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On Commutative Semigroup Rings

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I am now making a book on commutative semigroup rings. It will appear before long. This is an introduction to the book.

Thus let $G$ be an abelian additive group which is torsion-free. A sub-semigroup $S$ of $G$ which contains 0 is called a grading monoid (or a g-monoid). Let $R$ be a commutative ring, and let $R[X; S] = \{ \sum_{\text{finite}} a_iX^{s_i} | a_i \in R, s_i \in S \}$ be the semigroup ring of $S$ over $R$. Let $\Pi$ be a ring-theoretical property. We will determine conditions for $R[X; S]$ to have property $\Pi$. For the present, within my knowledge and within my interest, there are 71 Theorems and 38 Propositions on $R[X; S]$ by a number of authors. We confer a number of references. The following is a part of them:

REFERENCES


This is an abstract and the details will appear elsewhere.


Now we will note some theorems on commutative semigroup rings. Let $G$ be a non-zero torsion-free abelian additive group, $S$ be a non-zero grading monoid, $R$ be a commutative ring, and $D$ be an integral domain. Let $q(S) = \{a - b \mid a, b \in S\}$. Then it is called the quotient group of
Let $\alpha \in \text{q}(S)$. If $n\alpha \in S$ for some positive integer $n$, then $\alpha$ is called integral over $S$. If each integral element of $\text{q}(S)$ belongs to $S$, then $S$ is called integrally closed.

**Theorem 1** The followings are equivalent.

(1) $D[X;S]$ is integrally closed.

(2) $D$ is integrally closed, and $S$ is integrally closed.

Let $\alpha \in \text{q}(S)$. Then $\alpha$ is called almost integral over $S$, if there exists $s \in S$ such that $s + n\alpha \in S$ for each positive integer $n$. If each almost integral element belongs to $S$, then $S$ is called completely integrally closed.

**Theorem 2** The followings are equivalent.

(1) $D[X;S]$ is completely integrally closed.

(2) $D$ is completely integrally closed, and $S$ is completely integrally closed.

A non-zero divisor of $R$ is also called a regular element. An ideal of $R$ which contains regular elements is called a regular ideal.

The total quotient ring of $R$ is denoted by $\text{q}(R)$.

If each finitely generated regular ideal of $R$ is invertible, then $R$ is called a Prüfer ring.

If each finitely generated ideal of $R$ is principal, then $R$ is called a Bezout ring.

If, for each $a \in R$, there exists $b \in R$ such that $a = a^2b$, then $R$ is called a von Neumann regular ring.

**Theorem 3** Let $\mathbb{Q}_0$ be the non-negative rational numbers. The followings are equivalent.

(1) $R[X;S]$ is a Prüfer ring.

(2) $R$ is a von Neumann regular ring, and $S$ is isomorphic onto either a subgroup of $\mathbb{Q}$ or a subsemigroup $S'$ of $\mathbb{Q}_0$ such that $\text{q}(S') \cap \mathbb{Q}_0 = S'$.

(3) $R[X;S]$ is a Bezout ring.
If \( G \) satisfies ascending chain condition on cyclic subgroups, then \( G \) is said to satisfy ACCC.

**Theorem 4** Let \( G = q(S) \). The followings are equivalent.
(1) \( D[X; S] \) is a unique factorization ring.
(2) \( D \) is a unique factorization ring, \( S \) is a unique factorization semigroup, and \( G \) satisfies ACCC.

If \( R \) satisfies ascending chain condition on regular ideals, then \( R \) is called an \( r \)-Noetherian ring.

**Theorem 5** The followings are equivalent.
(1) \( R[X; S] \) is a Noetherian ring.
(2) \( R[X; S] \) is an \( r \)-Noetherian ring.
(3) \( R \) is a Noetherian ring, and \( S \) is a finitely generated \( g \)-monoid.

Let \( I \) be a non-empty subset of \( q(R) \). We set \( I^{-1} = \{ x \in q(R) \mid xI \subset R \} \). We set \( I^v = (I^{-1})^{-1} \).

Let \( I \) be a fractional ideal of \( R \). If \( I^v = I \), then \( I \) is called divisorial.
If each divisorial ideal of \( D \) is principal, then \( D \) is called a pseudo-principal ring.
If each divisorial ideal of \( S \) is principal, then \( S \) is called a pseudo-principal semigroup.

