<table>
<thead>
<tr>
<th>Title</th>
<th>On Commutative Semigroup Rings (Algebraic Systems, Formal Languages and Conventional and Unconventional Computation Theory)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Matsuda, Ryuki</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2004), 1366: 153-164</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/25372">http://hdl.handle.net/2433/25372</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
On Commutative Semigroup Rings

松田 隆輝 (Ryūki Matsuda)
茨城大学理学部 (Faculty of Science, Ibaraki University)

I am now making a book on commutative semigroup rings. It will appear before long. This is an introduction to the book.

Thus let $G$ be an abelian additive group which is torsion-free. A subsemigroup $S$ of $G$ which contains 0 is called a grading monoid (or a g-monoid). Let $R$ be a commutative ring, and let $R[X;S] = \{ \sum_{finite} a_i X^{s_i} \mid a_i \in R, s_i \in S \}$ be the semigroup ring of $S$ over $R$. Let $\Pi$ be a ring-theoretical property. We will determine conditions for $R[X;S]$ to have property $\Pi$. For the present, within my knowledge and within my interest, there are 71 Theorems and 38 Propositions on $R[X;S]$ by a number of authors. We confer a number of references. The following is a part of them:

REFERENCES


This is an abstract and the details will appear elsewhere.
Now we will note some theorems on commutative semigroup rings. Let $G$ be a non-zero torsion-free abelian additive group, $S$ be a non-zero grading monoid, $R$ be a commutative ring, and $D$ be an integral domain. Let $q(S) = \{a - b \mid a, b \in S\}$. Then it is called the quotient group of
Let $\alpha \in \mathbb{q}(S)$. If $n\alpha \in S$ for some positive integer $n$, then $\alpha$ is called integral over $S$. If each integral element of $\mathbb{q}(S)$ belongs to $S$, then $S$ is called integrally closed.

**Theorem 1** The followings are equivalent.
(1) $D[X;S]$ is integrally closed.
(2) $D$ is integrally closed, and $S$ is integrally closed.

Let $\alpha \in \mathbb{q}(S)$. Then $\alpha$ is called almost integral over $S$, if there exists $s \in S$ such that $s + n\alpha \in S$ for each positive integer $n$. If each almost integral element belongs to $S$, then $S$ is called completely integrally closed.

**Theorem 2** The followings are equivalent.
(1) $D[X;S]$ is completely integrally closed.
(2) $D$ is completely integrally closed, and $S$ is completely integrally closed.

A non-zero divisor of $R$ is also called a regular element. An ideal of $R$ which contains regular elements is called a regular ideal.

The total quotient ring of $R$ is denoted by $\mathbb{q}(R)$.

If each finitely generated regular ideal of $R$ is invertible, then $R$ is called a Prüfer ring.

If each finitely generated ideal of $R$ is principal, then $R$ is called a Bezout ring.

If, for each $a \in R$, there exists $b \in R$ such that $a = a^2b$, then $R$ is called a von Neumann regular ring.

**Theorem 3** Let $\mathbb{Q}_0$ be the non-negative rational numbers. The followings are equivalent.
(1) $R[X;S]$ is a Prüfer ring.
(2) $R$ is a von Neumann regular ring, and $S$ is isomorphic onto either a subgroup of $\mathbb{Q}$ or a subsemigroup $S'$ of $\mathbb{Q}_0$ such that $\mathbb{q}(S') \cap \mathbb{Q}_0 = S'$.
(3) $R[X;S]$ is a Bezout ring.
If \( G \) satisfies ascending chain condition on cyclic subgroups, then \( G \) is said to satisfy ACCC.

**Theorem 4**  Let \( G = q(S) \). The followings are equivalent.
(1) \( D[X; S] \) is a unique factorization ring.
(2) \( D \) is a unique factorization ring, \( S \) is a unique factorization semigroup, and \( G \) satisfies ACCC.

If \( R \) satisfies ascending chain condition on regular ideals, then \( R \) is called an \( r \)-Noetherian ring.

**Theorem 5**  The followings are equivalent.
(1) \( R[X; S] \) is a Noetherian ring.
(2) \( R[X; S] \) is an \( r \)-Noetherian ring.
(3) \( R \) is a Noetherian ring, and \( S \) is a finitely generated \( g \)-monoid.

Let \( I \) be a non-empty subset of \( q(R) \). We set \( I^{-1} = \{ x \in q(R) \mid xI \subset R \} \). We set \( I^v = (I^{-1})^{-1} \).

