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Type II superstring field theory with cyclic L_∞ structure

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We construct a complete type II superstring field theory that includes all the NS–NS, R–NS, NS–R, and R–R sectors. As in the open and heterotic superstring cases, the R–NS, NS–R, and R–R string fields are constrained by using the picture-changing operators. In particular, we use a non-local inverse picture-changing operator for the constraint on the R–R string field, which seems to be inevitable due to the compatibility of the extra constraint with the closed string constraints. The natural symplectic form in the restricted Hilbert space gives a non-local kinetic action for the R–R sector, but it correctly provides the propagator expected from the first-quantized formulation. Extending the prescription previously obtained for the heterotic string field theory, we give a construction of general type II superstring products, which realizes a cyclic L_∞ structure, and thus provides a gauge-invariant action based on the homotopy algebraic formulation. Three typical four-string amplitudes derived from the constructed string field theory are demonstrated to agree with those in the first-quantized formulation. We also give the half-Wess–Zumino–Witten action defined in the medium Hilbert space whose left-moving sector is still restricted to the small Hilbert space.

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1. Introduction

In recent years there has been some significant progress in constructing gauge-invariant superstring field theories. First, a complete WZW-like action for an open superstring including both the Neveu–Schwarz (NS) and Ramond (R) sectors, representing space-time bosons and fermions, respectively, has been constructed [1] after several significant developments [2–13]. A crucial idea to successfully incorporate the Ramond sector is to impose an extra constraint on the Ramond string field, which can naturally be interpreted to come from the fermionic moduli integration over the super-Riemann surface. Shortly thereafter, this was extended to an alternative formulation in the small Hilbert space [14], in which the gauge symmetry is beautifully realized using a homotopy algebraic structure, the A_∞ algebra. Several interesting studies, such as on the general structure of the complete WZW formulation [15,16], on the space-time supersymmetry [17,18], and on some generalization toward a heterotic string field theory [19–21], have also been undertaken. Then, in a previous paper, the authors extended these constructions to the heterotic string field theory [22]. We first constructed a gauge-invariant action in the small Hilbert space by constructing string interactions realizing a homotopy algebraic structure of a closed string, the cyclic L_∞ algebra, and then also gave the WZW-like action through a field redefinition.

Independent of these developments, Ashoke Sen has developed a closed superstring field theory applicable to both the heterotic and type II theories [23,24]. He has provided a quantum master action in a rather abstract way by considering string off-shell amplitudes allowing a cell decomposition. In addition, instead of imposing a constraint as in the former two formulations, he introduced an extra string field, which becomes free and decouples from the physical sector, to incorporate the Ramond sector consistently. It has also been shown that it can also be extended to the open superstring field theory [25].

Thanks to these developments, we now have three independent formulations of superstring field theory: the homotopy algebraic, the WZW-like, and the Sen's formulations. Each of these formulations has advantages and disadvantages, and they seem to be complementary. So the aim of this paper is to fill the blank still remaining by constructing a complete field theory of the type II superstring based on the homotopy algebraic and WZW-like formulations to provide a solid foundation for non-perturbative studies of superstring theories.

This paper is organized as follows. In Sect. 2 we summarize how the type II superstring field theory is constructed based on the homotopy algebraic structure for the closed string, the cyclic L_∞ algebra. We impose constraints on the string fields in the R–NS, NS–R, and R–R sectors. In the R–R sector, in particular, we introduce a non-local inverse picture-changing operator, which seems to be inevitable due to the compatibility of the extra constraint with the closed string constraints. We construct the free theory and explain that it provides the correct R–R propagator even though the kinetic term is non-local. We show that it can be replaced with the local action if an extra string field is introduced following Sen's formulation. Then, it is shown that we can construct a gauge-invariant action if the string products have the cyclic L_∞ structure. Such string products are explicitly constructed in Sect. 3. The prescription is an extension of the asymmetric construction proposed in Ref. [3] for the NS–NS sector, and is obtained by repeating twice the one proposed for the heterotic string products in Ref. [22]. The operators with non-zero picture number are inserted first for the left-moving sector and then for the right-moving sector, following the procedure used for the heterotic string field theory. We confirm that the action constructed in this way actually reproduces typical four-point amplitudes in Sect. 4. We explicitly calculate three on-shell four-point amplitudes with four R–R; two NS–R, two R–R; and NS–NS, R–NS, NS–R, R–R external states, and show that they agree with those obtained in the first-quantized formulation. In Sect. 5, we attempt to map the action and gauge transformation to those based on the WZW-like formulation. Unfortunately, however, we only obtain a half-WZW-like action defined not in the large Hilbert space but in the medium Hilbert space, the tensor product of the large Hilbert space for the left-moving sector and the small Hilbert space for the right-moving sector. Section 6 is devoted to the summary and discussion. We summarize how the string field is expanded with respect to the ghost zero modes for each sector in appendix A, which is useful in considering the Batalin–Vilkovisky (BV) quantization [26].

2. Type II string field theory in the homotopy algebraic formulation

We first summarize several basics of type II string field theory in the homotopy algebraic formulation. The free field theory is given and discussed in some detail. After confirming that the action of the R–R sector provides the propagator used in the first-quantized formulation, we show that it can also be written in the local form by introducing an extra R–R string field following Sen's formulation.

The gauge-invariant interacting action can be obtained if we assume multi-string products with the cyclic L_∞ structure.

2.1. *String field and constraints*

There are four sectors, the NS–NS, R–NS, NS–R, and R–R sectors, in the first-quantized Hilbert space of the type II superstring, $\mathcal{H}$,

$$\mathcal{H} = \mathcal{H}_{\text{NS–NS}} + \mathcal{H}_{\text{R–NS}} + \mathcal{H}_{\text{NS–R}} + \mathcal{H}_{\text{R–R}}, \tag{2.1}$$

corresponding to the combinations of the Neveu–Schwarz and Ramond boundary conditions for the left- and right-moving fermionic coordinates:

$$\begin{aligned} \mathcal{H}_{\text{NS–NS}} &= \mathcal{H}^{\text{NS}} \otimes \bar{\mathcal{H}}^{\text{NS}}, \qquad \mathcal{H}_{\text{R–NS}} = \mathcal{H}^{\text{R}} \otimes \bar{\mathcal{H}}^{\text{NS}}, \\ \mathcal{H}_{\text{NS–R}} &= \mathcal{H}^{\text{NS}} \otimes \bar{\mathcal{H}}^{\text{R}}, \qquad \mathcal{H}_{\text{R–R}} = \mathcal{H}^{\text{R}} \otimes \bar{\mathcal{H}}^{\text{R}}, \end{aligned} \tag{2.2}$$

where $\mathcal{H}$ and $\bar{\mathcal{H}}$ on the right-hand sides are the left-moving and right-moving small Hilbert spaces, respectively. Accordingly, the type II string field Φ has four components,

$$\Phi = \Phi_{\text{NS–NS}} + \Phi_{\text{R–NS}} + \Phi_{\text{NS–R}} + \Phi_{\text{R–R}} \in \mathcal{H}, \tag{2.3}$$

which is Grassmann even and has ghost number 2. The first and the last components, $\Phi_{\text{NS–NS}}$ and $\Phi_{\text{R–R}}$, have picture number $(-1,-1)$ and $(-1/2,-1/2)$, respectively, and represent space-time bosons. The second and the third components, $\Phi_{\text{R–NS}}$ and $\Phi_{\text{NS–R}}$, have picture number $(-1/2,-1)$ and $(-1,-1/2)$, respectively, and represent space-time fermions. All of these components satisfy the closed string constraints,

$$b_0^- \Phi = L_0^- \Phi = 0, \tag{2.4}$$

where $b_0^\pm = b_0 \pm \bar{b}_0$, $L_0^\pm = L_0 \pm \bar{L}_0$, and $c_0^\pm = (c_0 \pm \bar{c}_0)/2$. The first constraint imposes that the string field does not depend on the ghost-zero mode c_0^-. Therefore, the NS–NS component, in which only the bc ghosts have zero modes, is expanded with respect to the ghost zero mode as

$$\Phi_{\text{NS–NS}} = \phi_{\text{NS–NS}} - c_0^+ \psi_{\text{NS–NS}}. \tag{2.5}$$

As in the open and heterotic superstring field theories [1,14,22], we further restrict the dependence of the other components on the $\beta\gamma$ ghost zero modes. For the $\Phi_{\text{R–NS}}$ and $\Phi_{\text{NS–R}}$ components, we impose

$$XY\Phi_{\text{R–NS}} = \Phi_{\text{R–NS}}, \qquad \bar{X}\bar{Y}\Phi_{\text{NS–R}} = \Phi_{\text{NS–R}}, \tag{2.6}$$

respectively, where XY and $\bar{X}\bar{Y}$ are the projection operators defined by using the picture-changing operators and their inverses,

$$X = -\delta(\beta_0)G_0 + \delta'(\beta_0)b_0, \qquad Y = -2c_0^+\delta'(\gamma_0), \tag{2.7a}$$

$$\bar{X} = -\delta(\bar{\beta}_0)\bar{G}_0 + \delta'(\bar{\beta}_0)\bar{b}_0, \qquad \bar{Y} = -2c_0^+\delta'(\bar{\gamma}_0), \tag{2.7b}$$

which satisfy the relations

$$XYX = X, \qquad YXY = Y, \qquad [Q, X] = 0, \tag{2.8a}$$

$$\bar{X}\bar{Y}\bar{X} = \bar{X}, \qquad \bar{Y}\bar{X}\bar{Y} = \bar{Y}, \qquad [Q, \bar{X}] = 0. \tag{2.8b}$$