**Theorem 6** Let \( G = q(S) \). The followings are equivalent.
(1) \( D[X; S] \) is a pseudo-principal ring.
(2) \( D \) is a pseudo-principal ring, \( S \) is a pseudo-principal semigroup, and \( G \) satisfies ACCC.

Let \( I \) be an ideal of \( R \) such that \( I^{k+1} = 0 \) for some positive integer \( k \).
We set \( d(I^i/I^{i+1}) = \min \{ |X| \mid X \text{ is a set of generators of the } R\text{-module } I^i/I^{i+1} \} \) for each \( i \) (\( d(0) = 0 \)). We set \( \nu(I) = d(I/I^2) + \cdots + d(I^{k-1}/I^k) + d(I^k) \).
If each finitely generated ideal of $R$ is generated by $n$-elements, then $R$ is said to have $n$-generator property.

Let $S$ be a finitely generated subsemigroup of $Q_0$, and let $q(S) = Zr \ (r \in Q_0)$. Then $\min \{(1/r)S - \{0\}\}$ is called the order of $S$, and is denoted by $o(S)$.

**Theorem 7** Let $N$ be the nil radical of $R$. The followings are equivalent.

1. $R[X; S]$ has the $n$-generator property.
2. One of the followings holds.
   1. $S$ is isomorphic onto a subgroup of $Q$, and $\dim (R) = 0$. If $I$ is a finitely generated ideal contained in $N$, there exists a decomposition $R = R e_1 \oplus \cdots \oplus R e_h$ such that $\nu (I e_j) < n$ for each $j$.
   2. $S$ is isomorphic onto a subsemigroup of $Q_0$, $o(S) < \infty$, and $\dim (R) = 0$. If $I$ is a finitely generated ideal contained in $N$, there exists a decomposition $R = R e_1 \oplus \cdots \oplus R e_h$ such that $(\nu (I e_j) + 1)o(S) \leq n$ for each $j$.

If each finitely generated regular ideal is generated by $n$-elements, then $R$ is said to have $r$-$n$-generator property.

If, for each regular non-unit $a$ of $R$, $R/(a)$ has $n$-generator property, then $R$ is said to have $r$-$n(1/2)$-generator property.

If, for each non-zero and non-unit $a$ of $R$, $R/(a)$ has $n$-generator property, then $R$ is said to have $n(1/2)$-generator property.

**Theorem 8** The followings are equivalent.

1. $R[X; S]$ has $n(1/2)$-generator property.
2. $R[X; S]$ has $n$-generator property.
3. $R[X; S]$ has $r$-$n(1/2)$-generator property.
4. $R[X; S]$ has $r$-$n$-generator property.

If each ideal of $R$ is generated by $n$-elements, then $R$ is said to have rank $n$. 
Theorem 9 Let \( \mathbb{Z}_0 \) be the non-negative integers. The followings are equivalent.

(1) \( R[X; S] \) has rank \( n \).

(2) One of the followings holds.

(i) \( S \) is isomorphic onto \( \mathbb{Z} \), and there exists a decomposition \( R = R_1 \oplus \cdots \oplus R_h \) which satisfies the following condition: If \( N_i \) is the nil radical of \( R_i \), then \( \nu(N_i) < n \), and \( R_i \) is a Noetherian local ring with maximal ideal \( N_i \) for each \( i \).

(ii) \( S \) is isomorphic onto a subsemigroup of \( \mathbb{Z}_0 \), and there exists a decomposition \( R = R_1 \oplus \cdots \oplus R_h \) which satisfies the following condition: If \( N_i \) is the nil radical of \( R_i \), then \( (\nu(N_i) + 1)\sigma(S) \leq n \), and \( R_i \) is a Noetherian local ring with maximal ideal \( N_i \) for each \( i \).

Let \( K \) be a commutative ring with \( K = q(K) \), and let \( \Gamma \) be a totally ordered abelian additive group. A mapping \( \nu \) of \( K \) onto \( \Gamma \cup \{\infty\} \) is called a valuation on \( K \) if \( \nu(x + y) \geq \inf (\nu(x), \nu(y)) \), and \( \nu(xy) = \nu(x) + \nu(y) \) for all \( x, y \in K \). The subring \( V = \{x \in K \mid \nu(x) \geq 0\} \) of \( K \) is called a valuation ring on \( K \). t.f.r. \( (\Gamma) \) is called the rank of \( \nu \) (or of \( V \)), where t.f.r. \( (\Gamma) = \max \{|X| \mid X \text{ is a subset of } \Gamma \text{ which is linearly independent over } \mathbb{Z}\} \).

