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Power series and moment summability methods of finite order

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Abstract

We present a theory of moment summability methods that is particularly suited for application to formal power series.

1 Introduction

In this article we introduce the concept of *moment summability* corresponding to kernels of finite positive order k. One such method is *J.-P. Ramis'* k-summability [10, 11], but we shall present many more, among them some in terms of *J. Ecalle's* acceleration operators [7]. Although it is made clear that all summability methods of the same order are equivalent, it is worthwhile to allow general methods, instead of restricting oneself to the better known method of k-summability: For once, showing that a given series is summable can be considerably simplified by choosing a method that is well suited for this particular series. Moreover, one can easily use this general approach to prove that certain sequences form *summability factors* for k-summability. For a more complete presentation of this theory, and in particular for the proofs of the results presented here, see [3]. Also, compare [1] for a presentation of more general moment methods, including some of infinite order that, however, do not seem to have the same good properties as the ones presented here.

2 Kernel functions

A function $e(z)$ shall be called a *kernel function of order* $k > 1/2$, provided that it has the following properties:

- We require that $e(z)$ is holomorphic in $S_{k,+} = \{z \in \mathbb{C} \setminus \{0\} : 2k\,|\arg z| < \pi\}$, and $z^{-1}\,e(z)$ is integrable at the origin. For positive real $z = x$, we assume that the values $e(x)$ are positive real. Moreover, we demand that $e(z)$ is exponentially flat of order k in $S_{k,+}$, i. e., for every $\varepsilon > 0$ there exist constants $C, K > 0$ such that

$$|e(z)| \;\le\; C\;\exp[-(|z|/K)^k], \qquad 2k|\arg z| \;\le\; \pi - \varepsilon\,. \tag{2.1}$$

- In terms of the kernel function $e(z)$, we define the *corresponding moment function* by

$$m(u) \;=\; \int_0^\infty x^{u-1}\,e(x)\,dx, \quad \mathrm{Re}\,u \ge 0\,.$$

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Note that the integral converges absolutely and locally uniformly for these u, so that $m(u)$ is holomorphic for $\mathrm{Re}\, u > 0$ and continuous up to the imaginary axis, and the values $m(x)$ are positive real numbers for $x \geq 0$. In terms of this moment function, we then define

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{m(n)} \tag{2.2}$$

and require that the function $E(z)$ is entire and and *of exponential growth at most* k, meaning that for constants $C, K > 0$, not necessarily the same as above, we have

$$|E(z)| \leq C \exp[K\,|z|^k] \qquad \forall\, z \in \mathbb{C}.$$

Finally, in $S_{k,-} = \{z \in \mathbb{C} \setminus \{0\} : 2\,|\pi - \arg z| < \pi(2 - 1/k\} = \mathbb{C} \setminus \overline{S}_{k,+}$, the function $z^{-1}\, E(1/z)$ is required to be integrable at the origin.

Obviously, (2.1) implies $k\, m(n) \leq C\, K^n\, \Gamma(n/k)$ for $n \geq 1$. On the other hand, the fact that $E(z)$ is assumed to be of exponential growth at most k implies existence of $\hat{C}, \hat{K} > 0$ so that $m(n) \geq \hat{C}\, \hat{K}^n \Gamma(1 + n/k)$ for $n \geq 0$. Hence the moments $m(n)$ are *of order* $\Gamma(1 + n/k)$ as n tends to infinity in the sense that for suitable $C_\pm$ we have

$$0 < C_- \leq \left[\frac{m(n)}{\Gamma(1 + n/k)}\right]^{1/n} \leq C_+ \qquad \forall\, n \geq 1.$$

In particular, this shows that the order of a kernel function is uniquely defined, and that the entire function $E(z)$ is exactly of exponential growth k, or in other words, is of exponential order k and finite type. We consider the following first examples of kernel functions; other interesting ones shall follow later:

1. For $k > 0$, take $e(z) = k\, z^k \exp[-z^k]$; in this case $m(u) = \Gamma(1 + u/k)$, and $E(z) = E_{1/k}(z)$ is *Mittag-Leffler's function.* Using well-known properties of this function, it is easy to see that all requirements of above are satisfied for values of $k > 1/2$, while for smaller k the condition of integrability of $z^{-1}E(1/z)$ at the origin becomes meaningless, since the sector $S_{k,-}$ is empty.