Here, G_0 and $\bar{G}_0$ are the zero modes of the left- and right-moving (total) superconformal currents, respectively. Note that the inverse picture-changing operators in Eq. (2.7) are defined so that the additional constraints in Eq. (2.6) are compatible with the closed string constraints of Eq. (2.4). Since the picture-changing operators X and $\bar{X}$ are Becchi–Rouet–Stora–Tyutin (BRST) invariant, they can be written as the BRST exact form in the large Hilbert space:

$$X = [Q, \Xi], \qquad \bar{X} = [Q, \bar{\Xi}], \tag{2.9}$$

with[1]

$$\Xi = \xi + (\Theta(\beta_0)\eta\xi - \xi)P_{-3/2} + (\xi\eta\Theta(\beta_0) - \xi)P_{-1/2}, \tag{2.10a}$$

$$\bar{\Xi} = \bar{\xi} + (\Theta(\bar{\beta}_0)\bar{\eta}\bar{\xi} - \bar{\xi})\bar{P}_{-3/2} + (\bar{\xi}\bar{\eta}\Theta(\bar{\beta}_0) - \bar{\xi})\bar{P}_{-1/2}. \tag{2.10b}$$

The ghost zero-mode dependence of the components $\Phi_{\text{R–NS}}$ and $\Phi_{\text{NS–R}}$ is restricted to the form

$$\Phi_{\text{R–NS}} = \phi_{\text{R–NS}} - \frac{1}{2}(\gamma_0 + 2c_0^+ G)\psi_{\text{R–NS}}, \tag{2.11a}$$

$$\Phi_{\text{NS–R}} = \phi_{\text{NS–R}} - \frac{1}{2}(\bar{\gamma}_0 + 2c_0^+ \bar{G})\psi_{\text{NS–R}}, \tag{2.11b}$$

where $G = G_0 + 2\gamma_0 b_0$ and $\bar{G} = \bar{G}_0 + 2\bar{\gamma}_0\bar{b}_0$ are the ghost zero-mode independent part of G_0 and $\bar{G}_0$, respectively.

On the other hand, for the $\Phi_{\text{R–R}}$ component which depends on both the left- and right-moving $\beta\gamma$ zero modes, we cannot simultaneously impose two conditions,

$$XY\Phi_{\text{R–R}} = \Phi_{\text{R–R}}, \qquad \bar{X}\bar{Y}\Phi_{\text{R–R}} = \Phi_{\text{R–R}},$$

due to their non-commutativity: $[XY, \bar{X}\bar{Y}] \neq 0$. However, we should notice that the choices of inverse picture-changing operators are not unique. There is a possibility to use the non-local operators,

$$\mathcal{Y} = -2\frac{G}{L_0^+}\delta(\gamma_0), \qquad \bar{\mathcal{Y}} = -2\frac{\bar{G}}{L_0^+}\delta(\bar{\gamma}_0), \tag{2.12}$$

which also satisfy

$$X\mathcal{Y}X = X, \qquad \mathcal{Y}X\mathcal{Y} = \mathcal{Y}, \tag{2.13a}$$

$$\bar{X}\bar{\mathcal{Y}}\bar{X} = \bar{X}, \qquad \bar{\mathcal{Y}}\bar{X}\bar{\mathcal{Y}} = \bar{\mathcal{Y}}, \tag{2.13b}$$

as the inverse picture-changing operators [11]. We can impose the conditions

$$X\mathcal{Y}\Phi_{\text{R–R}} = \bar{X}\bar{\mathcal{Y}}\Phi_{\text{R–R}} = \Phi_{\text{R–R}}, \tag{2.14}$$

[1] In this paper, for notational simplicity we denote the zero modes of $\eta(z)$ and $\xi(z)$ as η and ξ, and those of $\bar{\eta}(\bar{z})$ and $\bar{\xi}(\bar{z})$ as $\bar{\eta}$ and $\bar{\xi}$, respectively.

which are now compatible with each other, and also with the closed string constraints of Eq. (2.4). It can be shown that the ghost zero-mode dependence of $\Phi_{\mathrm{R-R}}$ is restricted by the constraints in Eq. (2.14) as

$$\Phi_{\mathrm{R-R}} = \phi_{\mathrm{R-R}} - \frac{1}{2}(\gamma_0 \bar{G} - \bar{\gamma}_0 G + 2c_0^+ G\bar{G})\psi_{\mathrm{R-R}}. \tag{2.15}$$

Here we define $\psi_{\mathrm{R-R}}$ so that the expansion in Eq. (2.15) has a local form, which will be found to be natural shortly.

Here, if we define the operators $\mathcal{G}$ and $\mathcal{G}^{-1}$ by

$$\mathcal{G} = \mathbb{I}\pi_1^{(0,0)} + X\pi_1^{(1,0)} + \bar{X}\pi_1^{(0,1)} + X\bar{X}\pi_1^{(1,1)}, \tag{2.16a}$$

$$\mathcal{G}^{-1} = \mathbb{I}\pi_1^{(0,0)} + Y\pi_1^{(1,0)} + \bar{Y}\pi_1^{(0,1)} + \mathcal{Y}\bar{\mathcal{Y}}\pi_1^{(1,1)}, \tag{2.16b}$$

the relations in Eqs. (2.8) and (2.13) can collectively be written as

$$\mathcal{G}\mathcal{G}^{-1}\mathcal{G} = \mathcal{G}, \qquad \mathcal{G}^{-1}\mathcal{G}\mathcal{G}^{-1} = \mathcal{G}^{-1}, \qquad [Q, \mathcal{G}] = 0, \tag{2.17}$$

where $\pi_1^{(0,0)}$, $\pi_1^{(1,0)}$, $\pi_1^{(0,1)}$, and $\pi_1^{(1,1)}$ are the projection operators onto $\mathcal{H}_{\mathrm{NS-NS}}$, $\mathcal{H}_{\mathrm{R-NS}}$, $\mathcal{H}_{\mathrm{R-R}}$, and $\mathcal{H}_{\mathrm{R-R}}$ components of the Hilbert space $\mathcal{H}$, respectively. It is also useful to define

$$\mathcal{X} = \pi_1^{(0,*)}\mathbb{I} + \pi_1^{(1,*)}X, \qquad \bar{\mathcal{X}} = \pi_1^{(*,0)}\mathbb{I} + \pi_1^{(*,1)}\bar{X}, \tag{2.18}$$

with $\pi_1^{(r,*)} = \pi_1^{(r,0)} + \pi_1^{(r,1)}$ and $\pi_1^{(*,r)} = \pi_1^{(0,r)} + \pi_1^{(1,r)}$ $(r = 0, 1)$; then we can write $\mathcal{G} = \mathcal{X}\bar{\mathcal{X}}$. Note that $\mathcal{G}^{-1}$ is the inverse of $\mathcal{G}$ in this sense. Then we can define the projection operator $\mathcal{P}_{\mathcal{G}} = \mathcal{G}\mathcal{G}^{-1}$ and collectively write the extra constraints of Eqs. (2.6) and (2.14) as

$$\mathcal{P}_{\mathcal{G}}\Phi = \Phi. \tag{2.19}$$

We call the Hilbert space restricted by the constraints in Eqs. (2.4) and (2.19) the restricted Hilbert space, or frequently simply restricted space, in this paper. On the restricted Hilbert space, the BRST operator acts consistently:

$$\mathcal{P}_{\mathcal{G}} Q \mathcal{P}_{\mathcal{G}} = Q\mathcal{P}_{\mathcal{G}}. \tag{2.20}$$

A natural symplectic form in the restricted Hilbert space is defined as follows. First, the symplectic form in the space restricted by the closed string constraints of Eq. (2.4) is defined by using the Belavin–Polyakov–Zamolodchikov (BPZ) inner product as

$$\omega_s(\Phi_1, \Phi_2) = (-1)^{|\Phi_1|}\langle\Phi_1|c_0^-|\Phi_2\rangle, \tag{2.21}$$

where $\langle\Phi|$ is the BPZ conjugate of $|\Phi\rangle$. The symbol $|\Phi|$ denotes the Grassmann property of the string field Φ: $|\Phi| = 0$ or 1 if Φ is Grassmann even or odd, respectively. For later use, we also define the symplectic forms ω_m, $\omega_{\bar{m}}$, and ω_l in the Hilbert spaces

$$\mathcal{H}_m = \mathcal{H}_{\mathrm{large}} \otimes \bar{\mathcal{H}}, \qquad \mathcal{H}_{\bar{m}} = \mathcal{H} \otimes \bar{\mathcal{H}}_{\mathrm{large}}, \qquad \mathcal{H}_l = \mathcal{H}_{\mathrm{large}} \otimes \bar{\mathcal{H}}_{\mathrm{large}} \tag{2.22}$$

by

$$\omega_i(\varphi_1, \varphi_2) = (-1)^{|\varphi_1|}{}_i\langle\varphi_1|c_0^-|\varphi_2\rangle_i, \qquad i = m, \bar{m}, l, \tag{2.23}$$

where ${}_i\langle\varphi_1|$ and ${}_i\langle\varphi_2|$ are the BPZ conjugates of $|\varphi_1\rangle_i$ and $|\varphi_2\rangle_i$ in $\mathcal{H}_i$ $(i = m, \bar{m}, l)$, respectively. The symplectic form ω_s, ω_m, and $\omega_{\bar{m}}$ are related to ω_l as

$$\omega_s(\Phi_1, \Phi_2) = \omega_l(\xi\bar{\xi}\Phi_1, \Phi_2), \tag{2.24a}$$

$$\omega_m(\Phi_1, \Phi_2) = -\omega_l(\bar{\xi}\Phi_1, \Phi_2), \tag{2.24b}$$

$$\omega_{\bar{m}}(\Phi_1, \Phi_2) = \omega_l(\xi\Phi_1, \Phi_2), \tag{2.24c}$$

for $\Phi_1, \Phi_2 \in \mathcal{H}_i$ $(i = s, m, \bar{m})$. Then, a natural symplectic form in the restricted Hilbert space is defined by

$$\begin{aligned}\Omega(\Phi_1, \Phi_2) &= \omega_s(\Phi_1, \mathcal{G}^{-1}\Phi_2)\\ &= \omega_s(\Phi_{1\,\text{NS–NS}}, \Phi_{2\,\text{NS–NS}}) + \omega_s(\Phi_{1\,\text{R–NS}}, Y\Phi_{2\,\text{R–NS}})\\ &\quad + \omega_s(\Phi_{1\,\text{NS–R}}, \bar{Y}\Phi_{2\,\text{NS–R}}) + \omega_s(\Phi_{1\,\text{R–R}}, \mathcal{Y}\bar{\mathcal{Y}}\Phi_{2\,\text{R–R}}).\end{aligned} \tag{2.25}$$

This has the non-degenerate cross-diagonal form common in each sector

$$\Omega(\Phi_1, \Phi_2) = \sum_I \Big(\langle\langle\phi_{1I}|\psi_{2I}\rangle\rangle + \langle\langle\psi_{1I}|\phi_{2I}\rangle\rangle\Big), \tag{2.26}$$

after integrating out (or carrying out the inner product of) the ghost zero modes,[2] where the index I runs over each component, NS–NS, R–NS, NS–R, and R–R. The fields ϕ_I and ψ_I are subcomponents of each component Φ_I expanded with respect to the ghost zero modes as in Eqs. (2.5), (2.11), and (2.15). It should be noted that the $\psi_{\text{R–R}}$ in Eq. (2.15) is defined so that the non-locality of $\mathcal{Y}$ and $\bar{\mathcal{Y}}$ in the R–R sector disappears in the symplectic form in Eq. (2.26). In the following, we will see that this cross-diagonal form of the symplectic form Ω in the restricted space provides a free field theory which can properly be quantized via the BV formalism.

