Vanishing Theorems in Hyperasymptotic Analysis
and Applications to
Inhomogeneous Linear Differential Equations

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1 Introduction

The vanishing theorem of non-commutative case in asymptotic analysis is established by Sibuya([12],[13]) in 1970's to solve the so-called R-H-B problem. That was stated in terms of vector bundles, of which the origin is in a work on matricial functions of Birkhoff. Malgrange [6] translated Sibuya's theorem in terms of sheaves of germs of functions asymptotically developable on the $S^1$, the set of directions to a point in $C$. Malgrange proved also the vanishing theorem of commutative case in asymptotic analysis and, Malgrange and Deligne showed that it was useful to study the structure of formal solutions to inhomogenous linear differential equations by using solutions asymptotic to the series 0 of the associated homogeneous linear differential equations. These are successively extended to the Gevrey asymptotic case in one variable(Ramis [11],...,[7],...), to the general case of asymptotics in several variables (Majima [1],[2],[3]), the Gevrey case in several variables(Haraoka [7]), and some generalizations for these results (Mozo [8]).

These are also extended to the case of hyperasymptotics. The first attempt was done in [5](see also [4]).

In this paper, we give vanishing theorems in hyperasymptotic analysis of level 1 and 2.
2 Vanishing Theorems in Hyperasymptotic Analysis in the Commutative Case

In the following, we work at the infinity and for a real positive number $R$, real numbers $a$ and $b$, we denote by $S(R, a, b)$ the open sector at the infinity

$$S(R, a, b) = \{ z : |z| > R, a < \arg z < b \}.$$  \hfill (1)

Let $\{ S(R, a_\ell, b_\ell) \mid \ell = 1, \cdots, L \}$ be an open sectorial covering of the annulus

$$D(R, \infty) = \{ z \mid +\infty > |z| > R \}.$$  \hfill (2)

We say that $\{ S(R, a_\ell, b_\ell) \mid \ell = 1, \cdots, L \}$ is a good covering when the following condition is satisfied:

$$a_{L+1} = a_1, \quad a_\ell < b_{\ell-1} < a_{\ell+1} < b_\ell, \quad b_\ell - a_\ell < \pi, \quad \ell = 1, \cdots, L.$$  \hfill (3)

We set, for fixed $a_\ell, b_\ell$, $\ell = 1, \cdots, L$,

$$S_{\ell-1,\ell}(R) = S(R, a_{\ell-1}, b_{\ell-1}) \cap S(R, a_\ell, b_\ell) = S(R, a_\ell, b_{\ell-1}),$$  \hfill (4)

and take

$$\tau_\ell = \frac{a_\ell + b_{\ell-1}}{2}.$$  \hfill (5)

These will be the directions of the Stokes lines in the next theorem, and these Stokes lines will be denoted by

$$\gamma_\ell = \{ te^{i\tau_\ell} \mid t \in [0, \infty) \}, \quad \gamma'_\ell = \{ tRe^{i\tau_\ell} \mid t \in [1, \infty) \}.\hfill (6)$$

We will call $\{ \lambda_k \mid k = 1, \cdots, K \}$ an acceptable set of exponentials for our covering when for each $1 \leq k \leq K$ there exists an $\ell$ such that $\arg(-\lambda_k) = -\tau_\ell$, that is, $\lambda_k z < 0$ when $\arg z = \tau_\ell$. For each $\ell$ we define

$$\mathcal{K}_\ell = \{ k \in \{1, \cdots, K\} \mid \arg(-\lambda_k) = -\tau_\ell \}.$$  \hfill (7)

We will use the notation

$$\lambda_{jk} = \lambda_j - \lambda_k, \quad \mu_{jk} = \mu_j - \mu_k.$$  \hfill (8)