If there exists a family \( \{V_\lambda \mid \Lambda\} \) of valuation rings on \( q(R) \) which satisfies the following conditions, then \( R \) is called a Krull ring: \( R = \bigcap_\Lambda V_\lambda \), each \( V_\lambda \) is rank 1 and discrete, and each regular element of \( R \) is a unit of \( V_\lambda \) for almost all \( \lambda \).

Let \( \Gamma \) be a totally ordered abelian additive group. A mapping \( \nu \) of \( G \) onto \( \Gamma \) is called a valuation on \( G \), if \( \nu(x + y) = \nu(x) + \nu(y) \) for all \( x, y \in G \). The subsemigroup \( V = \{x \in G \mid \nu(x) \geq 0\} \) of \( G \) is called a valuation semigroup on \( G \). t.f.r. \( (\Gamma) \) is called the rank of \( \nu \) (or of \( V \)).

Theorem 10 Let \( G = q(S) \). The followings are equivalent.

(1) \( D[X; S] \) is a Krull ring.

(2) \( D \) is a Krull ring, \( S \) is a Krull semigroup, and \( G \) satisfies ACCC.

Let \( L \) be an abelian additive group, and let \( p \) be a prime number. The
Theorem 11 Let $H$ be the unit group of $S$, and let $F$ be a free subgroup of $H$ such that $H/F$ is torsion. Let $\Omega$ be the set of prime numbers $p$ such that $p1_R$ is a non-unit of $R$. The followings are equivalent.

(1) $R[X; S]$ is a locally Noetherian ring.
(2) t.f.r. $(H) < \infty$, $R$ is locally Noetherian, $S$ is of the form $H + \mathbb{Z}_0s_1 + \cdots + \mathbb{Z}_0s_n$, and the $p$-primary component of $H/F$ is finite for each $p \in \Omega$.

Theorem 12 Assume that $D[X; S]$ is a Krull ring. Then

$C(D[X; S]) \cong C(D) \oplus C(S)$,

where $C(\ )$ denotes the divisor class group.

$R$ is called a $v$-ring, if it satisfies the following condition: If $I, J_1, J_2$ are finitely generated ideals of $R$ with $I$ regular, and $(IJ_1)^v \subset (IJ_2)^v$, then $J_1^v \subset J_2^v$.

We may naturally define $v$-semigroup.

Theorem 13 The followings are equivalent.

(1) $D[X; S]$ is a $v$-ring.
(2) $D$ is a $v$-ring, and $S$ is a $v$-semigroup.

Theorem 14 Assume that $D$ is integrally closed, and $S$ is integrally closed. The followings are equivalent.

(1) For each finite number of finitely generated non-zero ideals $I_1, \cdots, I_n$ of $D[X; S]$, we have $(I_1 \cap \cdots \cap I_n)^v = I_1^v \cap \cdots \cap I_n^v$.
(2) For each finite number of finitely generated non-zero ideals $I_1, \cdots, I_n$ of $D$, we have $(I_1 \cap \cdots \cap I_n)^v = I_1^v \cap \cdots \cap I_n^v$, and for each finite number of finitely generated ideals $I_1, \cdots, I_m$ of $S$, we have $(I_1 \cap \cdots \cap I_m)^v = I_1^v \cap \cdots \cap I_m^v$.
(3) $D[X; S]$ is a $v$-ring.
If, for each finitely generated regular ideal $I$ of $R$, there exists a finitely generated regular fractional ideal $J$ such that $(IJ)^v = R$, then $R$ is called a Prüfer $v$-multiplication ring.

**Theorem 15** The followings are equivalent.
1. $D[X; S]$ is a Prüfer $v$-multiplication ring.
2. $D$ is a Prüfer $v$-multiplication ring, and $S$ is a Prüfer $v$-multiplication semigroup.

Let $I$ be a non-zero fractional ideal of $R$. We set $I^t = \cup \{ J^v \mid J$ is a finitely generated fractional ideal contained in $I \}.$

**Theorem 16** Assume that $D$ is integrally closed, and $S$ is integrally closed. The followings are equivalent.
1. For each finite number of non-zero ideals $I_1, \ldots, I_n$ of $D[X; S]$, we have $(I_1 \cap \cdots \cap I_n)^t = I_1^t \cap \cdots \cap I_n^t$.
2. For each finite number of non-zero ideals $I_1, \ldots, I_n$ of $D$, we have $(I_1 \cap \cdots \cap I_n)^t = I_1^t \cap \cdots \cap I_n^t$, and for each finite number of ideals $I_1, \ldots, I_m$ of $S$, we have $(I_1 \cap \cdots \cap I_m)^t = I_1^t \cap \cdots \cap I_m^t$.
3. $D[X; S]$ is a Prüfer $v$-multiplication ring.