Let \( I \) be a fractional ideal of \( R \). If \( I^v = I \), then \( I \) is called divisorial.
If each divisorial ideal of \( D \) is principal, then \( D \) is called a pseudo-principal ring.
If each divisorial ideal of \( S \) is principal, then \( S \) is called a pseudo-principal semigroup.

**Theorem 6**  Let \( G = q(S) \). The followings are equivalent.
(1) \( D[X; S] \) is a pseudo-principal ring.
(2) \( D \) is a pseudo-principal ring, \( S \) is a pseudo-principal semigroup, and \( G \) satisfies ACCC.

Let \( I \) be an ideal of \( R \) such that \( I^{k+1} = 0 \) for some positive integer \( k \). We set \( d(I^i/I^{i+1}) = \min \{|X| \mid X \text{ is a set of generators of the } R \text{-module } I^i/I^{i+1}\} \) for each \( i \) (\( d(0) = 0 \)). We set \( \nu(I) = d(I/I^2) + \cdots + d(I^{k-1}/I^k) + d(I^k) \).
If each finitely generated ideal of $R$ is generated by $n$-elements, then $R$ is said to have $n$-generator property.

Let $S$ be a finitely generated subsemigroup of $Q_0$, and let $q(S) = Zr$ $(r \in Q_0)$. Then $\min \{(1/r)S - \{0\}\}$ is called the order of $S$, and is denoted by $o(S)$.

Theorem 7 Let $N$ be the nil radical of $R$. The followings are equivalent.

(1) $R[X;S]$ has the $n$-generator property.
(2) One of the followings holds.

(i) $S$ is isomorphic onto a subgroup of $Q$, and $\dim (R) = 0$. If $I$ is a finitely generated ideal contained in $N$, there exists a decomposition $R = Re_1 \oplus \cdots \oplus Re_h$ such that $\nu(Ie_j) < n$ for each $j$.

(ii) $S$ is isomorphic onto a subsemigroup of $Q_0$, $o(S) < \infty$, and $\dim (R) = 0$. If $I$ is a finitely generated ideal contained in $N$, there exists a decomposition $R = Re_1 \oplus \cdots \oplus Re_h$ such that $(\nu(Ie_j) + 1)o(S) \leq n$ for each $j$.

Theorem 8 The followings are equivalent.

(1) $R[X;S]$ has $n(1/2)$-generator property.
(2) $R[X;S]$ has $n$-generator property.
(3) $R[X;S]$ has $r-n(1/2)$-generator property.
(4) $R[X;S]$ has $r-n$-generator property.

If each finitely generated regular ideal is generated by $n$-elements, then $R$ is said to have $r-n$-generator property.

If, for each regular non-unit $a$ of $R$, $R/(a)$ has $n$-generator property, then $R$ is said to have $r-n(1/2)$-generator property.

If, for each non-zero and non-unit $a$ of $R$, $R/(a)$ has $n$-generator property, then $R$ is said to have $n(1/2)$-generator property.

If each ideal of $R$ is generated by $n$-elements, then $R$ is said to have rank $n$. 
Theorem 9  Let $\mathbb{Z}_0$ be the non-negative integers. The followings are equivalent.

(1) $R[X;S]$ has rank $n$.

(2) One of the followings holds.

(i) $S$ is isomorphic onto $\mathbb{Z}$, and there exists a decomposition $R = R_1 \oplus \cdots \oplus R_h$ which satisfies the following condition: If $N_i$ is the nil radical of $R_i$, then $v(N_i) < n$, and $R_i$ is a Noetherian local ring with maximal ideal $N_i$ for each $i$.

(ii) $S$ is isomorphic onto a subsemigroup of $\mathbb{Z}_0$, and there exists a decomposition $R = R_1 \oplus \cdots \oplus R_h$ which satisfies the following condition: If $N_i$ is the nil radical of $R_i$, then $(\nu(N_i) + 1)\sigma(S) \leq n$, and $R_i$ is a Noetherian local ring with maximal ideal $N_i$ for each $i$.

Let $K$ be a commutative ring with $K = q(K)$, and let $\Gamma$ be a totally ordered abelian additive group. A mapping $v$ of $K$ onto $\Gamma \cup \{\infty\}$ is called a valuation on $K$ if $v(x + y) \geq \inf (v(x), v(y))$, and $v(xy) = v(x) + v(y)$ for all $x, y \in K$. The subring $V = \{x \in K \mid v(x) \geq 0\}$ of $K$ is called a valuation ring on $K$. t.f.r. $(\Gamma)$ is called the rank of $v$ (or of $V$), where t.f.r. $(\Gamma) = \max \{|X| \mid X \text{ is a subset of } \Gamma \text{ which is linearly independent over } \mathbb{Z}\}$.