**Theorem 1** Let $\{ S(R, a_\ell, b_\ell) \mid \ell = 1, \cdots, L \}$ be a good open sectorial covering of $D(R, \infty)$ and let $\{ \lambda_k \mid k = 1, \cdots, K \}$ be an acceptable set of exponentials for this covering. For $\ell = 1, \cdots, L$, let

$$U_{\ell-1,\ell}(z) = \sum_{k \in \mathcal{K}_\ell} \delta_k U_{\ell-1,\ell}^{(k)}(z)$$  \hfill (9)
be a finite sum of functions defined in $S_{\ell-1,\ell}(R)$ that are
in that sector asymptotically developable to the formal power-series

$$U_{\ell-1,\ell}^{(k)}(z) \sim e^{\lambda_k z} \sum_{s=0}^{\infty} u_{sk} z^{\mu_k-s},$$

where $\mu_k$ are complex constants. In (9) $\delta_k$ are constants that are either 1 or 0.

Then, there exist a positive number $R' (\geq R)$, a formal power-series $\tilde{V}(z) = \sum_{r=0}^{\infty} T_r z^{-r}$ and functions $V_\ell$ defined in $S_\ell(R')$, $\ell = 1, \cdots, L$, such that

(i) the relation

$$U_{\ell-1,\ell}(z) = V_\ell(z) - V_{\ell-1}(z)$$

holds for $z \in S_{\ell-1,\ell}(R')$.

(ii) $V_\ell$ is asymptotically developable to the formal power-series $\tilde{V}(z)$ in $S_\ell(R')$, and if we write

$$V_\ell(z) = \sum_{r=1}^{M-1} T_r z^{-r} + \tilde{R}_\ell^{(0)}(z, M),$$

then

$$\tilde{R}_\ell^{(0)}(z, M) = e^{-\alpha_0 z} \mathcal{O}(z^{1/2 + \varepsilon}),$$

as $|z| \to \infty$ in the sector $\tau_\ell \leq \arg z \leq \tau_{\ell+1}$, where we have taken the optimum number of terms

$M = \alpha_0 |z| + \mathcal{O}(1)$,

where

$$\alpha_0 = \min \{|\lambda_k| \mid k = 1 \cdots K, \delta_k \neq 0\},$$

$$\bar{\mu}_0 = \max \{\Re \mu_k \mid k = 1 \cdots K\}.$$

(iii) As $r \to \infty$

$$T_r \sim \frac{-1}{2\pi i} \sum_{k=1}^{K} \sum_{s=0}^{\infty} \delta_k u_{sk} \Gamma(r + \mu_k - s) (-\lambda_k)^{s - \mu_k - r},$$

Remark 1: The lines $\arg z = \tau_\ell, \tau_{\ell+1}$ are Stokes lines for the function $V_\ell(z)$.

Remark 2: The constant $\alpha_0$ defined in (15) is the distance from the origin to the nearest active $\lambda_k$ in the complex plane. By changing the values of $\delta_k$ the value of $\alpha_0$ might change.

The next theorem is the hyperasymptotic level 1 version. For this theorem we need some extra information in the asymptotic expansions of the functions $U_{\ell-1,\ell}^{(k)}(z)$. 
Theorem 2 In addition to the assumption of Theorem 1, we moreover assume that there exist constants $\tilde{\alpha}_k$ and $\nu_k$, $k = 1, \ldots, K$, such that

$$U_{k,\ell}^{(k)}(z) = e^{\lambda_k z} \sum_{s=0}^{N_k-1} u_{sk} z^{\mu_k-s} + R_k^{(0)}(z, N_k),$$

(18)

where for all $z$ 'near' $\gamma_t$ and large $N_k$ we have

$$R_k^{(0)}(z, N_k) = e^{\lambda_k z} z^{\nu_k} \sum_{s=0}^{N_k-1} u_{sk} z^{\mu_k-s} + O(1).$$

(19)

Define

$$\alpha_1 = \min \{ \tilde{\alpha}_k + |\lambda_k| : k = 1 \ldots K, \delta_k \neq 0 \},$$