2.2. *Free field theory*

Using the symplectic form Ω in the restricted Hilbert space, the free field action and gauge transformation of the type II superstring field theory are given by

$$S_0 = \frac{1}{2}\Omega(\Phi, Q\Phi), \qquad \delta\Phi = Q\Lambda, \tag{2.27}$$

where the gauge parameter also has four components,

$$\Lambda = \Lambda_{\text{NS–NS}} + \Lambda_{\text{R–NS}} + \Lambda_{\text{NS–R}} + \Lambda_{\text{R–R}}, \tag{2.28}$$

is Grassmann odd, and has ghost number 1. The equation of motion

$$0 = \mathcal{G}^{-1}Q\Phi, \tag{2.29}$$

derived from Eq. (2.27) can be written by using Eq. (2.20) as

$$0 = \mathcal{G}^{-1}Q\mathcal{P}_{\mathcal{G}}\Phi = \mathcal{G}^{-1}\mathcal{P}_{\mathcal{G}}Q\mathcal{P}_{\mathcal{G}}\Phi. \tag{2.30}$$

[2] The expansion with respect to the ghost zero mode is summarized in Appendix A.

Then, by multiplying by $\mathcal{G}$, we have

$$0 = \mathcal{P}_{\mathcal{G}} Q \mathcal{P}_{\mathcal{G}} \Phi = Q \mathcal{P}_{\mathcal{G}} \Phi = Q\Phi, \tag{2.31}$$

thanks to Eq. (2.17).

The action in Eq. (2.27) has superficially the same form as that of the bosonic string field theory, and its BV quantization can be carried out in a similar way [27]. The master action $\boldsymbol{S}_0$ can simply be given by removing the ghost number restriction on Φ in the classical action:

$$\boldsymbol{S}_0 = \frac{1}{2}\,\Omega(\boldsymbol{\Phi}, Q\boldsymbol{\Phi}), \tag{2.32}$$

where $\boldsymbol{\Phi} = \sum_{g=-\infty}^{\infty} \Phi^{(g)}$. Each $\Phi^{(g)}$ is the string field with the ghost number g. The component $\Phi^{(2)}$ is equal to the classical string field Φ, and the others are the space-time ghosts, anti-ghosts, and corresponding anti-fields. The BRST transformation, which keeps the master action in Eq. (2.32) invariant, is obtained by replacing the parameter Λ in the gauge transformation of Eq. (2.27) by the field $\boldsymbol{\Phi}$ as

$$\delta_{B0}\boldsymbol{\Phi} = Q\boldsymbol{\Phi}. \tag{2.33}$$

It is easy to show that the master action in Eq. (2.32) actually satisfies the BV master equation. Using the fact that the symplectic form Ω has the cross-diagonal form of Eq. (2.26), an arbitrary variation of the master action can be written as

$$\begin{aligned}\delta\boldsymbol{S}_0 &= \Omega(\delta\boldsymbol{\Phi}, \delta_{B0}\boldsymbol{\Phi}) \\ &= \sum_I \Big(\langle\!\langle \delta\boldsymbol{\phi}_I | \delta_{B0}\boldsymbol{\psi}_I \rangle\!\rangle + \langle\!\langle \delta\boldsymbol{\psi}_I | \delta_{B0}\boldsymbol{\phi}_I \rangle\!\rangle \Big),\end{aligned} \tag{2.34}$$

and thus we have

$$\frac{\partial \boldsymbol{S}_0}{\partial \boldsymbol{\phi}_I} = \delta_{B0}\boldsymbol{\psi}_I, \qquad \frac{\partial \boldsymbol{S}_0}{\partial \boldsymbol{\psi}_I} = \delta_{B0}\boldsymbol{\phi}_I. \tag{2.35}$$

The BRST invariance of the action implies that the classical BV master equation holds:

$$\begin{aligned}0 &= \sum_I \left(\frac{\partial \boldsymbol{S}_0}{\partial \boldsymbol{\phi}_I}\delta_{B0}\boldsymbol{\phi}_I + \frac{\partial \boldsymbol{S}_0}{\partial \boldsymbol{\psi}_I}\delta_{B0}\boldsymbol{\psi}_I \right) \\ &= 2\sum_I \left(\frac{\partial \boldsymbol{S}_0}{\partial \boldsymbol{\phi}_I}\frac{\partial \boldsymbol{S}_0}{\partial \boldsymbol{\psi}_I} \right).\end{aligned} \tag{2.36}$$

The components $\boldsymbol{\phi}_I$ and $\boldsymbol{\psi}_I$ are identified with the fields and anti-fields in the gauge-fixed basis in the BV formulation [28], respectively.[3] The gauge-fixed action in the Siegel gauge is obtained by setting $\boldsymbol{\psi}_I = 0$.

[3] Here, since the field $\boldsymbol{\phi}_I$ is Grassmann even, the anti-field $\boldsymbol{\psi}_I$ must be Grassmann odd. We can show that this is actually true for the Gliozzi–Scherk–Olive (GSO) projected string field, which we implicitly assume [28].

2.3. R–R action

Before incorporating the interactions, let us examine the action of the R–R sector,

$$S_0^{\mathrm{R-R}} = \frac{1}{2}\omega_s(\Phi_{\mathrm{R-R}}, \mathcal{Y}\bar{\mathcal{Y}}Q\Phi_{\mathrm{R-R}}), \tag{2.37}$$

in more detail since it is characteristic of type II superstring field theory. For simplicity we take the Siegel gauge $|\boldsymbol{\psi}_{\mathrm{R-R}}\rangle\rangle = 0$ in this discussion. After integrating out the ghost zero modes, the master action of Eq. (2.32) and the BRST transformation of Eq. (2.33) in the R–R sector become

$$\boldsymbol{S}_0^{\mathrm{R-R}} = \langle\langle\boldsymbol{\phi}_{\mathrm{R-R}}|\frac{2G\bar{G}}{L_0^+}|\boldsymbol{\phi}_{\mathrm{R-R}}\rangle\rangle, \qquad \delta_{B0}|\boldsymbol{\phi}_{\mathrm{R-R}}\rangle\rangle = \tilde{Q}|\boldsymbol{\phi}_{\mathrm{R-R}}\rangle\rangle. \tag{2.38}$$

Although this action is non-local, the propagator

$$-\frac{b_0^+b_0^-\delta(\bar{\beta}_0)\delta(\beta_0)G\bar{G}\delta(L_0^-)}{L_0^+} \tag{2.39}$$

agrees with that obtained in the first-quantized formulation [29].

If one wants to avoid the non-local action, one can replace it with the Sen-like action as an alternative by introducing an extra Grassmann even string field $\tilde{\Phi}_{\mathrm{R-R}}$, which is restricted by the closed string constraints in Eq. (2.4) and has ghost number 2 and picture number $-3/2$. Then the alternative action is given by

$$\begin{aligned}\tilde{S}_0^{\mathrm{R-R}} &= -\frac{1}{2}\omega_s(\tilde{\Phi}_{\mathrm{R-R}}, X\bar{X}Q\tilde{\Phi}_{\mathrm{R-R}}) + \omega_s(\tilde{\Phi}_{\mathrm{R-R}}, Q\Phi_{\mathrm{R-R}})\\ &= -\frac{1}{2}\Omega(X\bar{X}\tilde{\Phi}_{\mathrm{R-R}}, QX\bar{X}\tilde{\Phi}_{\mathrm{R-R}}) + \Omega(X\bar{X}\tilde{\Phi}_{\mathrm{R-R}}, Q\Phi_{\mathrm{R-R}}).\end{aligned} \tag{2.40}$$

The difference from Sen's original action, however, is that we can rewrite it using the method of completing the square as

$$\tilde{S}_0^{\mathrm{R-R}} = -\frac{1}{2}\Omega(X\bar{X}\tilde{\Phi}'_{\mathrm{R-R}}, QX\bar{X}\tilde{\Phi}'_{\mathrm{R-R}}) + \frac{1}{2}\Omega(\Phi_{\mathrm{R-R}}, Q\Phi_{\mathrm{R-R}}), \tag{2.41}$$

with $\tilde{\Phi}'_{\mathrm{R-R}} = \tilde{\Phi}_{\mathrm{R-R}} - \mathcal{Y}\bar{\mathcal{Y}}\Phi_{\mathrm{R-R}}$, thanks to the constraint in Eq. (2.14), where the equivalence is obvious. Since $\tilde{\Phi}_{\mathrm{R-R}}$ appears only in the form of $X\bar{X}\tilde{\Phi}_{\mathrm{R-R}}$ in the action of Eq. (2.40), we can restrict $\tilde{\Phi}_{\mathrm{R-R}}$ by the condition

$$\mathcal{Y}X\tilde{\Phi}_{\mathrm{R-R}} = \bar{\mathcal{Y}}\bar{X}\tilde{\Phi}_{\mathrm{R-R}} = \tilde{\Phi}_{\mathrm{R-R}}, \tag{2.42}$$

which is dual to the constraint in Eq. (2.14) on $\Phi_{\mathrm{R-R}}$ and restricts $\tilde{\Phi}_{\mathrm{R-R}}$ to the form of

$$\tilde{\Phi}_{\mathrm{R-R}} = \tilde{\phi}_{\mathrm{R-R}} - c_0^+\tilde{\psi}_{\mathrm{R-R}}. \tag{2.43}$$