2. Slightly more generally, the function $e(z) = k\, z^{k\alpha} \exp[-z^k]$, with $k > 1/2$, $\alpha > 0$, can also be seen to be a kernel function of order k. The corresponding moment function is $m(u) = \Gamma(\alpha + u/k)$.

3. In the above examples, the kernel of order k is obtained from that one of order 1 by a change of variable $z \mapsto z^k$. This generalizes to arbitrary kernels as follows: Let $e(z)$ be *any* kernel function of order $k > 1/2$, and let $0 < \alpha < 2k$. Then $e(z; \alpha) = e(z^{1/\alpha})/\alpha$ is a kernel function of order k/α, and the corresponding entire function $E(z) = E(z; \alpha)$ is given by

$$E(z; \alpha) = \frac{1}{2\pi i} \int_\gamma E(w) \frac{w^{\alpha - 1}}{w^\alpha - z}\, dw, \tag{2.3}$$

with a path of integration as in *Hankel's formula* for the inverse Gamma function [3]. Compare (2.3) to the integral representation of Mittag-Leffler's function!

3 Integral operators

With help of any kernel function $e(z)$ of order $k > 1/2$ and the corresponding entire function $E(z)$, we now define a pair of integral operators as follows:

- Let $S = S(d, \alpha) = \{z \in \mathbb{C} \setminus \{0\} : 2\,|d - \arg z| < \alpha\}$ be a sector of infinite radius, bisecting direction d, and opening α. Moreover, let $f \in A^{(k)}(S)$, by which we mean to say that f is holomorphic and of exponential growth of order at most k in S, meaning that for every $\varepsilon > 0$ there exist constants $C, K > 0$ with $|f(z)| \leq C \exp[K|z|^k]$ for $2\,|d - \arg z| \leq (\alpha - \varepsilon)$. Then for $2\,|d - \tau| < \alpha$, the integral

$$(Tf)(z) = \int_0^{\infty(\tau)} e(u/z)\, f(u)\, \frac{du}{u} \tag{3.1}$$

converges absolutely and locally uniformly for z in a sectorial region with bisecting direction τ and opening π/k and can, by variation of τ, be continued into a sectorial region $G = G(d, \alpha + \pi/k)$ of opening $\alpha + \pi/k$ and bisecting direction $\arg z = d$. In this region, Tf is holomorphic and bounded at the origin.

- If G is as above, and f is holomorphic in G and bounded at the origin, then we define

$$(T^- f)(u) \;=\; \frac{-1}{2\pi i}\int_{\gamma_k(\tau)} E(u/z)\, f(z)\, \frac{dz}{z} \tag{3.2}$$

with $2\,|\tau - d| < \alpha$, and $\gamma_k(\tau)$ denoting the path from the origin along $\arg z = \tau + (\varepsilon + \pi)/(2k)$ to some $z_1 \in G$ of modulus r, then along the circle $|z| = r$ to the ray $\arg z = \tau - (\varepsilon + \pi)/(2k)$, and back to the origin along this ray, for $\varepsilon, r > 0$ so small that $\gamma_k(\tau)$ fits into G. In other words, the path $\gamma_k(\tau)$ is the boundary of a sector in G with bisecting direction τ, finite radius, and opening slightly larger than $\pi/(2k)$, and its orientation is negative. The dependence of the path on ε and r will be inessential and therefore is not displayed. Then one can show that $T^- f \in A^{(k)}(S(d, \alpha))$.