If there exists a family $\{V_\lambda \mid \Lambda\}$ of valuation rings on $q(R)$ which satisfies the following conditions, then $R$ is called a Krull ring: $R = \cap_\lambda V_\lambda$, each $V_\lambda$ is rank 1 and discrete, and each regular element of $R$ is a unit of $V_\lambda$ for almost all $\lambda$.

Let $\Gamma$ be a totally ordered abelian additive group. A mapping $v$ of $G$ onto $\Gamma$ is called a valuation on $G$, if $v(x + y) = v(x) + v(y)$ for all $x, y \in G$. The subsemigroup $V = \{x \in G \mid v(x) \geq 0\}$ of $G$ is called a valuation semigroup on $G$. t.f.r. $(\Gamma)$ is called the rank of $v$ (or of $V$).

Theorem 10  Let $G = q(S)$. The followings are equivalent.

(1) $D[X;S]$ is a Krull ring.

(2) $D$ is a Krull ring, $S$ is a Krull semigroup, and $G$ satisfies ACCC.

Let $L$ be an abelian additive group, and let $p$ be a prime number. The
Theorem 11  Let $H$ be the unit group of $S$, and let $F$ be a free subgroup of $H$ such that $H/F$ is torsion. Let $\Omega$ be the set of prime numbers $p$ such that $pl_R$ is a non-unit of $R$. The followings are equivalent.

(1) $R[X;S]$ is a locally Noetherian ring.

(2) t.f.r. $(H) < \infty$, $R$ is locally Noetherian, $S$ is of the form $H + \mathbb{Z}_0s_1 + \cdots + \mathbb{Z}_0s_n$, and the $p$-primary component of $H/F$ is finite for each $p \in \Omega$.

Theorem 12  Assume that $D[X;S]$ is a Krull ring. Then $\mathcal{C}(D[X;S]) \cong \mathcal{C}(D) \oplus \mathcal{C}(S)$, where $\mathcal{C}( )$ denotes the divisor class group.

$R$ is called a v-ring, if it satisfies the following condition: If $I, J_1, J_2$ are finitely generated ideals of $R$ with $I$ regular, and $(IJ_1)^v \subset (IJ_2)^v$, then $J_1^v \subset J_2^v$.

We may naturally define v-semigroup.

Theorem 13  The followings are equivalent.

(1) $D[X;S]$ is a v-ring.

(2) $D$ is a v-ring, and $S$ is a v-semigroup.

Theorem 14  Assume that $D$ is integrally closed, and $S$ is integrally closed. The followings are equivalent.

(1) For each finite number of finitely generated non-zero ideals $I_1, \cdots, I_n$ of $D[X;S]$, we have $(I_1 \cap \cdots \cap I_n)^v = I_1^v \cap \cdots \cap I_n^v$.

(2) For each finite number of finitely generated non-zero ideals $I_1, \cdots, I_n$ of $D$, we have $(I_1 \cap \cdots \cap I_n)^v = I_1^v \cap \cdots \cap I_n^v$, and for each finite number of finitely generated ideals $I_1, \cdots, I_m$ of $S$, we have $(I_1 \cap \cdots \cap I_m)^v = I_1^v \cap \cdots \cap I_m^v$.

(3) $D[X;S]$ is a v-ring.
If, for each finitely generated regular ideal \( I \) of \( R \), there exists a finitely generated regular fractional ideal \( J \) such that \((IJ)^v = R\), then \( R \) is called a Prüfer \( v \)-multiplication ring.

**Theorem 15** The followings are equivalent.
1. \( D[X; S] \) is a Prüfer \( v \)-multiplication ring.
2. \( D \) is a Prüfer \( v \)-multiplication ring, and \( S \) is a Prüfer \( v \)-multiplication semigroup.

Let \( I \) be a non-zero fractional ideal of \( R \). We set \( I^v = \cup \{ J^v \mid J \text{ is a finitely generated fractional ideal contained in } I \} \).

**Theorem 16** Assume that \( D \) is integrally closed, and \( S \) is integrally closed. The followings are equivalent.
1. For each finite number of non-zero ideals \( I_1, \cdots, I_n \) of \( D[X; S] \), we have \((I_1 \cap \cdots \cap I_n)^v = I_1^v \cap \cdots \cap I_n^v \).
2. For each finite number of non-zero ideals \( I_1, \cdots, I_n \) of \( D \), we have \((I_1 \cap \cdots \cap I_n)^v = I_1^v \cap \cdots \cap I_n^v \), and for each finite number of ideals \( I_1, \cdots, I_m \) of \( S \), we have \((I_1 \cap \cdots \cap I_m)^v = I_1^v \cap \cdots \cap I_m^v \).
3. \( D[X; S] \) is a Prüfer \( v \)-multiplication ring.