The Sen-like master action in the generalized Siegel gauge $\boldsymbol{\psi}_{\mathrm{R-R}} = \tilde{\boldsymbol{\psi}}_{\mathrm{R-R}} = 0$ then becomes

$$\begin{aligned}\tilde{\boldsymbol{S}}_0^{\mathrm{R-R}} &= \frac{1}{2}\langle\langle\tilde{\boldsymbol{\phi}}_{\mathrm{R-R}}|L_0^+G\bar{G}|\tilde{\boldsymbol{\phi}}_{\mathrm{R-R}}\rangle\rangle - \langle\langle\tilde{\boldsymbol{\phi}}_{\mathrm{R-R}}|L_0^+|\boldsymbol{\phi}_{\mathrm{R-R}}\rangle\rangle\\ &= \frac{1}{2}\langle\langle\tilde{\boldsymbol{\phi}}'_{\mathrm{R-R}}|L_0^+G\bar{G}|\tilde{\boldsymbol{\phi}}'_{\mathrm{R-R}}\rangle\rangle + \frac{1}{2}\langle\langle\boldsymbol{\phi}_{\mathrm{R-R}}|\frac{4G\bar{G}}{L_0^+}|\boldsymbol{\phi}_{\mathrm{R-R}}\rangle\rangle,\end{aligned} \tag{2.44}$$

with $|\tilde{\boldsymbol{\phi}}'_{\mathrm{R-R}}\rangle\rangle = |\tilde{\boldsymbol{\phi}}_{\mathrm{R-R}}\rangle\rangle + 4\frac{G\bar{G}}{(L_0^+)^2}|\boldsymbol{\phi}_{\mathrm{R-R}}\rangle\rangle$ after integrating out the ghost zero modes. Although the extra sector is a higher-derivative theory, it stays free and is decoupled from the physical sector if the interaction part of the action does not include $\tilde{\Phi}_{\mathrm{R-R}}$.

2.4. Including interactions

The interactions of type II superstring field theory are described by the multi-string products,

$$L_n(\Phi_1, \dots, \Phi_n) \qquad (n \geq 1), \tag{2.45}$$

which make a string field from n string fields $\Phi_1, \dots, \Phi_n$. They are graded symmetric under interchange of the n string fields, and must carry proper ghost number and picture number. In addition, since the type II superstring field in this formulation is in the restricted small Hilbert space, the outputs of the string products must also satisfy the constraint in Eq. (2.19):

$$\mathcal{P}_{\mathcal{G}} L_n(\Phi_1, \dots, \Phi_n) = L_n(\Phi_1, \dots, \Phi_n). \tag{2.46}$$

By using these string products, the action of type II superstring field theory is given by

$$S = \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \Omega(\Phi, L_{n+1}(\underbrace{\Phi, \dots, \Phi}_{n+1})), \tag{2.47}$$

where L_1 is identified with the BRST operator: $L_1 = Q$. The action in Eq. (2.47) is invariant under the gauge transformation

$$\delta\Phi = \sum_{n=0}^{\infty} \frac{1}{n!} L_{n+1}(\underbrace{\Phi, \dots, \Phi}_{n}, \Lambda) \tag{2.48}$$

if the string products satisfy the L_∞ relations

$$\sum_{\sigma} \sum_{m=1}^{n} (-1)^{\epsilon(\sigma)} \frac{1}{m!(n-m)!} L_{n-m+1}(L_m(\Phi_{\sigma(1)}, \dots, \Phi_{\sigma(m)}), \Phi_{\sigma(m+1)}, \dots, \Phi_{\sigma(n)}) = 0 \tag{2.49}$$

and cyclicity with respect to the symplectic form Ω:

$$\Omega(\Phi_1, L_n(\Phi_2, \dots, \Phi_{n+1})) = -(-1)^{|\Phi_1|} \Omega(L_n(\Phi_1, \dots, \Phi_n), \Phi_{n+1}). \tag{2.50}$$

Here, the symbol σ denotes the permutation from $\{1, \dots, n\}$ to $\{\sigma(1), \dots, \sigma(n)\}$, and $\epsilon(\sigma)$ is the sign factor of the permutation of the string fields from $\{\Phi_1, \dots, \Phi_n\}$ to $\{\Phi_{\sigma(1)}, \dots, \Phi_{\sigma(n)}\}$. If the set of string products $\{L_n\}$ satisfies these conditions, it is called the string products with the cyclic L_∞ structure or simply the cyclic L_∞ algebra. The problem is how to construct such an L_∞ algebra.

3. Construction of string products with L_∞ structure

Let us construct a set of string products realizing a cyclic L_∞ algebra. We use a coalgebraic representation handling an infinite number of string products in the L_∞ algebra collectively. We follow the notation and convention in Ref. [22].

3.1. Cyclicity, Ramond numbers and picture number deficit

String products describing the interaction of type II superstrings must have a proper ghost number and picture number. Since the ghost number structure is the same as that of the bosonic closed string field theory, here it is enough to consider the picture number that the string products should have. Denote a coderivation corresponding to an $(n+2)$-string product ($n \geq 0$) with picture number (p, p') as $\boldsymbol{B}_{n+2}^{(p,p')}$. In order to describe the type II superstring interaction, the output string state must have the same picture number as that of the type II superstring field: the picture number of its NS–NS, R–NS, NS–R, and R–R components must be equal to $(-1,-1)$, $(-1/2,-1)$, $(-1,-1/2)$, and $(-1/2,-1,2)$, respectively. The string products are also characterized by their Ramond and cyclic Ramond number defined by

$$\begin{pmatrix} \text{Ramond} \\ \text{cyclic Ramond} \end{pmatrix} \text{number} = \#\text{ of Ramond inputs} \mp \#\text{ of Ramond outputs}, \tag{3.1}$$

which are also assigned for each of the left- and right-moving sectors. Since we can consider each sector separately, let us first consider the left-moving sector. Suppose that $2r$ of $n+2$ inputs are the R states in the left-moving sector. If we assume conservation of the space-time fermion number then the output must be the NS state, and thus

$$\left(-\frac{1}{2}\right) \times 2r + (-1) \times (n+2-2r) + p = -1 \tag{3.2a}$$

from the picture number conservation. Such a string product is characterized by the cyclic Ramond number $2r$ and the Ramond number $2r$. If $2r+1$ of the inputs are the R states, the output is the R state and

$$\left(-\frac{1}{2}\right) \times (2r+1) + (-1) \times (n+1-2r) + p = -\frac{1}{2}, \tag{3.2b}$$

which is the case characterized by the cyclic Ramond number $2r+2$ and the Ramond number $2r$. Both of these equations in Eq. (3.2) can be solved as $n = p + r - 1$. After repeating the same consideration for the right-moving sector, we can find that a candidate coderivation describing the type II superstring interaction is the one respecting the Ramond number:[4]

$$\begin{aligned}
\sum_{p,r,p',r'=0}^{\infty} \delta_{p+r,p'+r'} \Big(& \pi_1^{(0,0)} \boldsymbol{B}_{p+r+1}^{(p,p')}\big|_{(2r,2r')}^{(2r,2r')} + \pi_1^{(1,0)} \boldsymbol{B}_{p+r+1}^{(p,p')}\big|_{(2r,2r')}^{(2r+2,2r')} \\
& + \pi_1^{(0,1)} \boldsymbol{B}_{p+r+1}^{(p,p')}\big|_{(2r,2r')}^{(2r,2r'+2)} + \pi_1^{(1,1)} \boldsymbol{B}_{p+r+1}^{(p,p')}\big|_{(2r,2r')}^{(2r+2,2r'+2)} \Big) \\
= & \sum_{p,r,p',r'=0}^{\infty} \delta_{p+r,p'+r'} \pi_1 \boldsymbol{B}_{p+r+1}^{(p,p')}\big|_{(2r,2r')},
\end{aligned} \tag{3.3}$$

with $\boldsymbol{B}_1 \equiv 0$, which we call the string products with no picture number deficit. However, this is not suitable for considering the cyclicity since the Ramond number is not invariant under the cyclic

[4] We take the convention that the quantity with the (cyclic) Ramond number out of range is identically equal to zero, as in Ref. [22].

permutation as in Eq. (2.50). So, instead we consider the string products,

$$\begin{aligned}\pi_1\boldsymbol{B} &\equiv \sum_{p,,r,r'=0}^{\infty} \pi_1 \boldsymbol{B}^{(p,p')}_{p+r+1}\Big|^{(2r,2r')}\\ &= \sum_{p,r,r'=0}^{\infty}\Big(\pi_1^{(0,0)}\boldsymbol{B}^{(p,p')}_{p+r+1}\Big|^{(2r,2r')}_{(2r,2r')} + \pi_1^{(1,0)}\boldsymbol{B}^{(p,p')}_{p+r+1}\Big|^{(2r,2r')}_{(2r-2,2r')}\\ &\qquad + \pi_1^{(0,1)}\boldsymbol{B}^{(p,p')}_{p+r+1}\Big|^{(2r,2r')}_{(2r,2r'-2)} + \pi_1^{(1,1)}\boldsymbol{B}^{(p,p')}_{p+r+1}\Big|^{(2r,2r')}_{(2r-2,2r'-2)}\Big),\end{aligned} \tag{3.4}$$

respecting the cyclic Ramond number that is invariant under the permutation. While it becomes easy to consider the cyclicity, this combination of string products $\boldsymbol{B}$ cannot be used as it is since its NS–NS, R–NS, NS–R, and R–R components have picture number deficit $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, respectively. In the heterotic string field theory, similar string products can naturally appear in a non-linear extension of the combination of the operator (one-string product) $\boldsymbol{Q}-\boldsymbol{\eta}$ [22]. In type II superstring theory, however, the analogous combination $\boldsymbol{Q}-\boldsymbol{\eta}-\bar{\boldsymbol{\eta}}$ has no counterpart with picture number deficit $(1,1)$, and thus we cannot directly extend the prescription in Ref. [22] to construct the required L_∞ algebra. We take an alternative way that is a generalization of the asymmetric construction used in Ref. [3] to give those restricted into the NS–NS sector.

