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Kyoto University
A generalization of a non-symmetric numerical semigroup generated by three elements

神奈川工科大学 米田 二良 (Jiryo Komeda)  
Kanagawa Institute of Technology

§1. Non-symmetric numerical semigroups generated by three elements.

Let \( \mathbb{N} \) be the additive semigroup of non-negative integers. Let \( H \) be a numerical semigroup of genus \( g \), i.e., a subsemigroup of \( \mathbb{N} \) whose complement \( \mathbb{N} \setminus H \) consists of \( g \) elements. We denote by \( g(H) \) the genus of \( H \). We set

\[ c(H) = \min \{ c \in \mathbb{N} | c + \mathbb{N} \subseteq H \} \]

which is called the conductor of \( H \). Then \( c(H) \leq 2g(H) \). A numerical semigroup \( H \) is said to be symmetric if \( c(H) = 2g(H) \). Let \( M(H) = \{ a_1, a_2, \ldots, a_n \} \) be the minimal set of generators for \( H \). We set

\[ \alpha_i = \min \{ \alpha | \alpha a_i \in \langle a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \rangle \} \]

where for any non-negative integers \( b_1, \ldots, b_m \) the set \( \langle b_1, \ldots, b_m \rangle \) means the semigroup generated by \( b_1, \ldots, b_m \).

**Example 1.**

i) Let \( H = \langle 4, 5, 6 \rangle \). Then \( g(H) = 4 \) and \( c(H) = 8 \). Hence \( H \) is symmetric. If we set \( a_1 = 4 \), \( a_2 = 5 \) and \( a_3 = 6 \), then \( \alpha_1 = 3 \), \( \alpha_2 = 2 \) and \( \alpha_3 = 2 \).

ii) Let \( H = \langle 4, 5, 7 \rangle \). Then \( g(H) = 4 \) and \( c(H) = 7 \). Hence \( H \) is non-symmetric. If we set \( a_1 = 4 \), \( a_2 = 5 \) and \( a_3 = 7 \), then \( \alpha_1 = 3 \), \( \alpha_2 = 3 \) and \( \alpha_3 = 2 \).

**Remark 2** (Herzog [1]). Let \( H \) be a non-symmetric numerical semigroup with \( M(H) = \{ a_1, a_2, a_3 \} \). Then

\[ \alpha_1 a_1 = \alpha_{12} a_2 + \alpha_{13} a_3, \quad \alpha_2 a_2 = \alpha_{21} a_1 + \alpha_{23} a_3 \quad \text{and} \quad \alpha_3 a_3 = \alpha_{31} a_1 + \alpha_{32} a_2 \]

where \( \alpha_1 = \alpha_{21} + \alpha_{31}, \alpha_2 = \alpha_{12} + \alpha_{32}, \alpha_3 = \alpha_{13} + \alpha_{23} \) and \( 0 < \alpha_{ij} < \alpha_j \), all \( i, j \). In this case \( \alpha_{ij} \)'s are uniquely determined.

**Proposition 3.** Let the notation be as in Remark 2. Then we have

\[
\begin{vmatrix}
\alpha_1 & -\alpha_{12} \\
-\alpha_{21} & \alpha_2
\end{vmatrix}
= a_3.
\]

**Example 4.** Let \( H = \langle a_1 = 4, a_2 = 5, a_3 = 7 \rangle \). Then

\[ 3a_1 = a_2 + a_3, \quad 3a_2 = 2a_1 + a_3, \quad 2a_3 = a_1 + 2a_2 \]

and

\[
\begin{vmatrix}
\alpha_1 & -\alpha_{12} \\
-\alpha_{21} & \alpha_2
\end{vmatrix}
= \begin{vmatrix}
3 & -1 \\
-2 & 3
\end{vmatrix}
= 7 = a_3.
\]

In this section we introduce the notion of a numerical semigroup of quasi-toric type and give several examples.

