \leq \max(|A| - 2, 0). \)

**Proof:** Take out of consideration the trivial cases. Thus we assume \( |A| > 2 \).

Consider \( a \in A, x_1, \ldots, x_{m+2} \in X \) having \( \delta(a, x_1 \cdots x_{m}x_{m+1}) \neq \delta(a, x_1 \cdots x_{m}x_{m+2}). \) If \( a, \delta(a, x_1), \delta(a, x_1x_2), \ldots, \delta(a, x_1 \cdots x_m), \delta(a, x_1 \cdots x_{m}x_{m+1}), \delta(a, x_1 \cdots x_{m}x_{m+2}) \) are not distinct states then \( \mathcal{A} \) satisfies either Letichevsky's criterion or the semi-Letichevsky criterion, a contradiction. Hence, \( m \leq |A| - 2 \). Thus \( n_{\mathcal{A}} \leq |A| - 2. \)

We also note the next direct consequence of Proposition 2.

**Proposition 4** If \( \mathcal{A} \) is a strongly connected automaton without any Letichevsky criteria then \( \mathcal{A} \) is autonomous.

By this observation, we get immediately the following

**Proposition 5** Suppose that \( \mathcal{A} = (A, X, \delta) \) is a strongly connected automaton without any Letichevsky criteria. There exists a \( k > 0 \) such that for every \( a, b \in A \), \( a = b \) if and only if there exists a pair \( p, q \in X^* \) with \( |p| \equiv |q|(\text{mod} k) \) and \( \delta(a, p) = \delta(b, q). \)

**Lemma 6** Given an automaton \( \mathcal{A} = (A, X, \delta) \) be without any Letichevsky criteria, \( a \in A \) is a state of a strongly connected state-subautomaton of \( \mathcal{A} \) if and only if there exists a nonempty word \( p \in X^* \) with \( \delta(a, p) = a. \)

**Proof:** Let \( a \in A \) be a state of a strongly connected state-subautomaton of \( \mathcal{A} \). By definition, for every nonempty word \( q \in X^* \), there exists a word \( r \in X^* \) with \( \delta(a, qr) = a. \) Conversely, suppose that \( \delta(a, p) = a \) for some \( a \in A \) and \( p \in X^*, \ p \neq \lambda. \) Then for every prefix \( p' \) of \( p \) and input letters \( x, y \in X \), \( \delta(a, px) = \delta(a, p'y). \) Therefore, for every \( q \in X^* \), \( \delta(a, q) = \delta(a, r) \), where \( r \) is a prefix of \( p \) with \( |q| \equiv |r| (\text{mod} |p|) \). But then \( a \) generates a strongly connected state-subautomaton of \( \mathcal{A}. \)

We shall use the following consequence of the above statement.
Proposition 7 Let $A = (\mathcal{A}, X, \delta)$ be an automaton without any Letichevsky criteria. Moreover, suppose that $a \in \mathcal{A}$ is a strongly connected state-subautomaton of $\mathcal{A}$. If $\delta(b, p) = a$ for some $b \in \mathcal{A}$ and nonempty $p \in X^*$ then $\delta(a, q) \neq b, q \in X^*$. Conversely, if $\delta(a, r) = c$ for some $c \in \mathcal{A}$ and nonempty $r \in X^*$ then $\delta(c, q) \neq a, q \in X^*$.

Lemma 8 Let $A = (\mathcal{A}, X, \delta)$ be a automaton without any Letichevsky's criteria. If there are $a \in \mathcal{A}, q', q' \in X^*, |q'| \geq |\mathcal{A}|-1, \delta(a, q') \neq \delta(a, q')$ then for every pair of words $r, r' \in X^*, |r| = |r'|$ we have $\delta(a, qr) \neq \delta(a, q'r')$.

Proof: Suppose that our statement does not hold, i.e., there are $a \in \mathcal{A}, q, q'r, r' \in X^*, |q'| \geq |\mathcal{A}|-1, |r| = |r'|$ having $\delta(a, q) \neq \delta(a, q')$ and $\delta(a, qr) = \delta(a, q'r')$. Then, of course, $|r| = |r'| > 0$. We distinguish the following three cases.