If \( R \) satisfies the following condition, then \( R \) is called a root closed ring: If \( x \in q(R) \) and \( x^n \in R \) for some positive integer \( n \), then \( x \in R \).

**Theorem 17** The followings are equivalent.
1. \( D[X; S] \) is a root closed ring.
2. \( D \) is a root closed ring, and \( S \) is an integrally closed semigroup.

If \( R \) satisfies the following condition, then \( R \) is called a seminormal ring: If \( x \in q(R) \) and \( x^2, x^3 \in R \), then \( x \in R \).

If \( S \) satisfies the following condition, then \( S \) is called a seminormal semigroup: If \( x \in q(S) \) and \( 2x, 3x \in S \), then \( s \in S \).
Theorem 18  The followings are equivalent.
(1) $D[X;S]$ is seminormal.
(2) $D$ is seminormal, and $S$ is seminormal.

$R$ is called a u-closed ring, if it satisfies the following condition: If $x \in q(R)$, and $x^2 - x \in R$, $x^3 - x^2 \in R$, then $x \in R$.

Theorem 19  If $D$ is u-closed, then $D[X;S]$ is u-closed.

An ideal of $R$ (resp. $S$) is also called an integral ideal.

If $D$ satisfies the ascending chain condition on divisorial integral ideals of $D$, then $D$ is called a Mori-ring.

If $D$ is a Mori-ring, and if, for all $a, b \in D - \{0\}$, the ideal $(a, b)$ is divisorial, then $D$ is called an M-ring.

We may naturally define Mori-semigroup and M-semigroup.

Theorem 20  The followings are equivalent.
(1) $D[X;S]$ is an M-ring.
(2) $D$ is a field, and $S$ is isomorphic onto an M-subsemigroup of $Z$.

Let $F(R)$ be the set of non-zero fractinal ideals of $R$. A mapping $*$ of $F(R)$ to $F(R)$ is called a star operation on $R$, if, for regular $a \in q(R)$ and $I, J \in F(R)$,

(a) $^* = (a)$.
(b) $(aI)^* = aI^*$.
(c) $I \subset I^*$.
(d) If $I \subset J$, then $I^* \subset J^*$.
(e) $(I^*)^* = I^*$.

The mapping $I \mapsto I^* = (I^{-1})^{-1}$ is a star operation called v-operation.

Assume that $R$ is integrally closed, and let $\{V_\Lambda \mid \Lambda \}$ be the set of valuation overrings of $R$. The mapping $I \mapsto I^b = \cap_\Lambda IV_\Lambda$ is a star operation called b-operation.

A star operation $*$ is called an e.a.b., if it satisfies the following condition: If $I, J_1, J_2$ are finitely generated non-zero ideals of $R$ with $I$ regular,
and \((IJ_1)^* \subset (IJ_2)^*\), then \(J_1^* \subset J_2^*\).

Let \(F'(R)\) be the set of non-zero \(R\)-submodules of \(q(R)\). A mapping \(*\) of \(F'(R)\) to \(F'(R)\) is called a semistar operation on \(R\), if it satisfies the following condition: For regular \(a \in q(R)\) and \(I, J \in F'(R)\),
\[(aI)^* = aI^*.
\]

If \(I \subset J\), then \(I^* \subset J^*\).
\[(I^*)^* = I^*.
\]

A semistar operation \(*\) of \(R\) is called e.a.b., if it satisfies the following condition: If \(I, J_1, J_2\) are non-zero finitely generated ideals with \(I\) regular, and \((IJ_1)^* \subset (IJ_2)^*\), then \(J_1^* \subset J_2^*\).

The mapping \(I \mapsto I^*\) of \(F'(R)\) is a semistar operation called \(v\)-operation.

Let \(\{V_\lambda \mid \Lambda\}\) be the set of valuation overrings of \(R\). The mapping \(I \mapsto I^v = \cap_\lambda IV_\lambda\) of \(F'(R)\) is a semistar operation called \(b\)-operation.

If each finitely generated regular ideal is principal, then \(R\) is called an \(r\)-Bezout ring.

Let \(f = \sum a_i X^{s_i}\), where each \(a_i \neq 0\), and \(s_i \neq s_j\) for \(i \neq j\). We set \(\sum R a_i = c(f)\).