If $R$ satisfies the following condition, then $R$ is called a root closed ring: If $x \in \mathfrak{q}(R)$ and $x^n \in R$ for some positive integer $n$, then $x \in R$.

**Theorem 17** The followings are equivalent.
1. $D[X; S]$ is a root closed ring.
2. $D$ is a root closed ring, and $S$ is an integrally closed semigroup.

If $R$ satisfies the following condition, then $R$ is called a seminormal ring: If $x \in \mathfrak{q}(R)$ and $x^2, x^3 \in R$, then $x \in R$.

If $S$ satisfies the following condition, then $S$ is called a seminormal semigroup: If $x \in \mathfrak{q}(S)$ and $2x, 3x \in S$, then $s \in S$. 
Theorem 18  The followings are equivalent.
(1) $D[X;S]$ is seminormal.
(2) $D$ is seminormal, and $S$ is seminormal.

$R$ is called a u-closed ring, if it satisfies the following condition: If $x \in q(R)$, and $x^2 - x \in R$, $x^3 - x^2 \in R$, then $x \in R$.

Theorem 19  If $D$ is u-closed, then $D[X;S]$ is u-closed.

An ideal of $R$ (resp. $S$) is also called an integral ideal.
If $D$ satisfies the ascending chain condition on divisorial integral ideals of $D$, then $D$ is called a Mori-ring.
If $D$ is a Mori-ring, and if, for all $a, b \in D - \{0\}$, the ideal $(a, b)$ is divisorial, then $D$ is called an M-ring.
We may naturally define Mori-semigroup and M-semigroup.

Theorem 20  The followings are equivalent.
(1) $D[X;S]$ is an M-ring.
(2) $D$ is a field, and $S$ is isomorphic onto an M-subsemigroup of $\mathbb{Z}$.

Let $F(R)$ be the set of non-zero fractional ideals of $R$. A mapping $*$ of $F(R)$ to $F(R)$ is called a star operation on $R$, if, for regular $a \in q(R)$ and $I, J \in F(R)$,
(a) $^* = (a)$.
(al)$^* = aI^*$.
$I \subset I^*$.
If $I \subset J$, then $I^* \subset J^*$.
$(I^*)^* = I^*$.
The mapping $I \mapsto I^* = (I^{-1})^{-1}$ is a star operation called v-operation.
Assume that $R$ is integrally closed, and let $\{V_\lambda \mid \Lambda\}$ be the set of valuation overrings of $R$. The mapping $I \mapsto I^b = \cap_\lambda IV_\lambda$ is a star operation called b-operation.
A star operation $*$ is called an e.a.b., if it satisfies the following condition: If $I, J_1, J_2$ are finitely generated non-zero ideals of $R$ with $I$ regular,
and $(IJ_1)^* \subset (IJ_2)^*$, then $J_1^* \subset J_2^*$.

Let $F'(R)$ be the set of non-zero $R$-submodules of $q(R)$. A mapping $*$ of $F'(R)$ to $F'(R)$ is called a semistar operation on $R$, if it satisfies the following condition: For regular $a \in q(R)$ and $I, J \in F'(R)$,

$$(aI)^* = aI^*.$$  

If $I \subset J$, then $I^* \subset J^*$.  

$(I^*)^* = I^*$.  

A semistar operation $*$ of $R$ is called e.a.b., if it satisfies the following condition: If $I, J_1, J_2$ are non-zero finitely generated ideals with $I$ regular, and $(IJ_1)^* \subset (IJ_2)^*$, then $J_1^* \subset J_2^*$.

The mapping $I \mapsto I^*$ of $F'(R)$ is a semistar operation called $\nu'$-operation.

Let $\{V_\lambda | \Lambda\}$ be the set of valuation overrings of $R$. The mapping $I \mapsto I^\prime = \cap_\lambda IV_\lambda$ of $F'(R)$ is a semistar operation called $\phi'$-operation.

If each finitely generated regular ideal is principal, then $R$ is called an r-Bezout ring.

Let $f = \sum a_i X^{s_i}$, where each $a_i \neq 0$, and $s_i \neq s_j$ for $i \neq j$. We set $\sum R a_i = c(f)$.