Case 1. There are $q_1, r_1, q_2, r_2, q'_1, r'_1, q'_2, r'_2$ with $q = q_1r_1 = q_2r_2, q' = q'_1r'_1 = q'_2r'_2, |q_1| < |q_2|, |q'_1| < |q'_2|$ such that $\delta(a, q_1) = \delta(a, q_2), \delta(a, q'_1) = \delta(a, q'_2)$. 2 But then, by Proposition 2, $\delta(a, q_1w) = \delta(a, q'_1w')$ and $\delta(a, q_1w') = \delta(a, q'_1w)$ for every $w, w' \in X^*, |w| = |w'|$. Thus, because of $\delta(a, q_1) = \delta(a, q_2)$ and $\delta(a, q'_1) = \delta(a, q'_2)$, we obtain that, for every $w, w' \in X^*$ there are $z, z' \in X^*$ with $\delta(a, q_1wz) = \delta(a, q_2)$ and $\delta(a, q'_1w'z') = \delta(a, q'_2)$. Thus $q_1r_1 = q, q'_1r'_1 = q'$ imply that $\delta(a, qrz) = \delta(a, q_1)$ and $\delta(a, q'r'z') = \delta(a, q'_1)$ hold for some $z, z' \in X^*$. This means that $\delta(a, qrzr_1) = \delta(a, q)$ and $\delta(a, q'r'z'r_1) = \delta(a, q')$. Put $b = \delta(a, qr)(= \delta(a, q'r')) = \delta(a, q), c = \delta(a, q), c' = \delta(a, q')$. Then $\delta(b, zr_1) = c \neq c' = \delta(b, z'r'_1)$ and $\delta(c, r) = \delta(c', r') = b$. But then $|r| = |r'| > 0$ implies $|zr_1r_1|, |z'r'_1r'_1| > 0$. Therefore, by Proposition 1, $A$ satisfies Letichevsky's criterion, a contradiction.

Case 2. There are $q_1, r_1, q_2, r_2$ with $q = q_1r_1 = q_2r_2, |q_1| < |q_2|, |q'_1| < |q'_2|$ such that $\delta(a, q_1) = \delta(a, q_2)$, but $\delta(a, q'_1) \neq \delta(a, q'_2)$ holds for every distinct prefixes $q'_1, q'_2$ of $q$. Then, because of $|q_1| = |q'_1| \geq |\mathcal{A}|-1$, we necessarily have $|q| = |q'| = |\mathcal{A}|-1$, moreover, we also have that for every $d \in \mathcal{A}$ there exists a prefix $q'_1$ of $q'$ with $\delta(a, q'_1) = d$. (Indeed, we assumed $\delta(a, q'_1) \neq \delta(a, q'_2)$ for every distinct prefixes $q'_1, q'_2$ of $q'$, where $|q'| = |\mathcal{A}|-1$.) And then for every $d \in \mathcal{A}$ there exists an $r'_1 \in X^*$ having $\delta(d, r'_1) = \delta(a, q')$. On the other hand, we may assume $\delta(a, qrzr_1) = \delta(a, q)$ as in the previous case.

Now we suppose again $\delta(a,qr) = \delta(a, q'r')$ as before. Substituting $d$ for $\delta(a, qrzr_1)$, there exists an $r'_1 \in X^*$ holding $\delta(a, q'r_1r'_1) = \delta(a, q'_1)$. Put $b = \delta(a, qr), c = \delta(a, q), c' = \delta(a, q')$. But then $|r| = |r'| > 0$ implies $|zr_1r_1|, |zr_1r_1r'1| > 0$. Therefore, by Proposition 1 we obtain again that $A$ satisfies Letichevsky's criterion contrary of our assumptions.