In the case of $e(z) = k\,z^k \exp[-z^k]$, the two integral operators coincide with Laplace resp. Borel operators. Even in general they have many properties in common with those classical operators:

1. For $f(u) = u^\lambda$ with $\operatorname{Re}\lambda > 0$, so that $f(u)$ is continuous at the origin, we have $(Tf)(z) = m(\lambda)\, z^\lambda$.
2. For $f(u) = \sum_0^\infty f_n\, u^n$ being entire and of exponential growth at most k, the function Tf is holomorphic for $|z| < \rho$, with sufficiently small $\rho > 0$, and $(Tf)(z) = \sum_0^\infty f_n\, m(n)\, z^n$, $|z| < \rho$.
3. For $w \neq 0$ and $z \neq 0$ so that $|z/w|$ is sufficiently small, it follows from the above fact that

$$\frac{w}{w - z} \;=\; \int_0^{\infty(\tau)} e(u/z)\, E(u/w)\, \frac{du}{u}\,. \tag{3.3}$$

   This formula extends to values $w \neq 0$ and $z \neq 0$ for which both sides are defined. In particular, this is so for $\arg w \neq \arg z$ modulo 2π, since then we can choose τ so that $|\tau - \arg z| < \pi/(2k)$ and $2\,|\pi - \tau + \arg w| < \pi(2 - 1/k)$, implying absolute convergence of the integral, according to the properties of kernel functions.
4. For a sectorial region $G = G(d, \alpha)$ of opening more than π/k, and f holomorphic in G and continuous at the origin, the composition $h = (T \circ T^-) f$ is defined. Interchanging the order of integration and then evaluating the inner integral with help of (3.3) implies

$$h(z) \;=\; \frac{-1}{2\pi i}\int_{\gamma_k(\tau)} \frac{f(w)}{w - z}\, dw \;=\; f(z),$$

   since $\gamma_k(\tau)$ has negative orientation. Hence we conclude that T^- *is an injective integral operator, and T is its inverse.* Note, however, that this does not yet show that either operator is bijective; for this, see Theorem 4.
5. For $\operatorname{Re}\lambda > 0$ and $f(z) = z^\lambda$, we conclude from (3.2) by a change of variable $u/z = w$, and using Cauchy's theorem to deform the path of integration:

$$(T^- f)(u) \;=\; u^\lambda \frac{1}{2\pi i}\int_\gamma E(w)\, w^{-\lambda-1}\, dw\,,$$

   with γ as in Hankel's formula. Hence $T^- f$ equals u^λ times a constant. Using that T is the inverse operator, we conclude that this constant equals $1/m(\lambda)$. In particular, this shows $m(u) \neq 0$ for $\operatorname{Re} u > 0$. Moreover, we have the following integral representation for the reciprocal moment function:

$$\frac{1}{m(u)} \;=\; \frac{1}{2\pi i}\int_\gamma E(w)\, w^{-u-1}\, dw\,.$$

   Compare this to Hankel's formula for the reciprocal Gamma function, and note that the integral also converges for u on the imaginary axis.

It is convenient to say that the operators T, T^-, as well as the moment function $m(u)$, corresponding to a kernel of order $k > 1/2$, are also of this order.

4 Kernels of small order

In the previous section we restricted ourselves to kernels and corresponding operators of order $k > 1/2$. Here we generalize these notions to smaller orders.

- A function $e(z)$ will be called *a kernel function of order* $k > 0$ if we can find a kernel function $\tilde{e}(z)$ of order $\tilde{k} > 1/2$ so that
$$e(z) = \tilde{e}(z^{k/\tilde{k}})k/\tilde{k}, \quad z \in S(0, \pi/k). \tag{4.1}$$
Note that the sector $S(0, \pi/k)$ may have opening larger than 2π, in which case $e(z)$ will have a branch point at the origin.

