The finite basis problem
for monoids of triangular boolean matrices

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Abstract
We show that the monoid of all $n \times n$ upper triangular boolean matrices has no finite identity basis whenever $n > 3$. The identities of its submonoid consisting of matrices in which all diagonal entries are 1 possess a finite basis if and only if $n \leq 4$.

1 The finite basis problem for matrix monoids

An algebra $A$ is said to be finitely based if all identities holding in $A$ follow from a finite set of such identities (an identity basis of $A$); otherwise $A$ is called nonfinitely based. While every finite group is finitely based (Oates and Powell [5]), among finite monoids there exist nonfinitely based ones. The first example of a nonfinitely based finite monoid (due to Perkins [6]) was the Brandt monoid $B_2^1$ formed by the six matrices

\[
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 \\
0 & 0 \\
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 \\
0 & 0 \\
\end{pmatrix}, \quad \begin{pmatrix}
0 & 1 \\
0 & 0 \\
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 \\
1 & 0 \\
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 \\
0 & 1 \\
\end{pmatrix}
\]

under the usual multiplication of $n \times n$-matrices:

\[
(\alpha_{ij}) \cdot (\beta_{ij}) = \left( \sum_{k=1}^{n} \alpha_{ik} \beta_{kj} \right).
\]
We have not yet specified where 0 and 1, the entries of the matrices in (1), come from and how they add and multiply. In fact, there is a certain degree of freedom here: since the above matrices have at most one non-zero entry in each row, the multiplication (2) works just the same provided that the symbols 0 and 1 obey the following rules:

\[
0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0 + 0 = 0, \quad 1 \cdot 1 = 1 + 0 = 0 + 1 = 1, \tag{3}
\]

whereas the value of the sum \(1 + 1\) does not matter at all. Therefore, one may conveniently think of \(B_2^1\) as of a submonoid of the monoid \(M_2(F)\) of all \(2 \times 2\)-matrices over an arbitrary field \(F\) and, in the same time, as of a submonoid of the monoid \(B_2\) of all binary relations over the 2-element set because the latter monoid can be identified with the monoid of all \(2 \times 2\)-matrices over the boolean semiring \(\mathcal{B} = \langle \{0, 1\}; +, \cdot \rangle\) in which the operations satisfy the rules (3) and in addition \(1 + 1 = 1\).

The twofold nature of \(B_2^1\) is important in view of Sapir's powerful result [8] revealing that this monoid is inherently nonfinitely based. We recall that a finite monoid is said to be \emph{inherently nonfinitely based} if it is contained in no locally finite finitely based variety. Since the variety generated by a finite monoid is locally finite, every inherently nonfinitely based finite monoid automatically is nonfinitely based; moreover, if the variety \(\text{var} M\) generated by a finite monoid \(M\) contains an inherently nonfinitely based monoid, then \(M\) itself is inherently nonfinitely based and so \(M\) is nonfinitely based. Therefore the fact that \(B_2^1\) is inherently nonfinitely based immediately solves the finite basis problem for both the monoid \(M_n(F)\) of all \(n \times n\)-matrices over a finite field \(F\) and the monoid \(B_n\) of all binary relations over the \(n\)-element set: for \(n \geq 2\) these monoids are nonfinitely based. (Indeed, one only has to take into account that if \(n \geq 2\) then \(M_n(F)\) and \(B_n\) contain submonoids isomorphic to \(M_2(F)\) and respectively \(B_2\), whence they both possess submonoids isomorphic to \(B_2^1\)—then Sapir’s theorem applies.) An alternative proof of this result which does not depend on the notion of an inherently nonfinitely based monoid has been published in [12]; again, this proof is a byproduct of a more general approach.