(20)

$$\tilde{\mu}_1 = \max \{ \nu_k + \Re \mu_k : k = 1 \ldots K \}.$$  

(21)

Then

$$T_r \sim \frac{-1}{2\pi i} \sum_{k=1}^{K} \sum_{s=0}^{N_k-1} \delta_k u_{sk} \Gamma(r + \mu_k - s) (\lambda_k)^{r+\mu_k-r} + R^{(0)}(r, N_{p}),$$

(22)

where, when we take the optimal choice

$$N_k = \frac{\max(\alpha_1 - |\lambda_k|, 0)}{\alpha_1} r + O(1),$$

(23)

we have

$$R^{(0)}(r, N_{p}) = \frac{\Gamma(r)}{(\alpha_1)^r} O(r^{\tilde{\mu}_1+1/2}),$$

(24)

as $r \to \infty$. For the remainder in (12) we have

$$\tilde{R}_r^{(0)}(z, M) = -\frac{z^{1-M}}{2\pi i} \sum_{k=1}^{K} \sum_{s=0}^{N_k-1} \delta_k u_{sk} F^{(1)}(z; M + \mu_k - s) + \tilde{R}_r^{(1)}(z, M, N_{p}),$$

(25)

where, when we take the optimal choice

$$M = \alpha_1 |z| + O(1), \quad N_k = \max(\alpha_1 - |\lambda_k|, 0)|z| + O(1),$$

(26)

we have

$$\tilde{R}_r^{(1)}(z, M, N_{p}) = e^{-\alpha_1 |z|} O(|z|^{\tilde{\mu}_1+1}),$$

(27)

as $|z| \to \infty$ in the sector $\tau_\ell \leq \phi z \leq \tau_{\ell+1}$. 
In the definition we shall use the notation
\[
\int_{\lambda}^{\infty \eta} = \int_{\lambda}^{\infty e^{j\eta}}
\] for \( \eta \in \mathbb{R} \).

Let \( l \) be a nonnegative integer, \( \Re M_j > 1, \sigma_j \in \mathbb{C}, \sigma_j \neq 0, j = 0, \cdots, l \). Then
\[
F^{(0)}(z) = 1,
\]
\[
F^{(1)}(z; M_0, \sigma_0) = \int_{0}^{\pi} \frac{e^{\sigma_0 t_0 M_0 - 1}}{z - t_0} dt_0,
\]
\[
F^{(l+1)}(z; M_0, \cdots, M_l, \sigma_0, \cdots, \sigma_l) = \int_{0}^{\pi} \cdots \int_{0}^{\pi} \frac{e^{\sigma_0 t_0 + \cdots + \sigma_l t_l M_0 - 1 \cdots M_l - 1}}{(z - t_0)(t_0 - t_1) \cdots (t_{l-1} - t_l)} dt_l \cdots dt_0,
\]
where \( \theta_j = \arg \sigma_j, j = 0, 1, \cdots, l \). In the case \( \arg \sigma_j = \arg \sigma_{j+1} \) (mod \( 2\pi \)) we have to make the choice between the \( t_j \)-contour being on the 'left' or 'right' of the \( t_{j+1} \)-contour. We make the choice via the definition
\[
F^{(l+1)}(z; M_0, \cdots, M_l, \sigma_0, \cdots, \sigma_l) = \lim_{\epsilon \downarrow 0} F^{(l+1)}(z; \sigma_0 e^{-l\epsilon}, \sigma_1 e^{-(l-1)\epsilon}, \cdots, \sigma_{l-1} e^{-\epsilon}, \sigma_l M_1, \cdots, M_{l-1}, M_l),
\]
which means that once again we prefer 'right' over 'left'.

The multiple integrals converge when \( -\pi - \theta_0 < \arg z < \pi - \theta_0 \).

The next theorem is the hyperasymptotic level 2 version. For this theorem we need some extra information on the re-expansions of the functions \( U_{l-1,k}(z) \).