3.2. *Construction of string products*

The prescription we propose is simply repeating that used in heterotic string field theory twice: the first time is for getting the correct structure of the left-moving sector by inserting X and/or ξ in the bosonic string products, which we assume to be known [30–32], and the second time is for getting the correct structure of the right-moving sector by inserting $\bar{X}$ and/or $\bar{\xi}$ in the (heterotic) string products obtained in the first step.

We start from the combined coderivation

$$\boldsymbol{D}-\boldsymbol{C} = \boldsymbol{Q}-\boldsymbol{\eta}+\boldsymbol{B}, \tag{3.5}$$

with

$$\boldsymbol{B} = \sum_{p,r=0}^{\infty}\boldsymbol{B}^{(p)}_{p+r+1}\Big|^{2r}, \tag{3.6}$$

where p and $2r$ are the picture and cyclic Ramond numbers of the left-moving sector, respectively. This can be decomposed to $\boldsymbol{D}$ and $\boldsymbol{C}$ by picture number deficit as

$$\pi_1\boldsymbol{D} = \pi_1\boldsymbol{Q} + \sum_{p,r=0}^{\infty}\pi_1\boldsymbol{B}^{(p)}_{p+r+1}\Big|^{2r}_{2r} = \pi_1\boldsymbol{Q} + \pi_1^{(0,*)}\boldsymbol{B}, \tag{3.7}$$

$$\pi_1\boldsymbol{C} = \pi_1\boldsymbol{\eta} - \sum_{p,r=0}^{\infty}\pi_1\boldsymbol{B}^{(p)}_{p+r+1}\Big|^{2r}_{2r-2} = \pi_1\boldsymbol{\eta} - \pi_1^{(1,*)}\boldsymbol{B}. \tag{3.8}$$

Suppose $\boldsymbol{B}$ has zero right-moving picture number and is independent of the right-moving Ramond and cyclic Ramond numbers. The left-moving picture number deficit of $\boldsymbol{D}$ is equal to zero and that of $\boldsymbol{C}$ is equal to one. As was shown in Ref. [22], the L_∞ relation for the coderivation $\boldsymbol{D}-\boldsymbol{C}$,

$$[\boldsymbol{D}-\boldsymbol{C}, \boldsymbol{D}-\boldsymbol{C}] = 0, \tag{3.9}$$

following from the equations

$$[\boldsymbol{Q}, \boldsymbol{B}(s,t)] + \frac{1}{2}[\boldsymbol{B}(s,t), \boldsymbol{B}(s,t)]^1 + \frac{s}{2}[\boldsymbol{B}(s,t), \boldsymbol{B}(s,t)]^2 = 0, \tag{3.10a}$$

$$[\boldsymbol{\eta}, \boldsymbol{B}(s,t)] - \frac{t}{2}[\boldsymbol{B}(s,t), \boldsymbol{B}(s,t)]^2 = 0, \tag{3.10b}$$

for the generating function

$$\boldsymbol{B}(s,t) = \sum_{m,n,r=0}^{\infty} s^m t^n \boldsymbol{B}^{(n)}_{m+n+r+1}\big|^{2r} = \sum_{n=0}^{\infty} t^n \boldsymbol{B}^{(n)}(s), \tag{3.11}$$

from which we obtain the required string products as $\boldsymbol{B} = \boldsymbol{B}(0,1)$. The operations $[\boldsymbol{l}, \boldsymbol{l}']^1$ amd $[\boldsymbol{l}, \boldsymbol{l}']^2$ in Eq. (3.10) are defined by splitting the commutator into the pieces with definite left-moving cyclic Ramond number: if $\boldsymbol{l} = \sum_r \boldsymbol{l}|^{2r}$ and $\boldsymbol{l}' = \sum_{r'} \boldsymbol{l}'|^{2r'}$, then

$$[\boldsymbol{l}, \boldsymbol{l}']^1 = \sum_{r,r'} [\boldsymbol{l}|^{2r}, \boldsymbol{l}'|^{2r'}]\big|^{2r+2r'}, \qquad [\boldsymbol{l}, \boldsymbol{l}']^2 = \sum_{r,r'} [\boldsymbol{l}|^{2r}, \boldsymbol{l}'|^{2r'}]\big|^{2r+2r'-2}. \tag{3.12}$$

We can show that $[\boldsymbol{l}, \boldsymbol{l}'] = [\boldsymbol{l}, \boldsymbol{l}']^1 + [\boldsymbol{l}, \boldsymbol{l}']^2$. It was also shown in Ref. [22] that the equations in Eq. (3.10) are satisfied if we postulate the differential equations

$$\partial_t \boldsymbol{B}(s,t) = [\boldsymbol{Q}, \boldsymbol{\lambda}(s,t)] + [\boldsymbol{B}(s,t), \boldsymbol{\lambda}(s,t)]^1 + s[\boldsymbol{B}(s,t), \boldsymbol{\lambda}(s,t)]^2, \tag{3.13a}$$

$$[\boldsymbol{\eta}, \boldsymbol{\lambda}(s,t)] = \partial_s \boldsymbol{B}(s,t) + t[\boldsymbol{B}(s,t), \boldsymbol{\lambda}(s,t)]^2 \tag{3.13b}$$

by introducing (a generating function of) the gauge products represented by a degree-even coderivation,

$$\boldsymbol{\lambda}(s,t) = \sum_{m,n,r=0}^{\infty} s^m t^n \boldsymbol{\lambda}^{(n+1)}_{m+n+r+2}\big|^{2r} = \sum_{n=0}^{\infty} t^n \boldsymbol{\lambda}^{(n+1)}(s). \tag{3.14}$$

The differential equations in Eq. (3.13) can be recursively solved as

$$\boldsymbol{\lambda}^{(n+1)}(s) = \xi \circ \left(\partial_s \boldsymbol{B}^{(n)}(s) + \sum_{n'=0}^{n-1} [\boldsymbol{B}^{(n-n')}(s), \boldsymbol{\lambda}^{(n'+1)}(s)]^2 \right), \tag{3.15a}$$

$$\begin{aligned}(n+1)\boldsymbol{B}^{(n+1)}(s) = {} & [\boldsymbol{Q}, \boldsymbol{\lambda}^{(n+1)}(s)] \\ & + \sum_{n'=0}^{n} [\boldsymbol{B}^{(n-n')}(s), \boldsymbol{\lambda}^{(n'+1)}(s)]^1 + \sum_{n'=0}^{n} s[\boldsymbol{B}^{(n-n')}(s), \boldsymbol{\lambda}^{(n'+1)}(s)]^2,\end{aligned} \tag{3.15b}$$

with the initial condition

$$\boldsymbol{B}^{(0)}(s) = \boldsymbol{L}_B(s), \tag{3.16}$$

given by using the interacting part of the bosonic products (string products with no non-zero picture number operator insertion) [22]:

$$\boldsymbol{L}_B(s) = \sum_{m,r=0}^{\infty} s^m \boldsymbol{L}^B_{m+r+1}\big|^{2r}, \qquad (\boldsymbol{L}^B_1\big|^0 \equiv 0). \tag{3.17}$$

The operation $\xi\circ$ in Eq. (3.15a) is defined as that inserting ξ cyclically. Then, by construction, all the $\boldsymbol{B}(s,t)$ and $\boldsymbol{\lambda}(s,t)$ are cyclic with respect to the symplectic form ω_m. They provide a cyclic L_∞ algebra $(\mathcal{H}_m, \omega_m, \boldsymbol{D}-\boldsymbol{C})$. After decomposing this combined L_∞ algebra into $\boldsymbol{D}$ and $\boldsymbol{C}$, we can obtain a *heterotic* L_∞ algebra in the small Hilbert space, satisfying $[\boldsymbol{\eta}, \boldsymbol{L}_H] = [\bar{\boldsymbol{\eta}}, \boldsymbol{L}_H] = 0$, by similarity transformation generated by the cohomomorphism

$$\pi_1 \hat{\boldsymbol{F}}^{-1} = \pi_1 \mathbb{I}_{\mathcal{SH}} - \Xi \pi_1^{(1,*)} \boldsymbol{B} \tag{3.18}$$

as

$$\pi_1 \boldsymbol{L}_H \equiv \pi_1 \hat{\boldsymbol{F}}^{-1} \boldsymbol{D} \hat{\boldsymbol{F}} = \pi_1 \boldsymbol{Q} + \mathcal{X} \boldsymbol{b}_H, \tag{3.19}$$

with $\boldsymbol{b}_H = \boldsymbol{B}\hat{\boldsymbol{F}}$. This $\boldsymbol{L}_H$ has the required picture number structure for the left-moving sector but the right-moving picture number is still equal to zero:

$$\boldsymbol{L}_H = \boldsymbol{Q} + \sum_{p,r=0}^{\infty} (\boldsymbol{L}_H)^{(p,0)}_{p+r+1}\big|_{2r} = \boldsymbol{Q} + \sum_{p,r=0}^{\infty} \mathcal{X} (\boldsymbol{b}_H)^{(p,0)}_{p+r+1}\big|^{2r}. \tag{3.20}$$

Here, the subscript or superscript $2r$ after the vertical line is the left-moving Ramond or cyclic Ramond number, respectively. It is easy to see that $\boldsymbol{b}_H$ is cyclic with respect to ω_m in the same way as in Ref. [22].