Case 3. Let $\delta(a, q_1) \neq \delta(a, q_2)$ and $\delta(a, q'_1) \neq \delta(a, q'_2)$ for every distinct prefixes $q_1, q_2$ of $q$ and $q'_1, q'_2$ of $q'$, respectively. Then for every $d \in \mathcal{A}$ there are $r_1, r'_1 \in X^*$ having $\delta(d, r_1) = \delta(a, q)$ and $\delta(d, r'_1) = \delta(a, q')$. Therefore, assuming $\delta(a, qr) = \delta(a, q'r')$ for some $r, r' \in X^*$, and substituting $d$ for $\delta(a, qr) = \delta(a, q'r')$, we obtain $\delta(a, qr_{r_1}) = \delta(a, q), \delta(a, qrr_{r_1}) = \delta(a, q')$ (with $\delta(a, qr) = \delta(a, q'r')$). Put $c = \delta(a, q), c' = \delta(a, q')$. **This holds automatically if $|q| = |q'| \geq |\mathcal{A}|$.**
Then $\delta(d,r_1) = c, \delta(d,r'_1) = c', \delta(c,r) = \delta(c',r') = d$ such that, by $|r| = |r'| > 0$, $|r_1r|, |r'_1r'| > 0$. By Proposition 1, this implies that $A$ satisfies Letichevsky's criterion, a contradiction again.

\[ \square \]

**Theorem 9** Let $A = (A, X, \delta)$ be a automaton without any Letichevsky's criteria. For every state $a \in A$ we have one of the following two possibilities:

(i) there exist $q, q' \in X^*, |q| = |q'| \geq |A| - 1$ such that $\delta(a, qr) \neq \delta(a, q'r')$ for every $r, r' \in X^*$, $|r| = |r'|$,

(ii) $\delta(a, q) = \delta(a, q')$ for every $q, q' \in X^*, |q| = |q'| \geq |A| - 1$.

**Proof:** Suppose that (i) does not hold. Then for every $q, q' \in X^*, |q| = |q'| \geq |A| - 1$ there exist $r, r' \in X^*$, $|r| = |r'|$ having $\delta(a, qr) = \delta(a, q'r')$. Using Lemma 8, $\delta(a, qr) = \delta(a, q'r'), |r| = |r'|$ and $|q| = |q'| \geq |A| - 1$ implies $\delta(a, q) = \delta(a, q')$. Thus (ii) holds whenever (i) does not hold.

The following statement is obvious.

**Lemma 10** Given a digraph $D = (V, E)$, let $v \in V, p_1, p_2, p_3, p_4 \in V^*$ such that $p_1p_2p_3v$ and $p_1p_2p_3vp_4$ are walks and $vp_4v$ is a cycle. $|p_2| \equiv |p_2'| (\text{mod } |p_4v|)$ if and only if there are positive integers $k, \ell$ having $|p_2p_3v(p_4v)^k| = |p_2p_3v(p_4v)\ell|$.

\[ \square \]

We finish the paper studying both types of states given in Theorem 9.

**Proposition 11** Let $A = (A, X, \delta)$ be an automaton without any Letichevsky's criteria. Consider a state $a \in A$ and suppose that there are $q, q' \in X^*, |q| = |q'| \geq |A| - 1$, $\delta(a, q) \neq \delta(a, q')$. Then there are $q, q'$ having this property for which $q = uv$ and $q' = uv'$ for some $u, v, v' \in X^*$ such that for every prefixes $r$ of $v$ and $r'$ of $v'$ with $|r| = |r'| > 0$ we have $\delta(a, ur) \neq \delta(a, ur')$, and simultaneously, for every $w, z_1, z_2, w', z'_1, z'_2, |w| > 0$ we obtain $z_1 = z'_1$ whenever $\delta(a, uw) = \delta(a, uw')$, and $|z_1| = |z'_1|$.

**Proof:** Consider $a \in A$ and suppose that our conditions hold, i.e., there are $q, q' \in X^*$ having $|q| = |q'| \geq |A| - 1, \delta(a, q) \neq \delta(a, q')$. Then Proposition 3 implies that $\delta(a, q)$ and $\delta(a, q')$ generate autonomous state subautomata of $A$. We will distinguish the following cases (omitting some of the analogous cases):

Case 1. There are $u, u', v, v', \in X^*$ such that $q = uv, q' = u'v', \delta(a, u) = \delta(a, u')$ and for every nonempty prefixes $r$ of $v$ and $r'$ of $v'$, $\delta(a, u) \neq \delta(a, u'v')$, $\delta(a, u') \neq \delta(a, ur)$, and $\delta(a, ur) \neq \delta(a, u'v')$.