**Theorem 3** In addition to the assumption of Theorem 1, we moreover assume that there exist constants \( \tilde{\alpha}_{kj} \) and \( \nu_{kj} \), \( k, j = 1, \cdots, K, j \neq k \) such that
\[
R_k^{(0)}(z, N_k) = \sum_{j \neq k} e^{\lambda_k z - N_k + \mu_k} \frac{K_{jk}}{2\pi i} \sum_{s=0}^{\tilde{N}_{kj}} u_{skj} F^{(1)}(z; \mu_{kj} - s + \tilde{N}_{kj}, K_{jk}),
\]
where for all \( z \) 'near' \( \gamma'_{\ell} \) and large \( N_k - \tilde{N}_{kj} \) and large \( \tilde{N}_{kj} \) we have
\[
R_k^{(1)}(z, N_k, \tilde{N}_{kj}) = e^{\lambda_k z - \mu_k - N_k + 1} \frac{\Gamma(N_k - \tilde{N}_{kj} + \mu_{kj}) \Gamma(\tilde{N}_{kj} + \nu_{kj})}{|\lambda_{kj}|^{N_k - \tilde{N}_{kj} \tilde{N}_{kj}} (\tilde{\alpha}_{kj})^{\tilde{N}_{kj}}} \mathcal{O}(1).
\]

Define
\[
\alpha_2 = \min \{ \tilde{\alpha}_{kj} + |\lambda_{kj}| + |\lambda_k| \mid k, j = 1, \cdots, K, j \neq k, \delta_k \neq 0, K_{jk} \neq 0 \},
\]
\[
\tilde{\mu}_2 = \max \{ \nu_{kj} + \Re \mu_k \mid k, j = 1, \cdots, K, j \neq k \}.
\]
Then
\[ T_r = \frac{-1}{2\pi i} \sum_{k=1}^{K} \left\{ \sum_{s=0}^{N_k-1} \delta_k u_{sk} \Gamma(r + \mu_k - s) (-\lambda_k)^{s-\mu_k-r} \right\} \]
\[ + \sum_{j \neq k} \frac{K_{jk}}{2\pi i} \sum_{s=0}^{\tilde{N}_{lj}-1} u_{skj} F^{(2)} \left( 0; r + \mu_k - N_k + 2, N_k + \mu_{jk} - s \right) \]
\[ + \tilde{R}^{(1)}(r, N_{\mathbf{p}}, \tilde{N}_{\mathbf{pq}}), \]
where, when we take the optimal choice
\[ N_k = \frac{\max(\alpha_2 - |\lambda_k|, 0)}{\alpha_2} r + O(1), \quad \tilde{N}_{kj} = \frac{\max(\alpha_2 - |\lambda_k| - |\lambda_{kj}|, 0)}{\alpha_2} r + O(1), \]
we have
\[ \tilde{R}^{(1)}(r, N_{\mathbf{p}}, \tilde{N}_{\mathbf{pq}}) = \frac{\Gamma(r)}{\alpha_2} O \left( r^{\tilde{\mu}_2+1} \right), \]
as \( r \to \infty \). For the remainder in (12) we have
\[ \tilde{R}^{(0)}_{\ell}(z, M) = -\frac{z^{1-M}}{2\pi i} \sum_{k=1}^{K} \left\{ \sum_{s=0}^{N_k-1} \delta_k u_{sk} F^{(1)} \left( z; M + \mu_k - s \right) \right\} \]
\[ + \sum_{j \neq k} \frac{K_{jk}}{2\pi i} \sum_{s=0}^{\tilde{N}_{kj}-1} u_{skj} F^{(2)} \left( z; M + \mu_k - N_k + 1, N_k + \mu_{jk} - s \right) \]
\[ + \tilde{R}^{(2)}_{\ell}(z, M, N_{\mathbf{p}}, \tilde{N}_{\mathbf{pq}}), \]
where, when we take the optimal choice
\[ M = \alpha_2 |z| + O(1), \quad N_k = \max(\alpha_2 - |\lambda_k|, 0) |z| + O(1), \quad \tilde{N}_{kj} = \max(\alpha_2 - |\lambda_k| - |\lambda_{kj}|, 0) |z| + O(1), \]
we have
\[ \tilde{R}^{(2)}_{\ell}(z, M, N_{\mathbf{p}}, \tilde{N}_{\mathbf{pq}}) = e^{-\alpha_2 |z|} O \left( z^{\tilde{\mu}_3+3/2} \right), \]
as \( |z| \to \infty \) in the sector that is bounded (from the right) by the Stokes line \( \arg z = \tau_\ell \) and (on the left) by the Stokes line \( \arg z = \tau_{\ell+1} \) or one of the other Stokes lines \( \arg(\lambda_{kj}z) = 0 \), such that this sector doesn't contain any of these Stokes lines.