**Definition 5.** i) Let $H$ be a numerical semigroup with $M(H) = \{a_1, \ldots, a_n\}$. A system of relations

$$
\begin{align*}
\alpha_1a_1 &= \alpha_{12}a_2 + \cdots + \alpha_{1n}a_n \\
& \quad \vdots \\
\alpha_na_n &= \alpha_{n1}a_1 + \cdots + \alpha_{nn-1}a_{n-1}
\end{align*}
$$

satisfying

$$\alpha_j = \alpha_{1j} + \cdots + \alpha_{j-1j} + \alpha_{j+1j} + \cdots + \alpha_{nj}$$

for any $j$ and $0 \leq \alpha_{ij} < \alpha_j$ for all $i, j$ is said to be neat.

ii) A numerical semigroup $H$ is said to be neat if it has a neat system of relations.

**Example 6.** i) Any non-symmetric numerical semigroup $H$ with $\#M(H) = 3$ is neat by Remark 2.

ii) Let $H = \langle a_1 = 20, a_2 = 24, a_3 = 25, a_4 = 31 \rangle$. We have a unique neat system of relations

$$4a_1 = a_2 + a_3 + a_4, \quad 4a_2 = 2a_1 + a_3 + a_4, \quad 3a_3 = a_1 + a_2 + a_4 \quad \text{and} \quad 3a_4 = a_1 + 2a_2 + a_3,$$

which implies that $H$ is neat.

**Definition 7.** Let $H$ be a numerical semigroup with $M(H) = \{a_1, \ldots, a_n\}$. The $\mathbb{Z}$--module

$$R = \{(r_1, r_2, \ldots, r_n) \in \mathbb{Z}^n | \sum_{i=1}^{m} r_ia_i = 0\}$$

is called a relation module for $H$.

**Lemma 8.** Let the notation be as in Definition 7. Then a relation module for $H$ is a free $\mathbb{Z}$--module of rank $n - 1$.

**Definition 9.** Let $S$ be a subsemigroup of $\mathbb{Z}^n$. $S$ is said to be saturated if the condition $nr \in S$, where $n$ is a positive integer and $r$ an element of $\mathbb{Z}^n$, implies that $r \in S$.

**Definition 10.** Consider an order on the set

$$I = \{(i, j)|1 \leq i \leq n, 1 \leq j \leq n, i \neq j\},$$

which is fixed. Let $H$ be a neat numerical semigroup with $M(H) = \{a_1, \ldots, a_n\}$. 
Take a neat system of relations

\[
\begin{align*}
\alpha_{2a} a_1 + \cdots + \alpha_{1n} a_1 &= \alpha_1 a_1 = \alpha_{12} a_2 + \cdots + \alpha_{1n} a_n \\
\alpha_{2n-1} a_{n-1} + \cdots + \alpha_{n-1} a_{n-1} &= \alpha_{n-1} a_{n-1} + \alpha_{n-1} a_n \\
\alpha_{1n} a_n &= \alpha_{n1} a_1 + \cdots + \alpha_{nn-1} a_{n-1}
\end{align*}
\]

Assume that

\[
\begin{vmatrix}
\alpha_1 & -\alpha_{12} & \cdots & -\alpha_{1n-1} \\
-\alpha_{21} & \alpha_2 & \cdots & -\alpha_{2n-1} \\
\vdots & \vdots & \ddots & \vdots \\
-\alpha_{n-11} & -\alpha_{n-12} & \cdots & \alpha_n
\end{vmatrix} \neq 0.
\]

We set \( N = \# \{(i,j) | \alpha_{ij} \neq 0 \} - (n - 1) \). We associate the vector in \( \mathbb{Z}^N \) with \( \alpha_{ij} a_j \) by induction on \( i \) which means the \( i \)-th relation in the neat system of relations. Let \( i \) be fixed. Let \( (k_i, l_i) \) be the maximum of the set

\[
L_i = \{(j, i) | \alpha_{ji} \neq 0 \} \cup \{(i, j) | \alpha_{ij} \neq 0 \} \cap \left( I \setminus \bigcup_{p=1}^{n-1} L_p \right).
\]

We number successively the elements \( (i, j) \) of the set \( L_i \) by \( \sigma(i, j) \) in the given order if \( (i, j) \neq (k_i, l_i) \). We associate the vector \( b_{\sigma(i,j)} = e_{\sigma(i,j)} \) with \( \alpha_{ij} a_j \) if \( (i, j) \neq (k_i, l_i) \). For \( \alpha_{k_\sigma l_\sigma} a_{l_\sigma} \) we consider

\[
\alpha_{k_\sigma l_\sigma} a_{l_\sigma} = \cdots \pm \alpha_{pq} a_p \cdots
\]

from the \( i \)-th relation in the neat system. Using the relation we can associate the vector \( b_{N+i} \) with \( \alpha_{k_i l_i} a_{l_i} \), because we already have associated some vector with \( \alpha_{pq} a_p \). Thus, we can construct the subsemigroup \( S = \langle b_1, \ldots, b_{N+n-1} \rangle \) of \( \mathbb{Z}^N \). The neat numerical semigroup \( H \) is said to be of quasi-toric type if the semigroup \( S \) is saturated.