If each regular ideal of \(R\) is generated by regular elements, then \(R\) is called a Marot ring. If \(R\) satisfies the following condition, then \(R\) is said to have Property (A): If \(f\) is a regular element of \(R[X]\), then \(c(f)\) is a regular ideal of \(R\).

\(A\) denotes a Marot ring with Property (A).

**Theorem 21** Let \(*\) be an e.a.b. star operation on \(A\).

Set \(A_* = \{f/g \in q(A[X; S]) \mid f, g \in A[X; S] - \{0\}, g \text{ is regular, and } c(f)^* \subset c(g)^*\} \cup \{0\}\). Then,
\[(1) A_* \text{ is an overring of } A[X; S], \text{ and } A_* \cap K = A, \text{ where } K = q(A).
\]
\[(2) A_* \text{ is an } r\text{-Bezout ring}.
\]
\[(3) \text{If } I \text{ is a finitely generated regular ideal of } A, \text{ then } IA_* \cap K = I^* \text{ and } IA_* = I^*A_*.
\]

A multiplicative subset \(T\) of \(R\) is called a regular multiplicative subset,
if each element of \( T \) is regular.

**Theorem 22** Assume that \( A \) is integrally closed. Let \( T = \{ f \in A[X;S] \mid c(f) = A \} \). The followings are equivalent.

1. \( A \) is a Prüfer ring.
2. \( A[X;S]_T = A_b \).
3. \( A[X;S]_T \) is a Prüfer ring.
4. \( A_b \) is a quotient ring of \( A[X;S] \) with respect to a regular multiplicative subset.
5. Each prime ideal of \( A[X;S]_T \) is the contraction of a prime ideal of \( A_b \).
6. Each regular prime ideal of \( A[X;S]_T \) is the contraction of a prime ideal of \( A_b \).
7. Each regular prime ideal of \( A[X;S]_T \) is the extension of a prime ideal of \( A \).

If each regular ideal is the product of prime ideals, then \( R \) is called a Dedekind ring.

If each regular ideal of \( R \) is principal, then \( R \) is called an r-principal ideal ring.

**Theorem 23** Assume that \( A \) is integrally closed. Let \( T = \{ f \in A[X;S] \mid c(f) = A \} \). The followings are equivalent.

1. \( A \) is a Dedekind ring.
2. \( A[X;S]_T \) is a Dedekind ring.
3. \( A_b \) is a Dedekind ring.
4. \( A_b \) is an r-Noetherian ring.
5. \( A_b \) is a Krull ring.
6. \( A_b \) is an r-principal ideal ring.

Let \( * \) be a star operation on \( R \). If, for each finitely generated regular ideal \( I \) of \( R \), there exists a finitely generated regular fractional ideal \( J \) such that \( (IJ)^* = R \), then \( R \) is called a Prüfer \( * \)-multiplication ring.

Let \( P \) be a prime ideal of \( R \). Then we set \( R_{[P]} = \{ x \in q(R) \mid sx \in R \} \).
Theorem 24  Let $*$ be an e.a.b. star operation on $A$. Let $N = \{g \in A[X; S] \mid g \text{ is regular, and } c(g)^* = A\}$. The followings are equivalent.

1. $A$ is a Prüfer $*$-multiplication ring.
2. $A_*$ is a quotient ring of $A[X; S]$ with respect to a regular multiplicative subset.
3. If $V$ is a valuation overring of $A_*$, there exists a prime ideal $P$ of $A$ which satisfies the following condition: $A_{[P]}$ is a valuation overring of $A$, and $V = A[X; S]_{[P; A[X; S]]}$.
4. $A_*$ is a flat $A[X; S]$-module.
5. $A[X; S]_N$ is a Prüfer ring.

Let $f = \sum a_i X^{s_i}$, where each $a_i \neq 0$ and $s_i \neq s_j$ for $i \neq j$. We set $e(f) = \cup(S + s_i)$.

Theorem 25  Let $*$ be an e.a.b. star operation on $S$, $G = q(S)$, and let $K$ be a field. We set $S_* = \{f/g \mid f, g \in K[X; S] - \{0\}, e(f)^* \subset e(g)^*\} \cup \{0\}$.

1. $S_*$ is an overring of $K[X; S]$, and $S_* \cap G = S$.
2. $S_*$ is a Bezout ring.
3. If $I$ is a finitely generated ideal of $S$, then $(IS_*) \cap G = I^*$, and $IS_* = I^* S_*$.

For an e.a.b. semiatar operation $*$ on $A$ (or on $S$), we may naturally define Kronecker function ring $A_*$ (or $S_*$). Moreover, we may show the similar results to those for star operations.