The proof of Theorem 1 is given in [MHO](5) except for estimate (13). We can prove these theorems by using the integral representation:
\[ V_{\ell}(z) = \sum_{j=1}^{L} \frac{1}{2\pi i} \int_{\gamma_j} \frac{U_{j-1,j}(\zeta)}{\zeta-z} d\zeta = \sum_{j=1}^{L} \sum_{k \in \mathcal{K}_j} \frac{\delta_k}{2\pi i} \int_{\gamma_j} \frac{U_{j-1,j}^{(k)}(\zeta)}{\zeta-z} d\zeta, \]
where \( z \in S(R', \tau_{\ell}, \tau_{\ell+1}) \). Hence,
\[
T_r = \sum_{j=1 \atop j \neq k}^{L} \sum_{\gamma_j} \frac{-\delta_k}{2\pi i} \int_{\gamma_j} U_{j-1,j}^{(k)}(\zeta) \zeta^{r-1} d\zeta
\]
(43)

and
\[
\tilde{R}_{\ell}^{(0)}(z, M) = \sum_{j=1 \atop j \neq k}^{L} \sum_{\gamma_j} \delta_k \frac{z^{1-M}}{2\pi i} \int_{\gamma_j} \frac{U_{j-1,j}^{(k)}(\zeta)}{\zeta - z} d\zeta,
\]
(44)

where, again, \( z \in S(R', \tau_{\ell}, \tau_{\ell+1}) \).

3 Application: inhomogeneous linear ordinary differential equations

Let
\[
Pw := \frac{d^K w}{dz^K} + f_{K-1}(z) \frac{d^{K-1} w}{dz^{K-1}} + \cdots + f_0(z) w = 0,
\]
(45)

be a linear differential equation with a singularity of rank one at infinity, and let
\[
\hat{u}_k(z) = e^{\lambda_k z^\mu_k} \sum_{s=0}^{\infty} u_{sk} z^{-s}, \quad k = 1, \ldots, K,
\]
be all the formal solutions. We assume that all \( \lambda_k \) are nonzero and \( \lambda_j \neq \lambda_k \), if \( j \neq k \). With these exponentials we can construct the covering.

The complete hyperasymptotic expansions of solutions of (45) are given in [9], and with the theory and proofs in that paper it can be checked that all assumptions of Theorems 2 and 3 are satisfied when we take \( U_{\ell-1,\ell}^{(k)}(z) \) as follows.

For the moment we fix \( k \in \{1, \ldots, K\} \) take \( \ell \) such that \( k \in K_\ell \) and let \( U_{\ell-1,\ell}^{(k)}(z) \) be the solution of (45) with asymptotic behaviour \( \hat{u}_k(z) \) as its complete asymptotic expansion in a sector that either contains \( \arg z = \tau_\ell \), or has this line as its boundary on the 'right-hand side'. In other words, \( U_{\ell-1,\ell}^{(k)}(z) \) is supposed to be the Borel-Laplace transform of \( \hat{u}_k(z) \).