The reason why \( H \) is said to be of quasi-toric type is that if the associated semigroup \( S \) is saturated then the affine scheme \( \text{Spec } k[S] \) becomes an affine toric variety where \( k \) is an algebraically closed field.

**Remark 11.** Let a neat system of relations be fixed. Then the property of"quasi-toric type" does not depend on the choices of the numbering of the elements of \( M(H) \) and the order on the set \( I = \{(i,j)| 1 \leq i \leq n, 1 \leq j \leq n, i \neq j \} \).

**Example 12.** Let \( H \) be a non-symmetric numerical semigroup with \( M(H) = \{a_1, a_2, a_3\} \). We have the neat system of relations

\[
\begin{align*}
\alpha_1 a_1 &= (\alpha_{21} + \alpha_{31}) a_1 = \alpha_{12} a_2 + \alpha_{13} a_3 \\
\alpha_2 a_2 &= (\alpha_{12} + \alpha_{22}) a_2 = \alpha_{21} a_1 + \alpha_{23} a_3 \\
\alpha_3 a_3 &= (\alpha_{13} + \alpha_{23}) a_3 = \alpha_{31} a_1 + \alpha_{32} a_2
\end{align*}
\]
We define the order on the set
\[
\{(i,j)|i,j = 1,2,3\text{ and } i \neq j\}
\]
as follows: \((i,j) \leq (i',j')\) if \("j < j'"\) or \("j = j', i \leq i'"\). The associated subsemigroup \(S\) of \(\mathbb{Z}^4\) is generated by \(b_1 = e_1, b_2 = e_2, b_3 = e_3, b_4 = e_4, b_5 = (1,1,-1,0)\) and \(b_6 = (-1,0,1,1)\). Then \(S\) is saturated, which implies that \(H\) is of quasi-toric type.

**Proposition 13.** Let \(H\) be a neat numerical semigroup with \(M(H) = \{a_1, \ldots, a_n\}\) such that it has a neat system of relations
\[
\begin{align*}
\alpha_1 a_1 &= \alpha_{12} a_2 + \cdots + \alpha_{1n} a_n \\
\alpha_{n} a_n &= \alpha_{n1} a_1 + \cdots + \alpha_{n-1} a_{n-1}
\end{align*}
\]
with
\[
\begin{vmatrix}
\alpha_1 & -\alpha_{12} & \cdots & -\alpha_{1n-1} \\
-\alpha_{21} & \alpha_2 & \cdots & -\alpha_{2n-1} \\
\vdots & \vdots & \ddots & \vdots \\
-\alpha_{n-11} & -\alpha_{n-12} & \cdots & \alpha_{n-1}
\end{vmatrix} \neq 0.
\]
i) If there is some \(j_0\) with \(1 \leq j_0 \leq n\) such that \(\alpha_{ij_0} > 0\) for all \(i\), then \(H\) is of quasi-toric type.
ii) If there is some \(i_0\) with \(1 \leq i_0 \leq n\) such that \(\alpha_{i_0j} > 0\) for all \(j\), then \(H\) is of quasi-toric type.

**Example 14.** Let \(H = <a_1 = 20, a_2 = 24, a_3 = 25, a_4 = 31>\). We have a neat system of relations
\[
\begin{align*}
4a_1 &= a_2 + a_3 + a_4, \\
4a_2 &= 2a_1 + a_3 + a_4, \\
3a_3 &= a_1 + a_2 + a_4 \text{ and } 3a_4 = a_1 + 2a_2 + a_3.
\end{align*}
\]
Since we have
\[
\begin{vmatrix}
4 & -1 & -1 \\
-2 & 4 & -1 \\
-1 & -1 & 3
\end{vmatrix} = 31 \neq 0,
\]
by Proposition 13 \(H\) is of quasi-toric type (Cf. Example 6 ii)).