We repeat the same procedure for the right-moving sector. Let us consider the combined coderivation

$$\bar{\boldsymbol{D}} - \bar{\boldsymbol{C}} = \boldsymbol{Q} - \bar{\boldsymbol{\eta}} + \bar{\boldsymbol{B}}, \tag{3.21}$$

with

$$\bar{\boldsymbol{B}} = \sum_{\bar{p},\bar{r}=0}^{\infty} \bar{\boldsymbol{B}}^{(\bar{p})}_{\bar{p}+\bar{r}+1}\big|^{2\bar{r}}, \tag{3.22}$$

which can be decomposed by the right-moving picture number deficit as

$$\pi_1 \bar{\boldsymbol{D}} = \pi_1 \boldsymbol{Q} + \sum_{\bar{p},\bar{r}=0}^{\infty} \pi_1 \bar{\boldsymbol{B}}^{(\bar{p})}_{\bar{p}+\bar{r}+1}\big|^{2\bar{r}}_{2\bar{r}} = \pi_1 \boldsymbol{Q} + \pi_1^{(*,0)} \bar{\boldsymbol{B}}, \tag{3.23}$$

$$\pi_1 \bar{\boldsymbol{C}} = \pi_1 \bar{\boldsymbol{\eta}} - \sum_{\bar{p},\bar{r}=0}^{\infty} \pi_1 \bar{\boldsymbol{B}}^{(\bar{p})}_{\bar{p}+\bar{r}+1}\big|^{2\bar{r}}_{2\bar{r}-2} = \pi_1 \bar{\boldsymbol{\eta}} - \pi_1^{(*,1)} \bar{\boldsymbol{B}}. \tag{3.24}$$

It is noted that only the right-moving quantum numbers, the picture, Ramond, and cyclic Ramond numbers, are specified. Those of the left-moving sector are implicit but determined properly in our construction below. The L_∞ relation of the coderivation $\bar{\boldsymbol{D}} - \bar{\boldsymbol{C}}$,

$$[\bar{\boldsymbol{D}} - \bar{\boldsymbol{C}}, \bar{\boldsymbol{D}} - \bar{\boldsymbol{C}}] = 0, \tag{3.25}$$

again follows from the equations

$$[\boldsymbol{Q}, \bar{\boldsymbol{B}}(s,t)] + \frac{1}{2}[\bar{\boldsymbol{B}}(s,t), \bar{\boldsymbol{B}}(s,t)]^{\bar{1}} + \frac{s}{2}[\bar{\boldsymbol{B}}(s,t), \bar{\boldsymbol{B}}(s,t)]^{\bar{2}} = 0, \tag{3.26a}$$

$$[\bar{\boldsymbol{\eta}}, \bar{\boldsymbol{B}}(s,t)] - \frac{t}{2}[\bar{\boldsymbol{B}}(s,t), \bar{\boldsymbol{B}}(s,t)]^{\bar{2}} = 0 \tag{3.26b}$$

for the generating function

$$\bar{\boldsymbol{B}}(s,t) = \sum_{\bar{m},\bar{n},\bar{r}=0}^{\infty} s^{\bar{m}} t^{\bar{n}} \bar{\boldsymbol{B}}^{(\bar{n})}_{\bar{m}+\bar{n}+\bar{r}+1}\Big|^{2\bar{r}} = \sum_{\bar{n}=0}^{\infty} t^{\bar{n}} \bar{\boldsymbol{B}}^{(\bar{n})}(s). \tag{3.27}$$

The string products in Eq. (3.22) are obtained as $\bar{\boldsymbol{B}} = \bar{\boldsymbol{B}}(0,1)$. The operations $[\boldsymbol{l}, \boldsymbol{l}']^{\bar{1}}$ and $[\boldsymbol{l}, \boldsymbol{l}']^{\bar{2}}$ are the right-moving counterpart of Eq. (3.12), which split the commutator into pieces with a definite right-moving cyclic Ramond number. Equations (3.26) are satisfied if we postulate the differential equations

$$\partial_t \bar{\boldsymbol{B}}(s,t) = [\boldsymbol{Q}, \bar{\boldsymbol{\lambda}}(s,t)] + [\bar{\boldsymbol{B}}(s,t), \bar{\boldsymbol{\lambda}}(s,t)]^{\bar{1}} + s[\bar{\boldsymbol{B}}(s,t), \bar{\boldsymbol{\lambda}}(s,t)]^{\bar{2}}, \tag{3.28a}$$

$$[\bar{\boldsymbol{\eta}}, \bar{\boldsymbol{\lambda}}(s,t)] = \partial_s \bar{\boldsymbol{B}}(s,t) + t[\bar{\boldsymbol{B}}(s,t), \bar{\boldsymbol{\lambda}}(s,t)]^{\bar{2}} \tag{3.28b}$$

by introducing (a generating function of) the gauge products represented by a degree-even coderivation

$$\bar{\boldsymbol{\lambda}}(s,t) = \sum_{\bar{m},\bar{n},\bar{r}=0}^{\infty} s^{\bar{m}} t^{\bar{n}} \bar{\boldsymbol{\lambda}}^{(\bar{n}+1)}_{\bar{m}+\bar{n}+\bar{r}+2}\Big|^{2\bar{r}} = \sum_{\bar{n}=0}^{\infty} t^{\bar{n}} \bar{\boldsymbol{\lambda}}^{(\bar{n}+1)}(s). \tag{3.29}$$

This time we solve the differential equations in Eq. (3.28) by starting from the initial condition $\bar{\boldsymbol{B}}^{(0)}(s) = \boldsymbol{L}_H(s)$ with

$$\boldsymbol{L}_H(s) = \sum_{\bar{m},\bar{r}=0}^{\infty} s^{\bar{m}} (\boldsymbol{L}_H)_{\bar{m}+\bar{r}+1}\Big|^{2\bar{r}}, \tag{3.30}$$

or $\bar{\boldsymbol{b}}^{(0)}(s) = \boldsymbol{b}_H(s)$ with $\bar{\boldsymbol{B}}^{(0)}(s) = \mathcal{X}\bar{\boldsymbol{b}}^{(0)}(s)$ and $\boldsymbol{L}_H(s) = \mathcal{X}\boldsymbol{b}_H(s)$. Solving Eq. (3.28b) explicitly, we first obtain

$$\bar{\boldsymbol{\lambda}}^{(1)}(s) = \bar{\xi} \circ \partial_s \boldsymbol{L}_H(s) = \mathcal{X}\left(\bar{\xi} \circ \partial_s \boldsymbol{b}_H\right) \equiv \mathcal{X}\boldsymbol{\mu}^{(1)}(s), \tag{3.31}$$

and then, from Eq. (3.28a),

$$\begin{aligned} \bar{\boldsymbol{B}}^{(1)}(s) &= [\boldsymbol{Q}, \bar{\boldsymbol{\lambda}}^{(1)}(s)] + [\boldsymbol{L}_H(s), \bar{\boldsymbol{\lambda}}^{(1)}(s)]^{\bar{1}} + s[\boldsymbol{L}_H(s), \bar{\boldsymbol{\lambda}}^{(1)}(s)]^{\bar{2}} \\ &= \mathcal{X}\left([\boldsymbol{Q}, \boldsymbol{\mu}^{(1)}(s)] + [\![\boldsymbol{b}_H(s), \boldsymbol{\mu}^{(1)}(s)]\!]^{\bar{1}} + s[\![\boldsymbol{b}_H(s), \boldsymbol{\mu}^{(1)}(s)]\!]^{\bar{2}}\right) \equiv \mathcal{X}\bar{\boldsymbol{b}}^{(1)}(s), \end{aligned} \tag{3.32}$$

where the double square brackets $[\![\boldsymbol{l}, \boldsymbol{l}']\!]^{\bar{1}}$ and $[\![\boldsymbol{l}, \boldsymbol{l}']\!]^{\bar{2}}$ are defined by

$$[\![\boldsymbol{l}, \boldsymbol{l}']\!]^{\bar{1}} = \sum_{\bar{r},\bar{s}=0}^{\infty} \left(\boldsymbol{l}|^{2\bar{r}}\, \mathcal{X}\boldsymbol{l}'|^{2\bar{s}} - (-1)^{|l||l'|}\boldsymbol{l}'|^{2\bar{s}}\, \mathcal{X}\boldsymbol{l}|^{2\bar{r}}\right)\Big|^{2\bar{r}+2\bar{s}}, \tag{3.33}$$

$$[\![\boldsymbol{l}, \boldsymbol{l}']\!]^{\bar{2}} = \sum_{\bar{r},\bar{s}=0}^{\infty} \left(\boldsymbol{l}|^{2\bar{r}} \mathcal{X}\boldsymbol{l}'|^{2\bar{s}} - (-1)^{|l||l'|}\boldsymbol{l}'|^{2\bar{s}} \mathcal{X}\boldsymbol{l}|^{2\bar{r}}\right)\Big|^{2\bar{r}+2\bar{s}-2}. \tag{3.34}$$

Similarly solving the higher-order products recursively, we obtain

$$\boldsymbol{\mu}^{(\bar{n}+1)}(s) = \bar{\xi} \circ \left(\partial_s \bar{\boldsymbol{b}}^{(\bar{n})}(s) + \sum_{\bar{n}'=0}^{\bar{n}-1} [\![\bar{\boldsymbol{b}}^{(\bar{n}-\bar{n}')}(s), \boldsymbol{\mu}^{(\bar{n}'+1)}(s)]\!]^{\bar{2}} \right), \tag{3.35a}$$

$$\begin{aligned}(\bar{n}+1)\bar{\boldsymbol{b}}^{(\bar{n}+1)}(s) = {} & [\boldsymbol{Q}, \boldsymbol{\mu}^{(\bar{n}+1)}(s)] \\ & + \sum_{\bar{n}'=0}^{\bar{n}} [\![\bar{\boldsymbol{b}}^{(\bar{n}-\bar{n}')}(s), \boldsymbol{\mu}^{(\bar{n}'+1)}(s)]\!]^{\bar{1}} + \sum_{\bar{n}'=0}^{\bar{n}} s [\![\bar{\boldsymbol{b}}^{(\bar{n}-\bar{n}')}(s), \boldsymbol{\mu}^{(\bar{n}'+1)}(s)]\!]^{\bar{2}}, \end{aligned} \tag{3.35b}$$

where

$$\bar{\boldsymbol{\lambda}}^{(\bar{n}+1)}(s) = \mathcal{X}\boldsymbol{\mu}^{(\bar{n}+1)}(s), \qquad \bar{\boldsymbol{B}}^{(\bar{n}+1)}(s) = \mathcal{X}\bar{\boldsymbol{b}}^{(\bar{n}+1)}(s). \tag{3.36}$$