Since a complete solution of the finite basis problem for full matrix monoids has been found, the next logical step in the general programme of classifying finite monoids with respect to the finite basis property may consist in investigating certain distinguished submonoids in \(M_n(F)\) and \(B_n\). Here the monoids of triangular matrices naturally come into the play. Let \(T_n(F)\) denote the submonoid of all upper triangular matrices in \(M_n(F)\) and let \(T B_n\) be the monoid of all upper triangular boolean \(n \times n\)-matrices.
In [14] we have studied and partially solved the finite basis problem for the monoids \( \mathcal{I}_n(F) \) where \( F \) is a finite field. We note that these monoids generate varieties which do not contain the monoid \( B_2^1 \) so Sapir's result does not apply here; neither does the approach from [12]. Nevertheless, we have proved that \( \mathcal{I}_n(F) \) is nonfinitely based (and even inherently nonfinitely based) provided that \( |F| > 2 \) and \( n \geq 4 \). The cases when the field \( F \) has precisely 2 elements or when \( |F| > 2 \) but \( n = 2, 3 \) still remain open; we can only prove that under these conditions the monoid \( \mathcal{I}_n(F) \) is not inherently nonfinitely based but we do not know whether or not it is finitely based.

The first main result of the present note is to some extent similar to the cited result of [14]: we show that the monoid \( \mathcal{I}B_n \) is nonfinitely based (and even inherently nonfinitely based) provided that \( n \geq 4 \). The reason behind this fact is that in the boolean situation the monoid \( B_2^1 \) does belong to the varieties generated by the monoids of triangular matrices of sufficiently big size. Again, in the cases when \( n = 2, 3 \) we only know that \( \mathcal{I}B_n \) is not inherently nonfinitely based but we do not know whether it is finitely based.

Our second main result solves the finite basis problem for a closely related family of submonoids in \( B_n \), namely, for the monoids \( UTB_n \) of all upper unitriangular boolean \( n \times n \)-matrices, that is, upper triangular boolean matrices in which all diagonal entries are 1. We show that the monoids \( UTB_n \) are nonfinitely based whenever \( n \geq 5 \) and provide an explicit finite identity basis for each monoid \( UTB_n \) with \( n = 2, 3, 4 \).

For the sake of completeness we mention that the monoids of all upper unitriangular \( n \times n \)-matrices over a finite field \( F \) are finitely based. Indeed, these monoids are groups, and Oates–Powell’s theorem [5] applies.

We summarize this mini-survey of the current state of art in studying the finite basis problem for finite monoids of matrices in the following table.

**Table 1: The finite basis property of some finite monoids of \( n \times n \)-matrices**

<table>
<thead>
<tr>
<th>The underlying structure:</th>
<th>Type of matrices:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>arbitrary</td>
</tr>
<tr>
<td>a finite field ( F )</td>
<td>nonfinitely based if (</td>
</tr>
<tr>
<td>the boolean semiring ( B )</td>
<td>nonfinitely based if ( n \geq 4 ); unknown if ( n = 2, 3 )</td>
</tr>
</tbody>
</table>
The reader who wants to place the results listed in Table 1 into a broader context can find a comprehensive report on the finite basis problem for finite monoids in [13].

2 The identities of the monoid $\mathcal{J}_B_n$

Theorem 2.1. The monoid $\mathcal{J}_B_n$ is inherently nonfinitely based if and only if $n \geq 4$.

Proof. First observe that for each $n \geq 4$ the monoid $\mathcal{J}_B_n$ contains a submonoid isomorphic to $\mathcal{J}_B_4$. Therefore, as explained in Section 1, it suffices to verify that the Brandt monoid $B^1_2$ belongs to the variety $\text{var } \mathcal{J}_B_4$. Consider the submonoid $B$ in $\mathcal{J}_B_4$ generated by the following two matrices:

$$a = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  

Since for each matrix $(\gamma_{ij}) \in B$ one has $\gamma_{11} = \gamma_{44} = 1$, the set $I$ of all matrices $(\delta_{ij}) \in B$ such that $\delta_{14} = 1$ forms an ideal in $B$. A straightforward calculation shows that, besides $a$, $b$, and the identity matrix $e$, only the two matrices

$$ab = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad ba = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

belong to $B \setminus I$. This allows one to organize the following bijection between $B \setminus I$ and the set of non-zero matrices in $B^1_2$:

$$e \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad ab \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad a \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad ba \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and one easily checks that this bijection extends to an isomorphism between the Rees quotient $B/I$ and the monoid $B^1_2$. We see that $B^1_2$ turns out to be a morphic image of a submonoid in $\mathcal{J}_B_4$ whence $B^1_2$ belongs to the variety generated by $\mathcal{J}_B_4$.