**Proposition 15.** Let \(H\) be a neat numerical semigroup with \(M(H) = \{a_1, \ldots, a_n\}\) such that it has a neat system of relations
\[
\begin{align*}
\alpha_1 a_1 &= \alpha_{1n} a_n + \alpha_{12} a_2 \\
\alpha_2 a_2 &= \alpha_{23} a_3 + \alpha_{23} a_3 \\
\cdots & \cdots \cdots \\
\alpha_{i} a_{i} &= \alpha_{i-1} a_{i-1} + \alpha_{ii+1} a_{i+1} (2 \leq i \leq n-1) \\
\cdots & \cdots \cdots \\
\alpha_n a_n &= \alpha_{n-1} a_{n-1} + \alpha_{n1} a_1
\end{align*}
\]
with
\[
\begin{vmatrix}
\alpha_1 & -\alpha_{12} & \cdots & -\alpha_{1n-1} \\
-\alpha_{21} & \alpha_2 & \cdots & -\alpha_{2n-1} \\
\vdots & \vdots & \ddots & \vdots \\
-\alpha_{n-11} & -\alpha_{n-12} & \cdots & \alpha_{n-1}
\end{vmatrix} \neq 0
\]
where we set \( \alpha_{ij} = 0 \) if \( \alpha_{ij}a_j \) does not appear in the system of relations. Then \( H \) is of quasi-toric type.

Example 16 (Komeda [3]). For any \( n \geq 5 \), let \( H_n \) be the numerical semigroup with
\[
M(H_n) = \{a_1 = n, a_2 = n + 1, a_3 = 2n + 3, a_4 = 2n + 4, \ldots, a_{n-1} = 2n + n - 1\}.
\]
Then we have a neat system of relations
\[
\alpha_1a_1 = 4a_1 = a_2 + a_{n-1}, \quad \alpha_2a_2 = 3a_2 = a_1 + a_3, \quad \alpha_3a_3 = 2a_3 = 2a_2 + a_4,
\]
\[
\alpha_i a_i = 2a_i = a_{i-1} + a_{i+1} \quad (4 \leq i \leq n-2), \quad \alpha_{n-1}a_{n-1} = 2a_{n-1} = 3a_1 + a_{n-2}.
\]
By Proposition 15, \( H_n \) is a neat numerical semigroup of quasi-toric type.

Theorem 17. Let \( H \) be a neat numerical semigroup with \( M(H) = \{a_1, a_2, a_3, a_4\} \). Then \( H \) is of quasi-toric type.

Proof. Let
\[
\begin{align*}
\alpha_1a_1 &= \alpha_{12}a_2 + \alpha_{13}a_3 + \alpha_{14}a_4 \\
\alpha_2a_2 &= \alpha_{21}a_1 + \alpha_{23}a_3 + \alpha_{24}a_4 \\
\alpha_3a_3 &= \alpha_{31}a_1 + \alpha_{32}a_2 + \alpha_{34}a_4 \\
\alpha_4a_4 &= \alpha_{41}a_1 + \alpha_{42}a_2 + \alpha_{43}a_3
\end{align*}
\]
be a unique neat system of relations for \( H \). We note that
\[
D = \begin{vmatrix}
\alpha_1 & -\alpha_{12} & -\alpha_{13} \\
-\alpha_{21} & \alpha_2 & -\alpha_{23} \\
-\alpha_{31} & -\alpha_{32} & \alpha_3
\end{vmatrix} > 0.
\]
By Proposition 13 first we may assume that \( \alpha_1a_1 = \alpha_{12}a_2 + \alpha_{14}a_4 \), which implies that \( \alpha_3 = \alpha_{23} + \alpha_{43} \). Moreover, we have

"\( \alpha_2a_2 = \alpha_{21}a_1 + \alpha_{23}a_3 \) or \( \alpha_{23}a_3 + \alpha_{24}a_4 \)"

and

"\( \alpha_4a_4 = \alpha_{41}a_1 + \alpha_{43}a_3 \) or \( \alpha_{42}a_2 + \alpha_{43}a_3 \)."