3 Let, say, $|u| \geq |u'|$ and let $v''$ be a prefix of $v'$ with $|v''| = |v|$. Change $q'$ for $uv''$ and then we will have our requirements.

Case 2. There exist a prefix $u$ of $q$ having $\delta(a, u) = \delta(a, q')$. Let $t_2 \in X^*$ be a nonempty word with minimal length having $\delta(a, qt_1t_2) = \delta(a, q't_1)$ for some word $u = u' = \lambda$ is possible.
$t_1 \in X^*$ and assume that $t_2$ is minimal in the sense that for every nonempty $p \in X^*$, $\delta(a, q't_1p) = \delta(a, q't_1)$ implies $|t_2| \leq |p|$.\footnote{The finiteness of the state set of $\mathcal{A}$ implies the existence of $t_1$ and $t_2$.} Then, using that $\delta(a, q')$ generates an autonomous state subautomaton of $\mathcal{A}$, we have $q = uv$, where $v$ is a nonempty prefix of $t_1 t_2^k$ for a suitable $k \geq 0$.

Prove that in this case $u \equiv |q'| (\text{mod } |t_2|)$ is impossible. Assume the contrary. Recall again that $\delta(a, q')$ generates an autonomous state subautomaton of $\mathcal{A}$. But then, applying Lemma 10, there are words $r, r' \in X^*, |r| = |r'|$ having $\delta(a, qr) = \delta(a, q'r')$. By Lemma 8, then $|q| = |q'| < |A| - 1$ contrary of our assumptions. Thus we have the following cases.

Case 2.1. Suppose $u \equiv |q'| (\text{mod } |t_2|)$ such that for every prefixes $u_1$ of $u$ and $u'_1$ of $q'$ with $u_1 u'_1 \neq \lambda$, $\delta(a, u_1) = \delta(a, u'_1)$ implies $u_1 = u$ and $u'_1 = q'$. Then we obtain our requirements again (having $q = uv$, where $v$ is a nonempty prefix of $t_1 t_2^k$ for a suitable $k \geq 0$).

Case 2.2. Assume $u \equiv |q'| (\text{mod } |t_2|)$, and simultaneously, let for some prefixes $u_1$ of $u$ and $u'_1$ of $q'$, $\delta(a, u_1) = \delta(a, u'_1)$ such that $u = u_1 v_1$, $q' = u'_1 v'_1$, furthermore, $\lambda \in \{u_1, v_1\}$ implies $\lambda \notin \{u'_1, v'_1\}$ and $\lambda \in \{u'_1, v'_1\}$ implies $\lambda \notin \{u_1, v_1\}$. If $v_1 = \lambda$ and $v'_1 \neq \lambda$ then $\delta(a, u'_1) = \delta(a, u_1 v'_1) \neq \delta(a, u'_1 v')(= \delta(a, uv))$ such that $v$ is a nonempty suffix of $q$. But then $\mathcal{A}$ has either Letichevsky's criterion or the semi-Letichevsky criterion, a contradiction. Similarly, it also lead to a contradiction is we assume $v_1 \neq \lambda$ and $v'_1 = \lambda$. Thus $\lambda \notin \{v_1, v'_1\}$ can be assumed and we may also assume $\lambda \notin \{u_1, u'_1\}$ analogously.

By $u \equiv |q'| (\text{mod } |t_2|)$, either $|u_1| \equiv |u'_1| (\text{mod } |t_2|)$, or $|v_1| \equiv |v'_1| (\text{mod } |t_2|)$.\footnote{in Case 2a, of course, $w = \lambda$.} Take a prefix $v'$ of $t_1 t_2^k$ for a suitable $k \geq 0$ with $|u_1 v_1 v'| = |q|$ and let us consider $u'_1 v_1 v'$ instead of $q'$.

Case 2.2. Suppose $|u_1| \equiv |u'_1| (\text{mod } |t_2|)$ and let, say, $|v_1| \geq |v'_1|$. Take a prefix $v'$ of $t_1 t_2^k$ for a suitable $k \geq 0$ with $|u_1 v_1 v'| = |q|$ and change $u_1 v_1 v'$ for $q$.