It remains to verify that the monoids $\mathcal{J}_B_2$ and $\mathcal{J}_B_3$ are not inherently nonfinitely based. Since $\mathcal{J}_B_2$ embeds into $\mathcal{J}_B_3$, it suffices to consider the
latter monoid. With a little help of a computer programme, we have checked that $\mathcal{IB}_3$ satisfies the identities

$$x^5 = x^6 \quad \text{and} \quad (xy)^5(yx)^5(xy)^5 = (xy)^5.$$ 

Clearly, the first of these identities fails in any non-trivial group, and the second one can be shown to fail in the monoid $B^1_2$: just substitute $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for $x$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ for $y$. Hence the variety $\text{var} \mathcal{IB}_3$ contains neither $B^1_2$ nor non-trivial groups. From [9, Theorem 2] we conclude that the monoid $\mathcal{IB}_3$ is not inherently nonfinitely based.

Remark 2.1. The monoids $\mathcal{IB}_n$ have been extensively studied by Pin and Straubing [7]. Using the techniques of formal language theory, they have proved that the pseudovariety generated by these monoids coincides with the well known pseudovariety $\mathbf{PJ}$ generated by power monoids of $\mathcal{I}$-trivial finite monoids. Since the Brandt monoid $B^1_2$ is known to belong to $\mathbf{PJ}$ (see, for instance, [1, Section 11.6]), this result implies that $B^1_2$ belongs to the variety $\text{var} \mathcal{IB}_m$ for some $m$, and thus, the monoids $\mathcal{IB}_n$ with $n \geq m$ are inherently nonfinitely based. Our direct and rather simple approach gives the exact value of this $m$, namely, $m = 4$.

3 The identities of the monoid $\mathcal{UTB}_n$

We obtain the main result of this section by identifying the collection of all identities of the monoid $\mathcal{UTB}_n$ with a set of identities previously studied by Blanchet-Sadri [2-4]. Let us first define the latter set. We fix a countably infinite alphabet $\Sigma = \{x, y, z, t, x_1, x_2, \ldots, y_1, y_2, \ldots\}$; as usual, $\Sigma^*$ stands for the free monoid over $\Sigma$. If $u, v \in \Sigma^*$ are two words, we say that $u$ is a scattered subword of $v$ whenever there exist words $u_1, \ldots, u_n, v_0, v_1, \ldots, v_{n-1}, v_n \in \Sigma^*$ such that

$$u = u_1 \cdots u_n \quad \text{and} \quad v = v_0 u_1 v_1 \cdots v_{n-1} u_n v_n;$$

in other terms, this means that one can extract $u$ treated as a sequence of letters from the sequence $v$. We denote by $J_k$ the set of all identities $u = v$ such that the words $u$ and $v$ have the same set of scattered subwords of length $\leq k$. Clearly, we have

$$J_1 \supseteq J_2 \supseteq \cdots \supseteq J_k \supseteq \cdots.$$
Blanchet-Sadri [2–4] has established the following facts about the identity sets $J_k$:

**Proposition 3.1.** a) The identities

\[ x^2 = x, \ xy = yx \]  

(4) form a basis of the set $J_1$.

b) The identities

\[ xyzzx = xyzx, \ (xy)^2 = (yx)^2 \]  

(5) form a basis of the set $J_2$.

c) The identities

\[ xyx^2zx = xyzx, \ xyzz^2z = xyzx^2z, \ xyz^2zx = zx^2ztx, \]
\[ \quad (xy)^3 = (yx)^3 \]  

(6) form a basis of the set $J_3$.

d) The sets $J_k$ with $k \geq 4$ are not finitely based.

It should be stressed that speaking about identity bases we always mean bases in the sense of monoid identities. For instance, we have not included in the above base for $J_2$ the identity $x^3 = x^2$ (which certainly belongs to $J_2$) because in the monoid setting it can be deduced from the identity $xyzzx = xyzx$ by substituting 1 for $y$ and $z$. However, as observed in [13, p.173], the property of a monoid to be finitely based or nonfinitely based does not depend on using the semigroup or the monoid deduction rules.