Define
\[
W_k^{(+)}(z) = V_\ell(z) = \frac{1}{2\pi i} \int_{\gamma_\ell} \frac{U_{\ell-1,\ell}^{(k)}(\zeta)}{\zeta - z} d\zeta, \quad z \in S(R'' a_\ell, b_{\ell-1} + 2\pi),
\]
(47)

\[
W_k^{(-)}(z) = V_{\ell-1}(z) = \frac{1}{2\pi i} \int_{\gamma_{\ell-1}} \frac{U_{\ell-1,\ell}^{(k)}(\zeta)}{\zeta - z} d\zeta, \quad z \in S(R'' a_\ell - 2\pi, b_{\ell-1}).
\]
(48)

Compare (42). Thus we have taken all \( \delta_k \) zero except one. In (47) we integrate to the 'right' of \( z \) and in (48) we integrate to the 'left' of \( z \). Note that we have the relations
\[ W_{k}^{(+)}(z) = W_{k}^{(-)}(ze^{-2\pi i}) \quad \text{and} \quad W_{k}^{(+)}(z) - W_{k}^{(-)}(z) = U_{l-1,l}^{(k)}(z). \] (49)

Compare (11). Hence,
\[ PW_{k}^{(+)}(z) = PW_{k}^{(+)}(ze^{2\pi i}) \] (50)
Thus
\[ p_{k}(z) = PW_{k}^{(+)}(z) \] (51)
is analytic at infinity.

Let \( \hat{W}_{k}^{(+)}(z) \) be the asymptotic expansion of function \( W_{k}^{(+)}(z) \) for \( k = 1, \ldots, K \). Then, \( \hat{W}_{k}^{(+)}(z) \) is a formal solution of the inhomogeneous equations \( P\hat{W} = p_{k} \) for \( k = 1, \ldots, K \).

If we consider \( P \) as an operator on \( \mathcal{O} \), we see that \( \hat{W}_{1}^{(+)}(z), \ldots, \hat{W}_{K}^{(+)}(z) > \text{mod. O} \) form a basis of \( \text{Ker}(P; \mathcal{O}) \cong H^{1}(S^{1}, \text{Ker}(P; \mathcal{A}_{0})) \) (see, for example [4]).

Namely, for any analytic function \( p(z) \) at infinity and a formal solution of \( Pw = p \), there exist constants \( C_{k} \) and an analytic function \( h(z) \) at infinity, such that
\[ \hat{W}(z) = C_{1}W_{1}^{(+)}(z) + \cdots + C_{K}W_{K}^{(+)}(z) + h(z). \] (52)

Put
\[ \hat{W}(z) = \sum_{r=0}^{\infty} t_{r}z^{-r}. \] (53)

According to Theorem 1 we have
\[ t_{r} \sim \frac{-1}{2\pi i} \sum_{k=1}^{K} C_{k} \sum_{s=0}^{\infty} u_{sk} \Gamma(r + \mu_{k} - s) (-\lambda_{k})^{s} \mu_{k}^{-r}, \] (54)
as \( r \to \infty \), with re-expansions in Theorems 2 and 3. The constants \( C_{k} \) can be computed via this relation, or the higher level versions of this relation. For more details on the computation of the connection coefficients \( C_{k} \) see [10].

At the moment that we have these connection coefficients, for an actual solution \( w(z) \) of \( Pw = p \), we can use them in the approximation
\[ w(z) \sim \sum_{r=0}^{M-1} t_{r}z^{-r} - \frac{z^{1-M}}{2\pi i} \sum_{k=1}^{K} C_{k} \sum_{s=0}^{N_{k}-1} u_{sk} \Gamma^{(1)}(z; \frac{M + \mu_{k} - s}{\lambda_{k}}), \] (55)
or higher order level versions of this approximation.

NOTE: that in (55) only the Poincaré part depends on \( p(z) \). The re-expansions are the same for any \( p(z) \), and once we have computed these re-expansions for one function \( p(z) \), we can use them for any other function!
参考文献