In both of the above Case 2.2.1 and Case 2.2.2, we have words $w, w_1, w_2, w'_1, w'_2 \in X^*, \lambda \notin \{w_1, w'_1\}, w_1 \equiv |w'_1| (\text{mod } |t_2|)$, $w'_2$ is a prefix of $w_2$ (or, in the opposite case, $w_2$ is a prefix of $w'_2$), $q = ww_1 w_2, q' = w w'_1 w'_2$, such that $\delta(a, w w_1) = \delta(a, w w'_1)$. Then let $w, w_1, w_2, w'_1, w'_2 \in X^*$ be arbitrary having these properties for which $\min(|w_1|, |w_2|)$ is minimal.

If for every nonempty proper prefixes $z_1$ of $w_1$ and $z'_1$ of $w'_1$ we have $\delta(a, w) \notin \{\delta(a, wz'_1), \delta(a, w z'_1)\}$ and $\delta(a, w z_1) \neq \delta(a, wz'_1)$ then we are ready having our properties for $q = w w_1 w_2, q' = w w'_1 w'_2$.

Now we assume $|w_1| \equiv |w'_1| (\text{mod } |t_2|)$ such that for some prefixes $z_1$ of $w_1$ and $z'_1$ of $w'_1$, $\delta(a, z_1) = \delta(a, z'_1)$ such that $w_1 = z_1 z_2, w'_1 = z'_1 z'_2$, furthermore, $\lambda \in \{z_1, z_2\}$ implies $\lambda \notin \{z'_1, z'_2\}$ and $\lambda \in \{z'_1, z'_2\}$ implies $\lambda \notin \{z_1, z_2\}$. We can prove $\lambda \notin \{z_1, z'_1, z_2, z'_2\}$ similarly as before. Then either $|z_1| \neq |z'_1| (\text{mod } |t_2|)$ or $|z_2| \neq |z'_2| (\text{mod } |t_2|)$. It remains to prove that these cases are impossible.
If $z_{1} \not \equiv z'_{1}$(mod $t_{2}$) and, say, $z_{2} \geq |z'_{2}|$ then considering the prefix $w'_{2}$ of $w_{2}$ having $|z'_{1}w'_{2}| = |z_{1}w_{2}|$, we can take $w, z_{1}, z_{2}w_{2}, z_{1}, z_{2}w_{2}$ as $w, w_{1}, w_{2}, w'_{1}, w'_{2}$ contrary to the minimality of $\mathrm{min}|w_{1}|, |w_{2}|)\}$. 

If $|z_{1}| \equiv |z'_{1}|$(mod $t_{2}$) with $|z_{2}| \not \equiv |z'_{2}|$(mod $t_{2}$) and, say, $z_{1} \geq |z_{1}'|$ then considering the prefix $w'_{2}$ of $w_{2}$ having $|z'_{2}w'_{2}| = |z_{2}w_{2}|$, we can take $w, z_{1}, z_{2}, z_{2}', w_{2}, w''$ as $w, w_{1}, w_{2}, w_{2}'$ contradicting the minimality of $\mathrm{min}|w_{1}|, |w_{2}|)\}$. 

The proof is complete. □

**Proposition 12** Let $A = (A, X, \delta)$ be an automaton without any Letichevsky’s criteria. Consider $a, a_{0} \in A, p \in X^{*}$ with $\delta(a_{0}, p) = a$ and suppose that $\delta(a, r) = \delta(a, r')$ holds for every $r, r' \in X^{*}$, $|pr| = |pr'| \geq |A| - 1$. Assume that $\delta(a, q) \neq \delta(a, q')$ holds for some $q, q' \in X^{*}, |pq| = |pq'|(<|A| - 1)$ and let $q, q'$ be words of maximal length having this property. Then there are $q, q'$ with this property having 

(i) $q = uv$ and $q' = uv'$ for some $u, v, v' \in X^{*}$ such that for every prefixes $r$ of $v$ and $r'$ of $v'$ with $|r| = |r'| > 0$ we have $\delta(a, ur) \neq \delta(a, ur')$, and simultaneously, for every $u, z_{1}, z_{2}, w', z_{1}', z_{2}'$ with $v = wz_{1}z_{2}, v' = w'z_{1}'z_{2}'$ we obtain $z_{1} = z'_{1}$ whenever $\delta(a, uv) = \delta(a, uv')$, and $|z_{1}| = |z'_{1}|$;

(ii) for every distinct prefixes $p_{1}, p_{2}$ of $pq$, $\delta(a_{0}, p_{1}) \neq \delta(a_{0}, p_{2})$.