i) \( \alpha_2a_2 = \alpha_{21}a_1 + \alpha_{23}a_3, \quad \alpha_4a_4 = \alpha_{41}a_1 + \alpha_{43}a_3. \) Then \( \alpha_3a_3 = \alpha_{32}a_2 + \alpha_{34}a_4. \) This case is reduced to Proposition 15.
ii) $\alpha_2 a_2 = \alpha_1 a_1 + \alpha_3 a_3$, $\alpha_4 a_4 = \alpha_2 a_2 + \alpha_4 a_3$. Hence we get $\alpha_1 = \alpha_2 + \alpha_3, \alpha_2 = \alpha_1 + \alpha_4, \alpha_3 = \alpha_2 + \alpha_4$ and $\alpha_4 = \alpha_1 + \alpha_3$. We introduce the order on the set

$$I = \{(i, j) | 1 \leq i \leq 4, 1 \leq j \leq 4, i \neq j\}$$

as follows:

$(i, j) \leq (i', j')$ if "$j < j'"$ or "$j = j', i \leq i'"$. Then we get the associated subsemigroup

$$S = \langle b_1 = e_1, \ldots, b_5 = e_5, b_6, b_7, b_8 \rangle$$

of $\mathbb{Z}^5$ through the method in Definition 10 where $b_5 = e_1 + e_2 - e_3, b_7 = e_1 + e_4 - e_3$ and $b_8 = e_2 + e_5 - e_4$. We can show that $S$ is saturated.

iii) $\alpha_2 a_2 = \alpha_2 a_3 + \alpha_2 a_4, \alpha_4 a_4 = \alpha_4 a_1 + \alpha_4 a_3$. In the similar way to ii) we can show that $H$ is of quasi-toric type.

iv) $\alpha_2 a_2 = \alpha_2 a_3 + \alpha_2 a_4, \alpha_4 a_4 = \alpha_2 a_2 + \alpha_4 a_3$. In this case at most one $\alpha_{i1}$ appears. This contradicts the neatness of $H$.

Q.E.D.

**Problem 18.** Let $n \geq 5$. If $H$ is a neat numerical semigroup with $\# M(H) = n$, then is it of quasi-toric type?

**§3. Numerical semigroups of toric type.**

We study the relation between a 1-neat numerical semigroup and a numerical semigroup of toric type whose definitions are given in this section.

**Definition 19.** Let $H$ be a neat numerical with $M(H) = \{a_1, \ldots, a_n\}$ such that it has a neat system of relations

$$\begin{cases}
\alpha_1 a_1 = \alpha_1 a_2 + \cdots + \alpha_{1n} a_n. \\
\vdots \\
\alpha_n a_n = \alpha_{n1} a_1 + \cdots + \alpha_{nn} a_{n-1}
\end{cases}$$

It is said to be 1-neat if

$$\begin{vmatrix}
\alpha_1 & -\alpha_{12} & \cdots & -\alpha_{1n-1} \\
-\alpha_{21} & \alpha_2 & \cdots & -\alpha_{2n-1} \\
\vdots & \vdots & \ddots & \vdots \\
-\alpha_{n-11} & -\alpha_{n-12} & \cdots & \alpha_{n-1}
\end{vmatrix} = a_n.$$

**Example 20.** By Proposition 3 a non-symmetric numerical semigroup $H$ with $M(H) = \{a_1, a_2, a_3\}$ is 1-neat.

**Proposition 21.** Let $H$ be a numerical semigroup with $M(H) = \{a_1, \ldots, a_n\}$. Let $r_i = (r_{i1}, r_{i2}, \ldots, r_{in})$ be an element of the relation module $R$ for $H$ with
\[ i = 1, \ldots, n - 1. \] Assume that

\[
\begin{array}{cccc}
\tau_{11} & \tau_{12} & \cdots & \tau_{1n-1} \\
\tau_{21} & \tau_{22} & \cdots & \tau_{2n-1} \\
\vdots & \vdots & & \vdots \\
\tau_{n-11} & \tau_{n-12} & \cdots & \tau_{n-1n-1}
\end{array}
= \pm a_n.
\]

Then \( \tau_1, \ldots, \tau_{n-1} \) form a basis for the \( \mathbb{Z} \)-module \( R \).