From Example 3 we conclude that if a kernel function of *some* order $\tilde{k} > 1/2$ exists so that (4.1) holds, then there exists one for *any* such $\tilde{k}$. In particular, if k happens to be larger than $1/2$, then we can choose $\tilde{k} = k$, hence $e(z)$ is a kernel function in the earlier sense. Moreover, to verify that $e(z)$ is a kernel function of order k, we may always assume that $\tilde{k} = p\,k$, for a sufficiently large $p \in \mathbb{N}$. This then implies the following characterization of such kernel functions: *For arbitrary $k > 0$, $e(z)$ is a kernel function of order k if, and only if, it has the following properties:*

- The function $e(z)$ is holomorphic in $S_{k,+} = S(0, \pi/k)$, and $z^{-1}\,e(z)$ is integrable at the origin. Moreover, $e(z)$ is exponentially flat of order k in $S_{k,+}$.
- For positive real $z = x$, the values $e(x)$ are positive real.
- For some $p \in \mathbb{N}$ with $p\,k > 1/2$, the function $E_p(z) = \sum_0^\infty z^n/m(n/p)$ is entire and of exponential growth not more than $p\,k$. Moreover, in the sector $S_{pk,-}$ the function $z^{-1}\,E_p(1/z)$ is integrable at the origin.

Let a kernel function $e(z)$ of order k with $0 < k \le 1/2$ be given. Then we define the corresponding integral operator T as in (3.1). The definition of T^-, however, cannot be given as in (3.2): While we can define an entire function $E(z)$ by means of (2.2), this function does not have the same properties as for $k > 1/2$. Therefore, we define the operator T^- as follows:

- Let a kernel function $e(z)$ of order $0 < k \le 1/2$ be given. Choose $\tilde{k} > 1/2$ and let $\tilde{e}(z)$ and $\tilde{E}(z)$ be as above. For a sectorial region $G = G(d, \alpha)$ of opening larger than π/k, and any f holomorphic in G and bounded at the origin, we define:
$$(T^- f)(u) \;=\; \frac{-1}{2\alpha\pi i} \int_{\gamma_k(\tau)} \tilde{E}((u/z)^{k/\tilde{k}})\, f(z)\, \frac{dz}{z}\,.$$

This definition gives good sense, since the right-hand side can be shown not to depend upon the choice of $\tilde{k}$. However, observe that the operator T^- allows infinitely many different integral representations!

5 Properties of the integral operators

In this section, we consider fixed operators T, T^- of some order $k > 0$ and state results saying that both operators "behave well" with respect to Gevrey asymptotics:

Theorem 1 *Let $f \in A^{(k)}(S)$, for $k > 0$ and a sector $S = S(d, \alpha)$, and let $g = Tf$ be given by* (3.1), *defined in a corresponding sectorial region $G = G(d, \alpha + \pi/k)$. For $s_1 \ge 0$, assume $f(z) \cong_{s_1} \sum f_n\, z^n$ in S and set $s_2 = 1/k + s_1$. Then*
$$g(z) \;\cong_{s_2}\; \sum f_n\, m(n)\, z^n \quad \text{in } G.$$

Theorem 2 *Let $G = G(d, \alpha)$ be an arbitrary sectorial region of opening $\alpha > \pi/k$, let f be holomorphic in G, and for $s_1 > 0$ assume $f(z) \cong_{s_1} \sum f_n\, z^n$ in G. Then $T^- f$ is defined and holomorphic in $S = S(d, \alpha - \pi/k)$. For $s_2 = \max\{s_1 - k^{-1}, 0\}$ we then have*
$$(T^- f)(u) \;\cong_{s_2}\; \sum z^n\, f_n/m(n) \quad \text{in } S\,,$$
observing that a Gevrey asymptotic of order $s = 0$ is equivalent to saying that the power series converges to the function.

Roughly speaking, the following two theorems say that T and T^- are inverse to one another:

Theorem 3 *Let $G(d,\alpha)$ be a sectorial region of opening $\alpha > \pi/k$. For f holomorphic in G and continuous at the origin, let*

$$g(u) \;=\; (T^- f)(u), \quad u \in S = S(d, \alpha - \pi/k)\,.$$

Then $g \in A^{(k)}(S)$, so that $(Tg)(z)$ is defined and holomorphic in a sectorial region $\tilde{G} = \tilde{G}(d,\tilde{\alpha})$ of opening $\tilde{\alpha} > \pi/k$, and

$$f(z) \;=\; (Tg)(z) \quad \forall\, z \in \tilde{G} \cap G\,.$$

Theorem 4 *For a sector $S = S(d,\alpha)$ of infinite radius and $k > 0$, let $f \in A^{(k)}(S)$ and define $g(z) = (Tf)(z)$, $z \in G = G(d, \alpha + \pi/k)$. Then we have $f(z) = (T^- g)(z)$ in S.*

For the proofs of these and later theorems the reader may refer to [3].