**Proof:** Consider $a \in A$ and suppose that our conditions hold.

First we suppose that, whenever $u, u' \neq \lambda$, $\delta(a, u) = \delta(a, u')$ implies $u = q$ and $u' = q'$ for every prefixes $u$ of $q$ and $u'$ of $q'$. It is clear that then we are ready.

Assume the opposite case and let $q = uv, q' = uv'$ with $\lambda \notin \{uu', vv'\}$ such that $\delta(a, u) = \delta(a, u')$. 

Let $\min(|u|, |u'|)$ be maximal with the above property and prove that in this case $u = u'$ can be assumed. Indeed, if it true if $|u| = |u'|$ because we can consider, say, $uv$ instead of $uv'$.

Finally, prove that, say, $|u| > |u'|$ is impossible. Indeed, otherwise we could change $q$ for $uwv'$, where $v'$ is a prefix of $v'$ with $|v'| = |v'|$. This contradicts the maximality of $\min(|u|, |u'|)$.

Now we prove (ii) omitting some analogous cases. If there are no distinct prefixes $p_{1}' = p_{1}'$, $p_{2}' \in X^{*}$ of $pq$ with $\delta(a_{0}, p_{1}') = \delta(a_{0}, p_{2}')$ for $pq'$ and $pq$. Therefore, in this case, we are ready. Otherwise, we may suppose $\delta(a_{0}, p_{1}') = \delta(a_{0}, p_{2}')$ for some distinct prefixes $p_{1}', p_{2}' \in X^{*}$ of $pq'$. Let, say, $p_{1}' = p_{2}'r'$ for some nonempty $r' \in X$. By Lemma 2 and $\delta(a_{0}, pq) \neq \delta(a_{0}, pq')$, this implies that $\delta(a_{0}, p_{1}')$ generates an autonomous state-subautomaton $B$ of $A$. Moreover, $\delta(a_{0}, p_{1}') = \delta(a_{0}, p_{2}'r') = \delta(a_{0}, p_{2}')$, $r' \neq \lambda$ implies that this autonomous state-subautomaton is strongly connected. On the other hand, by the maximality of $|q| = |q'|$, $\delta(a_{0}, pqx) = \delta(a_{0}, pq'x')$ holds for every $x, x' \in X$. Thus, $\delta(a_{0}, pqx)$ is also a state of the state-subautomaton $B$ of $A$. Recall that by the maximality of $q$ and $q'$, we have $\delta(a_{0}, pqx) = \delta(a_{0}, pq'x)$, $x, x' \in X$. Then $\delta(a_{0}, pq) \neq \delta(a_{0}, pq')$ and $\delta(a_{0}, pqx) = \delta(a_{0}, pq'x')$ imply that $\delta(a_{0}, pq)$ is not a state of $B$. Therefore, for every prefix $p_{1}$ of $pq$, $\delta(a_{0}, p_{1})$ is not a state of $B$. 

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Suppose that, contrary of our assumptions, $\delta(a_0, p_1) = \delta(a_0, p_2)$ holds for distinct prefixes $p_1$ and $p_2$ of $pq$ and put, say, $p_1 = p_2r_1$ (where $r_1 \neq \lambda$ is assumed). In other words, $\delta(a_0, p_2r_1) = \delta(a_0, p_2)$ holds such that $\delta(a_0, p_2)$ is not a state of $B$. But $\delta(a_0, pqx) = \delta(a_0, pq'x'), x, x' \in X$ implies that there exists an $r_2 \in X^*$ such that $\delta(a_0, p_2r_2)$ is a state of $B$. Clearly, then $A$ satisfies either Letichevsky’s criterion or the semi-Letichevsky criterion, a contradiction. This completes the proof. \qed

References