Let \( H \) be a neat numerical semigroup with \( M(H) = \{a_1, \ldots, a_n\} \) with its fixed neat system of relations. We set \( N = \#\{(i, j) | a_{ij} \neq 0\} - (n - 1) \). Let \( S = \langle b_1, \ldots, b_{N+n-1} \rangle \) of \( \mathbb{Z}^N \) be the associated subsemigroup. Let \( k \) be a field. Let \( \varphi_H : k[X] = k[X_1, \ldots, X_n] \rightarrow k[H] = k[t^h]_{h \in H} \) be a \( k \)-algebra homomorphism sending \( X_i \) to \( t^{a_i} \), \( \pi : k[Y] = k[Y_1, \ldots, Y_{N+n-1}] \rightarrow k[S] = k[T^b]_{b \in S} \) a \( k \)-algebra homomorphism sending \( Y_i \) to \( T^{b_i} \), \( \eta : k[Y] \rightarrow k[X] \) a \( k \)-algebra homomorphism sending \( Y_i \) to \( g_i = X_i^{a_i} \) if \( b_i \) corresponds to \( a_\alpha \), and \( \zeta : k[N^N] = k[t_1, \ldots, t_n] \rightarrow k[H] \) a \( k \)-algebra homomorphism sending \( t_i \) to \( t^{w(c)} \) where the weight \( w \) on \( k[X] \) is defined by \( w(X_i) = a_i \) and \( w(c) = 0 \) for \( c \in k^\times \). By the definition of \( b_i \)'s, \( \zeta \) extends to \( \zeta' : k[S] \rightarrow k[H] \). Then we get \( \varphi_H \circ \eta = \zeta' \circ \pi \), which implies that \( \text{Ker} \varphi_H \supseteq \eta(\text{Ker} \pi) \).

**Definition 22.** A neat numerical semigroup \( H \) is said to be of toric type if it is of quasi-toric type and we have an isomorphism \( k[H] \cong k[S] \otimes_{k[Y]} k[X] \), that is to say, \( \text{Ker} \varphi_H = (\eta(\text{Ker} \pi)) \).

**Remark 23** (Komeda [2]). A numerical semigroup of toric type is Weierstrass, where a numerical semigroup \( H \) is said to be Weierstrass if there is a pointed non-singular complete curve \((C, P)\) over an algebraically closed field such that

\[ H = \{ n \in \mathbb{N} \mid \text{there is a rational function } f \text{ on } C \text{ with } (f)_\infty = nP \} \].

**Example 24.** Any non-symmetric numerical semigroup with \( M(H) = \{a_1, a_2, a_3\} \) is of toric type, because we know that the ideal \( \text{Ker} \varphi_H \) is generated by

\[ X_1^{a_1} - X_2^{a_2} X_3^{a_3}, X_2^{a_2} - X_1^{a_1} X_3^{a_3}, \text{ and } X_3^{a_3} - X_1^{a_1} X_2^{a_2} \] (Herzog [1]).

We note that \( H \) is 1-neat (Cf. Example 20).

**Example 25.** For any integer \( n \geq 5 \), let \( H_n \) be a numerical semigroup with

\[ M(H_n) = \{a_1 = n, a_2 = n + 1, a_3 = 2n + 3, a_4 = 2n + 4, \ldots, a_{n-1} = 2n + n - 1\} \]

(Cf. Example 16). Then the ideal \( \text{Ker} \varphi_{H_n} \) is generated by

\[ X_2^3 - X_1 X_3, X_2 X_j - X_1 X_{j+1} (3 \leq j \leq n - 2), X_2 X_{n-1} - X_4, \]
$$X_3X_j - X_2^2X_{j+1}(3 \leq j \leq n - 2), \ X_3X_{n-1} - X_2^2X_1^3,$$

$$X_iX_j - X_{i-1}X_{j+1}(4 \leq i \leq n - 2, \ i \leq j \leq n - 2), \ X_iX_{n-1} - X_{i-1}X_1^3(4 \leq i \leq n - 1).$$

It is proved that $H_n$ is of toric type. In this case $H_n$ is also 1-neat.

**Theorem 26.** A 1-neat numerical semigroup with $M(H) = \{a_1, a_2, a_3, a_4\}$ is of toric type.

**Problem 27.** Let $n \geq 5$. If $H$ is a 1-neat numerical semigroup with \#$M(H) = n$, is it of toric type?

**References**