6 Construction of kernels

In this section, we consider two kernels $e_1(z), e_2(z)$ of orders k_1, k_2 with corresponding moment functions $m_1(u), m_2(u)$, respectively. The following two theorems are concerned with existence of kernels corresponding to the product, resp. the quotient, of the two moment functions:

Theorem 5 *For $e_1(z), e_2(z)$ as above, there is a unique kernel function $e(z)$ of order $k = (1/k_1 + 1/k_2)^{-1}$ with corresponding moment function $m(u) = m_1(u)\, m_2(u)$. The function $e(1/z)$ can be viewed as given by applying the integral operator T_1 to the function $e_2(1/u)$. This can also be written as*

$$e(z) \;=\; \int_0^{\infty(\tau)} e_1(z/w)\, e_2(w)\, \frac{dw}{w}\,,$$

for $2k_2\,|\tau| < \pi$, and $2k_1\,|\tau - \arg z| < \pi$. For the integral operator T corresponding to the kernel $e(z)$ we have $T f = T_1(T_2(f))$, provided that both sides are defined and in the double integral on the right the order of integration may be interchanged.

Note that the above integral defining $e(z)$ can be viewed as a multiplicative version of a convolution of two functions.

Theorem 6 *For $e_1(z), e_2(z)$ as above, assume $k_1 > k_2$. Then there is a unique kernel function $e(z)$ of order $k = (1/k_2 - 1/k_1)^{-1}$ with corresponding moment function $m(u) = m_2(u)/m_1(u)$. The function $e(1/z)$ can be viewed as given by applying the integral operator T_1^- to the function $e_2(1/u)$. For $k_1 > 1/2$, this can also be written as*

$$e(z) \;=\; \frac{1}{2\pi i} \int_\beta E_1(w/z)\, e_2(w)\, \frac{dw}{w}\,,$$

for $2k\,|\arg z| < \pi$, and a path of integration β as follows: From infinity to the origin along the ray $\arg w = -(\pi/(2k_2) - \varepsilon)$, and back to infinity along $\arg w = \pi/(2k_2) - \varepsilon$, with $\varepsilon > 0$ sufficiently small. For the integral operator T corresponding to the kernel $e(z)$ we have $T f = T_1^-(T_2(f))$, provided that both sides are defined and in the double integral on the right the order of integration may be interchanged.

The above theorems shall be used in Section 8 to show existence of many more useful kernel functions.

7 Moment summability

Analogous to *J.-P. Ramis'* definition of k-summability, we now define ***moment-summability of formal power series*** as follows: Let a kernel function $e(z)$ of order $k > 0$, with corresponding moment function $m(u)$ and integral operator T, be given. We say that a formal power series $\hat{f}(z) = \sum f_n\, z^n$ ***is T-summable in a direction $d \in \mathbb{R}$***, if the following holds:

(S1) The series $g(z) = \sum z^n\, f_n/m(n)$ has positive radius of convergence.

(S2) For some $\varepsilon > 0$, the function g defined above can be holomorphically continued into $S = S(d, \varepsilon)$ and is of exponential growth at most k there; or in other words, $g \in A^{(k)}(S)$.