The following proposition relates the identity sets $J_k$ and the monoid $\mathcal{U}

For instance, if $w = xyx$ and $k = 3$, then

$$
\begin{align*}
\varphi_w & = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & y_3 \varphi_w & = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & z_3 \varphi_w & = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{align*}
$$

for all letters $z \neq x, y$.

**Lemma 3.3.** For each word $v \in \Sigma^*$, the word $w$ is a scattered subword of $v$ if and only if $(v_3 \varphi_w)_{1 m+1} = 1$.

**Proof.** Let $v = y_1 \cdots y_n$ for some $y_1, \ldots, y_n \in \Sigma$, and let $y_s \varphi_w = \alpha_{ij}^{(s)}$. Then, using the multiplication rule (2) $n-1$ times, we arrive at the following expansion of the element $(v_3 \varphi_w)_{1 m+1}$:

$$
(v_3 \varphi_w)_{1 m+1} = \sum_{1 \leq j_1 \leq \cdots \leq j_{n-1} \leq m+1} \alpha_{1 j_1}^{(1)} \alpha_{j_1 j_2}^{(2)} \cdots \alpha_{j_{n-1} m+1}^{(n)}. \tag{8}
$$

If $w = x_1 \cdots x_m$ is a scattered subword of $v$, then there exists a sequence $1 \leq s_1 < \cdots < s_m \leq n$ such that $x_i = y_{s_i}$ for every $i = 1, \ldots, m$. Then all factors in the product

$$
\alpha_{11}^{(1)} \cdots \alpha_{11}^{(s_1-1)} \alpha_{12}^{(s_1)} \cdots \alpha_{12}^{(s_2-1)} \alpha_{22}^{(s_2)} \cdots \alpha_{22}^{(s_{m-1})} \alpha_{23}^{(s_{m-1})} \cdots \alpha_{m m}^{(s_{m-1})} \alpha_{m m}^{(s_m)} \alpha_{m m+1}^{(s_m+1)} \alpha_{m+1 m+1} \cdots \alpha_{m+1 m+1}^{(n)} \tag{9}
$$

are equal to 1, whence the product itself equals 1 and so does the sum in the right hand side of (8) of which this product is one of the summands. Thus, $(v_3 \varphi_w)_{1 m+1} = 1$.

Conversely, if $(v_3 \varphi_w)_{1 m+1} = 1$, then at least one of the summands in the right hand side of (8) is equal to 1. Let

$$
\alpha_{1 j_1}^{(1)} \alpha_{j_1 j_2}^{(2)} \cdots \alpha_{j_{n-1} m+1}^{(n)} = 1.
$$

By the definition of the mapping $\varphi_w$, we have $\alpha_{ij}^{(s)} = 0$ whenever $i \neq j, j+1$. Therefore the increasing sequence $j_0 = 1 \leq j_1 \leq \cdots \leq j_{n-1} \leq m + 1 = j_n$ contains exactly $m$ "jumps" of the form $j_r < j_{r+1} = j_r + 1$. If we denote by $s_i$, $i = 1, \ldots, m$, the position of the $i^{th}$ such jump, then the product $\alpha_{1 j_1}^{(1)} \alpha_{j_1 j_2}^{(2)} \cdots \alpha_{j_{n-1} m+1}^{(n)}$ can be written in the form (9), and since all the factors $\alpha_{i i+1}^{(s_i)}$ are equal to 1, we conclude from (7) that $y_{s_i} = x_i$ for every $i$. Thus, $w = x_1 \cdots x_m$ is indeed a scattered subword of $v = y_1 \cdots y_n$. 

\[\blacksquare\]
Now in order to prove that $I_k \subseteq J_k$, take any identity $u = v$ holding in the monoid $\mathcal{U} \mathcal{T} \mathcal{B}_{k+1}$. Then $w \varphi = v \varphi$ under an arbitrary morphism $\varphi : \Sigma^* \to \mathcal{U} \mathcal{T} \mathcal{B}_{k+1}$. In particular, $w \varphi_w = v \varphi_w$ for every word $w \in \Sigma^*$ of length $m \leq k$, whence $(u \varphi_w)_1^m+1 = (v \varphi_w)_1^m+1$. By Lemma 3.3, the latter equality means that $w$ is a scattered subword of $u$ if and only if it is a scattered subword of $v$. Therefore the identity $u = v$ belongs to set $J_k$.