If each regular ideal of $R$ is generated by regular elements, then $R$ is called a Marot ring. If $R$ satisfies the following condition, then $R$ is said to have Property (A): If $f$ is a regular element of $R[X]$, then $c(f)$ is a regular ideal of $R$.

$A$ denotes a Marot ring with Property (A).

**Theorem 21** Let $*$ be an e.a.b. star operation on $A$.

Set $A_* = \{f/g \in q(A[X;S]) | f, g \in A[X;S] - \{0\}, g \text{ regular}, \ c(f)^* \subset c(g)^*\} \cup \{0\}$. Then,

1. $A_*$ is an overring of $A[X;S]$, and $A_* \cap K = A$, where $K = q(A)$.
2. $A_*$ is an $r$-Bezout ring.
3. If $I$ is a finitely generated regular ideal of $A$, then $IA_* \cap K = I^*$ and $IA_* = I^* A_*$. 

A multiplicative subset $T$ of $R$ is called a regular multiplicative subset,
if each element of $T$ is regular.

**Theorem 22**  Assume that $A$ is integrally closed. Let $T = \{ f \in A[X; S] \mid c(f) = A \}$. The followings are equivalent.

1. $A$ is a Prüfer ring.
2. $A[X; S]_T = A_b$.
3. $A[X; S]_T$ is a Prüfer ring.
4. $A_b$ is a quotient ring of $A[X; S]$ with respect to a regular multiplicative subset.
5. Each prime ideal of $A[X; S]_T$ is the contraction of a prime ideal of $A_b$.
5'. Each regular prime ideal of $A[X; S]_T$ is the contraction of a prime ideal of $A_b$.
6. Each regular prime ideal of $A[X; S]_T$ is the extension of a prime ideal of $A$.

If each regular ideal is the product of prime ideals, then $R$ is called a Dedekind ring.

If each regular ideal of $R$ is principal, then $R$ is called an r-principal ideal ring.

**Theorem 23**  Assume that $A$ is integrally closed. Let $T = \{ f \in A[X; S] \mid c(f) = A \}$. The followings are equivalent.

1. $A$ is a Dedekind ring.
2. $A[X; S]_T$ is a Dedekind ring.
3. $A_b$ is a Dedekind ring.
4. $A_b$ is an r-Noetherian ring.
5. $A_b$ is a Krull ring.
6. $A_b$ is an r-principal ideal ring.

Let $*$ be a star operation on $R$. If, for each finitely generated regular ideal $I$ of $R$, there exists a finitely generated regular fractional ideal $J$ such that $(IJ)^* = R$, then $R$ is called a Prüfer $*$-multiplication ring.

Let $P$ be a prime ideal of $R$. Then we set $R_{[P]} = \{ x \in q(R) \mid sx \in R \}$
for some $s \in R - P$.

**Theorem 24** Let $*$ be an e.a.b. star operation on $A$. Let $N = \{ g \in A[X;S] | g \text{ is regular, and } c(g)^* = A \}$. The followings are equivalent.

1. $A$ is a Prüfer $*$-multiplication ring.
2. $A_*$ is a quotient ring of $A[X;S]$ with respect to a regular multiplicative subset.
3. If $V$ is a valuation overring of $A_*$, there exists a prime ideal $P$ of $A$ which satisfies the following condition: $A_{[P]}$ is a valuation overring of $A$, and $V = A[X;S]_{[P,A[X;S]]}$.
4. $A_*$ is a flat $A[X;S]$-module.
5. $A[X;S]_N$ is a Prüfer ring.

Let $f = \sum a_iX^{s_i}$, where each $a_i \neq 0$ and $s_i \neq s_j$ for $i \neq j$. We set $e(f) = \cup (S + s_i)$.

**Theorem 25** Let $*$ be an e.a.b. star operation on $S$, $G = q(S)$, and let $K$ be a field. We set $S_* = \{ f/g | f, g \in K[X;S] - \{0\}, e(f)^* \subset e(g)^* \} \cup \{0\}$. 

1. $S_*$ is an overring of $K[X;S]$, and $S_* \cap G = S$.
2. $S_*$ is a Bezout ring.
3. If $I$ is a finitely generated ideal of $S$, then $(IS_*) \cap G = I^*$, and $IS_* = I^*S_*$. 

For an e.a.b. semiatar operation $*$ on $A$ (or on $S$), we may naturally define Kronecker function ring $A_*$ (or $S_*$). Moreover, we may show the similar results to those for star operations.