Obviously, (S1) holds if, and only if, $\hat{f}(z)$ has Gevrey order $s = 1/k$. Condition (S2) implies applicability of the integral operator T to g, and we call $f = T\,g$ *the T-sum of* $\hat{f}$, and write $f = \mathcal{S}_{T,d}\hat{f}$. For the special case of $e(z) = k\,z^k \exp[-z^k]$ the above definition of moment summability coincides with k-summability in Ramis' sense. Due to the results presented above, it is immediately seen that all moment summability methods of fixed order $k > 0$ are equivalent to k-summability in the following sense:

Theorem 7 *Let arbitrary kernel functions $e_1(z), e_2(z)$ of the same order $k > 0$ be given. Then $\hat{f}(z)$ is T_1-summable in a direction d if, and only if, it is T_1-summable in the direction d, and $(\mathcal{S}_{T_1,d}\hat{f})(z) = (\mathcal{S}_{T_2,d}\hat{f})(z)$ for every z in a sectorial region of bisecting direction d and opening more than π/k.*

The above result shows that using general moment methods of the above type does not lead to more power series that can be summed. On the other hand, we shall show in the next section that an appropriate choice of a kernel will help in showing that a concrete series is, indeed, k-summable.

8 Applications and examples

As an application of the results presented above, we can now construct many more kernels, some of which will turn out to be equal to *J. Ecalle's acceleration operators*. Moreover, we shall make clear that in many cases one can more easily prove k-summability of a given power series by choosing an appropriate kernel of order k.

- Using the above results, one can verify existence of a kernel $e(z)$ corresponding to the moment function
$$m(u) \;=\; \frac{\Gamma(\alpha_1 + s_1 u)\cdot\ldots\cdot\Gamma(\alpha_\nu + s_\nu u)}{\Gamma(\beta_1 + \sigma_1 u)\cdot\ldots\cdot\Gamma(\beta_\mu + \sigma_\mu u)}\,,$$
with α_j, β_j, s_j, $\sigma_j > 0$ satisfying $s := \sum_{j=1}^{\nu} s_j - \sum_{j=1}^{\mu}\sigma_j > 0$, and the order of the kernel equals $k = 1/s$. One can represent $e(z)$, as well as the corresponding entire function $E(z)$, as multiple integrals involving exponential resp. Mittag-Leffler functions, but we shall not attempt doing this here. For $s_j = \sigma_k = 1$ and $\nu = \mu + 1$, the corresponding entire function $E(z)$ is closely related to the generalized confluent hypergeometric function.

- As a special case of the operators considered above, we shall now take $\nu = \mu = 1$ and $\alpha_1 = \beta_1 = 1$: To represent the corresponding kernel, let $\alpha > 1$ and define an entire function by means of the integral
$$C_\alpha(z) \;=\; \frac{1}{2\pi i}\int_\gamma u^{1/\alpha-1}\exp[u - z\,u^{1/\alpha}]\,du\,,$$
with a path of integration γ as in Hankel's integral for the inverse Gamma function. By a change of variable $zu^{1/\alpha} = w^{-1}$, and then substituting $z = t^{-1}$, we see that $t^{-1}\,C_\alpha(t^{-1})$ is the Borel transform of index α of $z^{-1}e^{-1/z}$. From Theorem 6 we conclude that $e(t) = t\,C_\alpha(t)$ is a kernel function of order $\beta = \alpha/(\alpha-1)$, corresponding to the moment function $m(u) = \Gamma(1+u)/\Gamma(1+u/\alpha)$. For $\tilde{k} > k > 0$, set $\alpha = \tilde{k}/k$. Then the function $e_{\tilde{k},k}(t) = k\,t^k\,C_\alpha(t^k)$ is a kernel function of order $\kappa = k\,\beta = (1/k - 1/\tilde{k})^{-1}$, whose moment function equals
$$m(u) \;=\; \frac{\Gamma(1+u/k)}{\Gamma(1+u/\tilde{k})}\,.$$

The corresponding integral operator, or to be precise: one of a slightly modified form, as well as its inverse, have been introduced by *J. Ecalle* [5–7] under the names of *acceleration*, resp. *deceleration operator*. These operators played a central role in Ecalle's definition of multisummability resp. in *B. Braaksma's* [4] proof of multisummability of formal solutions of meromorphic ordinary differential equations.