The converse inclusion $J_k \subseteq I_k$ amounts to saying that for every identity $u = v$ in $J_k$ and for every morphism $\varphi : \Sigma^* \to \mathcal{U} \mathcal{T} \mathcal{B}_{k+1}$ one has

$$(u \varphi)_\ell^m = (v \varphi)_\ell^m$$

for each $\ell \leq m$. By symmetry, it suffices to verify that $(u \varphi)_\ell^m = 1$ whenever $(v \varphi)_\ell^m = 1$. Thus, consider $v = y_1 \ldots y_n$ for some $y_1, \ldots, y_n \in \Sigma$, and let $y_s \varphi = (\alpha_{ij}^{(s)})$. Then

$$(v \varphi)_\ell^m = \sum_{\ell \leq j_1 \leq \ldots \leq j_{n-1} \leq m} \alpha_{\ell j_1}^{(1)} \alpha_{j_1 j_2}^{(2)} \cdots \alpha_{j_{n-1} m}^{(n)}$$

and since the sum on the right hand side is equal to 1, one of its summands is equal to 1. Let the product $\alpha_{\ell j_1}^{(1)} \alpha_{j_1 j_2}^{(2)} \cdots \alpha_{j_{n-1} m}^{(n)}$ be such a summand. The number $p$ of "jumps" $j_r < j_{r+1}$ in the increasing sequence $j_0 = \ell \leq j_1 \leq \ldots \leq j_{n-1} \leq m = j_n$ does not exceed $m - \ell \leq (k + 1) - 1 = k$. If we denote the position of the $i^{th}$ such jump by $s_i$ and its increment $j_{r+1} - j_r$ by $d_i$, $i = 1, \ldots, p$, then the chosen product can be written as

$$\alpha_{\ell j_{s_1}}^{(1)} \cdots \alpha_{\ell j_{s_{p-1}}}^{(s_{p-1})-1} \cdots \alpha_{\ell j_{s_p}}^{(s_p)} \cdots \alpha_{j_{s_{p-1}} m_d}^{(n)} \alpha_{j_{s_{p-1}} m_d}^{(n)} \alpha_{j_{s_{p-1}} m_d}^{(n)} \cdots \alpha_{j_{s_{p-1}} m_d}^{(n)} \alpha_{j_{s_{p-1}} m_d}^{(n)} \alpha_{j_{s_{p-1}} m_d}^{(n)} \cdots \alpha_{j_{s_{p-1}} m_d}^{(n)}$$

whence we conclude that

$$\alpha_{\ell+1}^{(s_1)} \alpha_{\ell+1}^{(s_2)} \cdots \alpha_{\ell+1}^{(s_p)} = \alpha_{\ell+1}^{(s_1)} \alpha_{\ell+1}^{(s_2)} \cdots \alpha_{\ell+1}^{(s_p)} = \alpha_{\ell+1}^{(s_1)} \alpha_{\ell+1}^{(s_2)} \cdots \alpha_{\ell+1}^{(s_p)} = \alpha_{\ell+1}^{(s_1)} \alpha_{\ell+1}^{(s_2)} \cdots \alpha_{\ell+1}^{(s_p)} = \alpha_{\ell+1}^{(s_1)} \alpha_{\ell+1}^{(s_2)} \cdots \alpha_{\ell+1}^{(s_p)} = \alpha_{\ell+1}^{(s_1)} \alpha_{\ell+1}^{(s_2)} \cdots \alpha_{\ell+1}^{(s_p)} = \alpha_{\ell+1}^{(s_1)} \alpha_{\ell+1}^{(s_2)} \cdots \alpha_{\ell+1}^{(s_p)} = \alpha_{\ell+1}^{(s_1)} \alpha_{\ell+1}^{(s_2)} \cdots \alpha_{\ell+1}^{(s_p)} = \alpha_{\ell+1}^{(s_1)} \alpha_{\ell+1}^{(s_2)} \cdots \alpha_{\ell+1}^{(s_p)} = \alpha_{\ell+1}^{(s_1)} \alpha_{\ell+1}^{(s_2)} \cdots \alpha_{\ell+1}^{(s_p)}$$