Note that the factor $\mathcal{X}$ for the left-moving sector can still be pulled out after repeating the procedure for the right-moving sector. The initial condition $\bar{\boldsymbol{B}}^{(0)}(s) = \boldsymbol{L}_H^{(0)}(s)$ fixes the structure of the left-moving picture number as

$$\boldsymbol{\mu}^{(\bar{n}+1)}_{\bar{m}+\bar{n}+\bar{r}+2}\Big|^{2\bar{r}} = \sum_{n,r=0}^{\infty} \delta_{n+r,\bar{m}+\bar{n}+\bar{r}+1} \boldsymbol{\mu}^{(n,\bar{n}+1)}_{\bar{m}+\bar{n}+\bar{r}+2}\Big|^{(2r,2\bar{r})}, \tag{3.37a}$$

$$\bar{\boldsymbol{b}}^{(\bar{n})}_{\bar{m}+\bar{n}+\bar{r}+1}\Big|^{2\bar{r}} = \sum_{n,r=0}^{\infty} \delta_{n+r,\bar{m}+\bar{n}+\bar{r}} \bar{\boldsymbol{b}}^{(n,\bar{n})}_{\bar{m}+\bar{n}+\bar{r}+1}\Big|^{(2r,2\bar{r})}. \tag{3.37b}$$

All the $\bar{\boldsymbol{B}}(s,t)$ and $\bar{\boldsymbol{\lambda}}(s,t)$ are in $\mathcal{H}_{\bar{m}}$, satisfying $[\boldsymbol{\eta}, \bar{\boldsymbol{B}}(s,t)] = [\boldsymbol{\eta}, \bar{\boldsymbol{\lambda}}(s,t)] = 0$, and now cyclic with respect to the symplectic form $\omega_{\bar{m}}$ by construction. We have obtained a cyclic L_∞ algebra $(\mathcal{H}_{\bar{m}}, \omega_{\bar{m}}, \bar{\boldsymbol{D}} - \bar{\boldsymbol{C}})$. Finally, the cohomomorphism

$$\pi_1 \hat{\bar{\boldsymbol{F}}}^{-1} = \pi_1 \mathbb{I}_{\mathcal{SH}} - \bar{\Xi}\pi_1^{(*,1)}\bar{\boldsymbol{B}} \tag{3.38}$$

generates the similarity transformation which provides an L_∞ algebra $\boldsymbol{L}$ in the small Hilbert space, satisfying $[\boldsymbol{\eta}, \boldsymbol{L}] = [\bar{\boldsymbol{\eta}}, \boldsymbol{L}] = 0$:

$$\begin{aligned}\pi_1 \boldsymbol{L} \equiv \pi_1 \hat{\bar{\boldsymbol{F}}}^{-1} \bar{\boldsymbol{D}} \hat{\bar{\boldsymbol{F}}} &= \pi_1 \boldsymbol{Q} + \bar{\mathcal{X}} \bar{\boldsymbol{B}} \hat{\bar{\boldsymbol{F}}} \\ &= \pi_1 \boldsymbol{Q} + \mathcal{G}\boldsymbol{b}, \end{aligned} \tag{3.39}$$

where $\boldsymbol{b} = \bar{\boldsymbol{b}}\hat{\bar{\boldsymbol{F}}}$ with $\bar{\boldsymbol{b}} = \sum_{\bar{p},\bar{r}=0}^{\infty} \bar{\boldsymbol{b}}^{(\bar{p})}_{\bar{p}+\bar{r}+1}\Big|^{2\bar{r}}$. We can again show that $\boldsymbol{b}$ is cyclic with respect to ω_l in a similar way to Ref. [22], and thus the cyclic L_∞ algebra $(\mathcal{H}, \Omega, \boldsymbol{L})$ is obtained.

4. Four-point amplitudes

In this section we concretely calculate three typical on-shell physical amplitudes with four external strings in a similar way to Refs. [2,22] to demonstrate how the type II string field theory we have constructed reproduces the first-quantized amplitudes. We take the Siegel gauge defined by the conditions

$$\boldsymbol{\psi}_{\text{NS–NS}} = \boldsymbol{\psi}_{\text{R–NS}} = \boldsymbol{\psi}_{\text{NS–R}} = \boldsymbol{\psi}_{\text{R–R}} = 0. \tag{4.1}$$

From the action in Eq. (2.47) we can find that the propagators in this gauge are given by

$$\begin{aligned} \Pi_{\text{NS–NS}} &= -\frac{b_0^+ b_0^-}{L_0^+}\delta(L_0^-), & \Pi_{\text{R–NS}} &= -\frac{b_0^+ b_0^- X}{L_0^+}\delta(L_0^-), \\ \Pi_{\text{NS–R}} &= -\frac{b_0^+ b_0^- \bar{X}}{L_0^+}\delta(L_0^-), & \Pi_{\text{R–R}} &= -\frac{b_0^+ b_0^- X\bar{X}}{L_0^+}\delta(L_0^-), \end{aligned} \tag{4.2}$$

which agree with those appearing in the first-quantized formulation [29].

4.1. *Four-(R–R) amplitude*

Let us first consider the case that all the external strings are in the R–R sector. The first-quantized amplitude is given in the form

$$\mathcal{A}_4^{\text{1st}}(\Phi_1, \Phi_2, \Phi_3, \Phi_4) = \int d^2z \langle \Phi_1(0)(b_{-1}^+ b_{-1}^- \Phi_2(z))\Phi_3(1)\Phi_4(\infty)\rangle, \tag{4.3}$$

where $\Phi_1, \ldots, \Phi_4$ are on-shell physical R–R vertex operators, satisfying $Q\Phi = 0$, in a $(-1/2, -1/2)$ picture. The correlator is evaluated in the small Hilbert space on the complex z-plane. It is not necessary to add any picture-changing operators at all. Owing to the same moduli structure as the bosonic closed string, we can express this using the bosonic closed string products L_n^B as

$$\begin{aligned} \mathcal{A}_4^{\text{1st}}(\Phi_1, \Phi_2, \Phi_3, \Phi_4) = \omega_s\Bigg(\Phi_1, \Bigg(&L_3^B(\Phi_2, \Phi_3, \Phi_4) - L_2^B\bigg(\Phi_2, \frac{b_0^+}{L_0^+}L_2^B(\Phi_3, \Phi_4)\bigg) \\ &- L_2^B\bigg(\Phi_3, \frac{b_0^+}{L_0^+}L_2^B(\Phi_4, \Phi_2)\bigg) - L_2^B\bigg(\Phi_4, \frac{b_0^+}{L_0^+}L_2^B(\Phi_2, \Phi_3)\bigg)\Bigg)\Bigg). \end{aligned} \tag{4.4}$$

It should be noted that the moduli integral $b_0^-\delta(L_0^-) = \int_0^{2\pi}\frac{d\theta}{2\pi}b_0^- \exp(i\theta L_0^-)$ is hidden behind the definition of the string product. This amplitude can be regarded as a multi-linear map:

$$\langle \mathcal{A}_4| \,:\, \mathcal{H}_Q^{\text{R–R}} \otimes (\mathcal{H}_Q^{\text{R–R}})^{\wedge 3} \,\longrightarrow\, \mathbb{C}, \tag{4.5}$$

where $\mathcal{H}_Q \subset \mathcal{H}$ is the subspace of states annihilated by Q. Putting the string fields $\Phi_1, \ldots, \Phi_4$ out, we can express Eq. (4.4) as

$$\langle \mathcal{A}_4^{\text{1st}}| = \langle \omega_s| \, \mathbb{I} \otimes \left(L_3^B - L_2^B\left(\mathbb{I} \wedge \frac{b_0^+}{L_0^+}L_2^B\right)\right) \tag{4.6}$$

by introducing the bilinear map representation $\langle \omega_s|$ of the symplectic form ω_s defined by

$$\begin{array}{ccc} \mathcal{H} \otimes \mathcal{H} & \xrightarrow{\langle \omega_s|} & \mathbb{C} \\ \Cup & & \Cup \\ |\Phi_1\rangle \otimes |\Phi_2\rangle & \longmapsto & \omega_s(\Phi_1, \Phi_2). \end{array} \tag{4.7}$$

The expression in Eq. (4.6) can also be written by using the coderivations as

$$\langle \mathcal{A}_4^{\text{1st}}| = \langle \omega_s| \, \mathbb{I} \otimes \pi_1^{(1,1)}\left(\boldsymbol{L}_3^B\big|_{(2,2)}^{(4,4)} - \boldsymbol{L}_2^B\big|_{(0,0)}^{(2,2)} \frac{b_0^+}{L_0^+}\boldsymbol{L}_2^B\big|_{(2,2)}^{(2,2)}\right). \tag{4.8}$$

Here, $\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^B$ is the coderivation derived from $\frac{b_0^+}{L_0^+}L_2^B$.