- Let $m(u)$ resp. T be the moment function resp. integral operator corresponding to some kernel of order k. To check that the series $\hat{f}(z) = \sum_0^\infty f_n z^n$, with $f_n = m(n)$ for every n, is k-summable in the direction d, using Ramis' definition, requires holomorphic continuation of $g(z) = \sum_0^\infty z^n m(n)/\Gamma(1+n/k)$. Using the fact that k-summability is equivalent to T-summability we can immediately say that, since $\tilde{g}(z) = \sum_0^\infty z^n f_n/m(n)$ is the geometric series, $\hat{f}(z)$ is T-summable, and hence k-summable, in every direction $d \not\equiv 0$ modulo 2π. E. g., this can be applied to the divergent generalized hypergeometric series

$$F(\alpha_1, \ldots, \alpha_\nu; \beta_1, \ldots, \beta_\mu; z) = \sum_{n=0}^\infty \frac{(\alpha_1)_n \cdot \ldots \cdot (\alpha_\nu)_n}{(\beta_1)_n \cdot \ldots \cdot (\beta_\mu)_n} \frac{z^n}{n!}$$

for $\nu - 2 \geq \mu \geq 0$, reproving a result obtained by *K. Ichinobe* [8].

- Similarly to the situation discussed above, investigations of formal solutions of partial differential equations are facilitated by the use of moment summability methods instead of k-summability. For the case of the heat equation, compare articles of *Lutz, Miyake, and Schäfke* [9] and the author's [2].

- For $k > 0$, a sequence $(\lambda_n)_{n\geq 0}$ shall be called ***a summability factor*** for k-summability, if for every series $\sum f_n z^n$ that is k-summable in a direction d, we have that $\sum \lambda_n f_n z^n$ is again k-summable in the direction d. Let $e_1(z)$, $e_2(z)$ be kernel functions of the same order $\tilde{k} > 0$, with corresponding moment functions $m_1(u)$, $m_2(u)$. Then the sequence $m_1(n)/m_2(n)$ is a summability factor for k-summability. To see this, first assume that $\tilde{k} = k$, and let $\hat{f}(z) = \sum z^n f_n$ be k-summable in a direction d. Then $\hat{f}(z)$ also is T_2-summable in the direction d; hence, the function g, defined be the convergent series $g(z) = \sum z^n f_n/m_2(n)$, is so that the operator T_1 may be applied, and $T_1 g$ is holomorphic in a sectorial region with bisecting direction d and opening more than π/k, and has the series $\hat{h}(z) = \sum z^n f_n m_1(n)/m_2(n)$ as its Gevrey asymptotic of order $s = 1/k$. This fact, however, is equivalent to k-summability of $\hat{h}(z)$ in the direction d. If $\tilde{k} > k$, observe that there exist kernel functions $\tilde{e}_1(z), \tilde{e}_1(z)$ whose moment functions are $m_1(u)\,\Gamma(1+su)$ resp. $m_2(u)\,\Gamma(1+su)$, with $s = 1/k - 1/\tilde{k}$. The new kernels both are of order k, and the quotient of the two new moment functions is the same as that of the previous ones. For the remaining case of $\tilde{k} < k$ we may proceed analogously, with new kernels whose moment functions equal $m_1(u)/\Gamma(1+su)$ resp. $m_2(u)/\Gamma(1+su)$.

- More generally, let $e_j(z)$ be kernel functions of orders $k_j > 0$ and corresponding moment functions $m_j(u)$, for $1 \leq j \leq \mu$. If some μ exists for which

$$s := \sum_{j=1}^{\nu} 1/k_j = \sum_{j=\nu+1}^{\mu} 1/k_j$$

holds, then the sequence

$$\lambda_n = \frac{m_1(n) \cdot \ldots \cdot m_\nu(n)}{m_{\nu+1}(n) \cdot \ldots \cdot m_\mu(n)}$$

is a summability factor for k-summability. To prove this, use the above result, together with the fact that there exist kernels of order $\tilde{k} = 1/s$ whose moment functions are equal to the products $m_1(n) \cdot \ldots \cdot m_\nu(n)$ resp. $m_{\nu+1}(n) \cdot \ldots \cdot m_\mu(n)$.

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