Consider the scattered subword $y_{s_1} y_{s_2} \cdots y_{s_p}$ of the word $u$. Since the length of this subword is $\leq k$ and since the identity $u = v$ belongs to $J_k$, the word $u$ also has $y_{s_1} y_{s_2} \cdots y_{s_p}$ as a scattered subword. Thus, we can write $u$ as

$$u = u_0 y_{s_1} u_1 y_{s_2} u_2 \cdots u_{p-1} y_{s_p} u_p$$

for some suitable words $u_0, u_1, \ldots, u_p \in \Sigma^*$. Therefore the entry $(u \varphi)_\ell^m$ of the matrix $u \varphi$ has in its expansion the summand

$$(u_0)_{\ell}^{(s_1)} (u_1)_{\ell+1}^{(s_2)} (u_2)_{\ell+1}^{(s_2)} \cdots (u_{p-1})_{m-d_p}^{(s_p)} (u_p)_{m}^{(n)}$$
which is equal to 1 in view of (10) and of the fact that all diagonal entries of matrices in \( \mathcal{UTB}_{k+1} \) are 1. Hence \((w_0)_{lm} = 1\) as required.

From Propositions 3.1 and 3.2, we immediately obtain

**Theorem 3.4.** The monoid \( \mathcal{UTB}_n \) is nonfinitely based for all \( n \geq 5 \). The monoids \( \mathcal{UTB}_2, \mathcal{UTB}_3, \) and \( \mathcal{UTB}_4 \) are finitely based, and the identity systems (4), (5), and respectively (6) may serve as their identity bases.

**Remark 3.1.** It is fairly easy to see that for each \( n \) the variety \( \text{var} \mathcal{UTB}_n \) contains neither \( B^1_2 \) nor non-trivial groups whence none of the monoids \( \mathcal{UTB}_n \) are inherently nonfinitely based by [9, Theorem 2].

**Remark 3.2.** The approach used in our proof of Theorem 3.4 can be also applied in order to provide a partial answer to Problem 6.3 in [13], namely, to solve the finite basis problem for the monoids \( C_n \) of order preserving and decreasing transformations of a chain with \( n \) elements. (Recall that a transformation \( \alpha \) of the chain \( \langle X, \leq \rangle \) is called order preserving if \( i \leq j \) implies \( i.\alpha \leq j.\alpha \) for all \( i, j \in X \) and decreasing if \( i.\alpha \leq i \) for every \( i \in X \).) It can be shown that the monoid \( C_n \) satisfies precisely the same identities as the monoid \( \mathcal{UTB}_n \), and therefore, \( C_n \) is finitely based if and only if \( n \leq 4 \). The proof of this result will be published elsewhere.

**Remark 3.3.** Using Simon's celebrated characterization of piecewise testable languages, Straubing [10] has shown that the monoids \( \mathcal{UTB}_n \) generate the pseudovariety \( J \) of all \( J \)-trivial finite monoids. This nice result, however, does not seem to shed any extra light on the finite basis problem for the monoids \( \mathcal{UTB}_n \) (compare with Remark 2.1).

**Remark 3.4.** Straubing and Thérien [11] have observed that the monoid \( \mathcal{UTB}_n \) admits the following compatible partial order: \((\alpha_{ij}) \leq (\beta_{ij})\) if for \( i, j \) either \( \alpha_{ij} = \beta_{ij} \) or \( \alpha_{ij} = 1 \) while \( \beta_{ij} = 0 \). On the other hand, from their elegant proof of Simon's theorem [11, p.397] one can extract the following

**Proposition 3.5.** Let \( M \) be a monoid with a compatible partial order \( \leq \) under which 1 is the greatest element. If \( k+1 \) is the length of the longest chain in \( \langle M, \leq \rangle \), then \( M \) satisfies every identity in the set \( J_k \).

It is easy to calculate that the length of the longest chain in the monoid \( \mathcal{UTB}_n \) with the above ordering is \( \frac{n(n-1)}{2} + 1 \) so the application of Proposition 3.5 yields that \( \mathcal{UTB}_n \) satisfies all identities in \( J_{\frac{n(n-1)}{2}} \). Our Proposition 3.2 shows that in fact \( \mathcal{UTB}_n \) satisfies all identities in \( J_{n-1} \) and this is the best possible result.
References