From the type II superstring field theory, on the other hand, the four-(R–R) amplitude is calculated as

$$\begin{aligned}\mathcal{A}_4(\Phi_1,\Phi_2,\Phi_3,\Phi_4) = \omega_s\Big(\Phi_1, \big(b_3^{(0,0)}(\Phi_2,\Phi_3,\Phi_4) - b_2^{(0,0)}(\Phi_2, \Pi_{\text{NS–NS}}\, c_0^- b_2^{(0,0)}(\Phi_3,\Phi_4))\\ - b_2^{(0,0)}(\Phi_3, \Pi_{\text{NS–NS}}\, c_0^- b_2^{(0,0)}(\Phi_4,\Phi_2))\\ - b_2^{(0,0)}(\Phi_4, \Pi_{\text{NS–NS}}\, c_0^- b_2^{(0,0)}(\Phi_2,\Phi_3))\big)\Big)\\ = \omega_s\left(\Phi_1, \left(b_3^{(0,0)}(\Phi_2,\Phi_3,\Phi_4) - b_2^{(0,0)}\left(\Phi_2, \frac{b_0^+}{L_0^+} b_2^{(0,0)}(\Phi_3,\Phi_4)\right)\right.\right.\\ \left.\left. - b_2^{(0,0)}\left(\Phi_3, \frac{b_0^+}{L_0^+} b_2^{(0,0)}(\Phi_4,\Phi_2)\right) - b_2^{(0,0)}\left(\Phi_4, \frac{b_0^+}{L_0^+} b_2^{(0,0)}(\Phi_2,\Phi_3)\right)\right)\right).\end{aligned} \tag{4.9}$$

The second equality holds owing to the fact that the string field $b_2^{(0,0)}(\Phi_1,\Phi_2)$ satisfies the closed string constraints in Eq. (2.4). Rewriting by using the coderivations, we find

$$\langle\mathcal{A}_4| = \langle\omega_s|\mathbb{I}\otimes\pi_1^{(1,1)}\left(\boldsymbol{b}_3^{(0,0)}\big|^{(4,4)}_{(2,2)} - \boldsymbol{b}_2^{(0,0)}\big|^{(2,2)}_{(0,0)}\frac{b_0^+}{L_0^+}\boldsymbol{b}_2^{(0,0)}\big|^{(2,2)}_{(2,2)}\right). \tag{4.10}$$

Here, the string products without picture number $\boldsymbol{b}_n^{(0,0)}$ are equal to the bosonic string products $\boldsymbol{L}_n^B$ by construction.[5] Hence, the string field theory amplitude in Eq. (4.10) certainly agrees with the first-quantized amplitude in Eq. (4.8).

4.2. *Two-(NS–R)-two-(R–R) amplitude*

Next, we consider the amplitude with two NS–R strings and two R–R strings, which is given in the first-quantized formulation by

$$\langle\mathcal{A}_4| = \langle\omega_s|X_0\otimes\pi_1^{(0,1)}\left(\boldsymbol{L}_3^B\big|^{(2,4)}_{(2,2)} - \boldsymbol{L}_2^B\big|^{(0,2)}_{(0,0)}\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^B\big|^{(2,2)}_{(2,2)} - \boldsymbol{L}_2^B\big|^{(2,2)}_{(2,0)}\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^B\big|^{(2,2)}_{(0,2)}\right), \tag{4.11}$$

in a similar representation to the four-(R–R) amplitude. Here, $\langle\mathcal{A}_4|$ is a multi-linear map

$$\langle\mathcal{A}_4| \,:\, \mathcal{H}_Q^{\text{NS–R}}\otimes\left(\mathcal{H}_Q^{\text{NS–R}}\wedge(\mathcal{H}_Q^{\text{R–R}})^{\wedge 2}\right) \longrightarrow \mathbb{C}, \tag{4.12}$$

and $X_0 = \{Q,\xi\}$. In this case, the amplitude obtained from the type II string field theory is calculated as

$$\langle\mathcal{A}_4| = \langle\omega_s|\,\mathbb{I}\otimes\pi_1^{(0,1)}\left(\boldsymbol{b}_3^{(1,0)}\big|^{(2,4)}_{(2,2)} - \boldsymbol{b}_2^{(1,0)}\big|^{(0,2)}_{(0,0)}\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^B\big|^{(2,2)}_{(2,2)} - \boldsymbol{L}_2^B\big|^{(2,2)}_{(2,0)}\frac{b_0^+X}{L_0^+}\boldsymbol{L}_2^B\big|^{(2,2)}_{(0,2)}\right). \tag{4.13}$$

[5] This will be used without notice hereafter.

In our construction given in the previous section, the string products without right-moving picture number $\boldsymbol{b}_n^{(1,0)}$ are equal to the heterotic string products $(\boldsymbol{b}_H)_n^{(1)} = (\boldsymbol{B}\hat{\boldsymbol{F}})_n^{(1)}$ and given explicitly by

$$\sum_{\bar{r}=0,1} \boldsymbol{b}_2^{(1,0)}\big|^{(0,2\bar{r})} = (\boldsymbol{B}\hat{\boldsymbol{F}})_2^{(1)}\big|^0 = \boldsymbol{B}_2^{(1)}\big|^0$$

$$= [\boldsymbol{Q}, \boldsymbol{\lambda}_2^{(1)}\big|^0], \tag{4.14a}$$

$$\sum_{\bar{r}=0,1,2} \boldsymbol{b}_3^{(1,0)}\big|^{(2,2\bar{r})} = (\boldsymbol{B}\hat{\boldsymbol{F}})_3^{(1)}\big|^2 = \boldsymbol{B}_3^{(1)}\big|^2 + \boldsymbol{B}_2^{(0)}\big|^2\, \Xi\pi_1^{(1,*)}\boldsymbol{B}_2^{(0)}\big|^2$$

$$= [\boldsymbol{Q}, \boldsymbol{\lambda}_3^{(1)}\big|^2] + [\boldsymbol{B}_2^{(0)}\big|^2, \boldsymbol{\lambda}_2^{(1)}\big|^2]\big|^2 + \boldsymbol{B}_2^{(0)}\big|^2\, \Xi\pi_1^{(1,*)}\boldsymbol{B}_2^{(0)}\big|^2, \tag{4.14b}$$

where the last equalities follow from the recursion relation in Eq. (3.15b) with $n = 0$. If we further note that

$$\boldsymbol{B}_2^{(0)}\big|^2 = \sum_{\bar{r}=0,1} \boldsymbol{L}_2^B\big|^{(2,2\bar{r})}, \qquad \boldsymbol{\lambda}_2^{(1)}\big|^0 = \sum_{\bar{r}=0,1} \boldsymbol{\lambda}_2^{(1)}\big|^{(0,2\bar{r})}, \qquad \boldsymbol{\lambda}_3^{(1)}\big|^2 = \sum_{\bar{r}=0,1,2} \boldsymbol{\lambda}_3^{(1)}\big|^{(2,2\bar{r})}, \tag{4.15}$$

the relations in Eq. (4.14) can be decomposed with respect to the Ramond and cyclic Ramond numbers. In particular, we find that

$$\pi_1^{(0,1)}\boldsymbol{b}_2^{(1,0)}\big|^{(0,2)}_{(0,0)} = \pi_1^{(0,1)}[\boldsymbol{Q}, \boldsymbol{\lambda}_2^{(1)}\big|^{(0,2)}_{(0,0)}], \tag{4.16a}$$

$$\pi_1^{(0,1)}\boldsymbol{b}_3^{(1,0)}\big|^{(2,4)}_{(2,2)} = \pi_1^{(0,1)}\big([\boldsymbol{Q}, \boldsymbol{\lambda}_3^{(1)}\big|^{(2,4)}_{(2,2)}] - \boldsymbol{\lambda}_2^{(1)}\big|^{(0,2)}_{(0,0)}\boldsymbol{L}_2^B\big|^{(2,2)}_{(2,2)} + \boldsymbol{L}_2^B\big|^{(2,2)}_{(2,0)}\,\Xi\boldsymbol{L}_2^B\big|^{(2,2)}_{(0,2)}\big). \tag{4.16b}$$

Substituting this into the string field theory amplitude in Eq. (4.13) and then pulling $\boldsymbol{Q}$ out, we can rewrite it as

$$\begin{aligned}\langle\mathcal{A}_4| = \langle\omega_l|\,\xi\bar{\xi}\otimes\pi_1^{(0,1)}\Big(&[\boldsymbol{Q}, \boldsymbol{\lambda}_3^{(1)}\big|^{(2,4)}_{(2,2)}] - \boldsymbol{\lambda}_2^{(1)}\big|^{(0,2)}_{(0,0)}\boldsymbol{L}_2^B\big|^{(2,2)}_{(2,2)} + \boldsymbol{L}_2^B\big|^{(2,2)}_{(2,0)}\,\Xi\boldsymbol{L}_2^B\big|^{(2,2)}_{(0,2)} \\ &- [\boldsymbol{Q}, \boldsymbol{\lambda}_2^{(1)}\big|^{(0,2)}_{(0,0)}]\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^B\big|^{(2,2)}_{(2,2)} - \boldsymbol{L}_2^B\big|^{(2,2)}_{(2,0)}\frac{b_0^+\{\boldsymbol{Q}, \Xi\}}{L_0^+}\boldsymbol{L}_2^B\big|^{(2,2)}_{(0,2)}\Big) \\ = -\langle\omega_l|(\xi\bar{X}_0 - X_0\bar{\xi})& \\ \otimes\pi_1^{(0,1)}\Big(&\boldsymbol{\lambda}_3^{(1)}\big|^{(2,4)}_{(2,2)} - \boldsymbol{\lambda}_2^{(1)}\big|^{(0,2)}_{(0,0)}\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^B\big|^{(2,2)}_{(2,2)} - \boldsymbol{L}_2^B\big|^{(2,2)}_{(2,0)}\frac{b_0^+\,\Xi}{L_0^+}\boldsymbol{L}_2^B\big|^{(2,2)}_{(0,2)}\Big),\end{aligned} \tag{4.17}$$

except for the terms vanishing when they hit the states in $\mathcal{H}_Q$. Inserting $1 = \{\eta, \xi\}$ or $1 = [\bar{\eta}, \bar{\xi}]$, we can find that the amplitude in Eq. (4.17) agrees with the first-quantized one:

$$\begin{aligned}\langle\mathcal{A}_4| &= \langle\omega_s|X_0\otimes\pi_1^{(0,1)}\Big([\boldsymbol{\eta}, \boldsymbol{\lambda}_3^{(1)}\big|^{(2,4)}_{(2,2)}] - [\boldsymbol{\eta}, \boldsymbol{\lambda}_2^{(1)}\big|^{(0,2)}_{(0,0)}]\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^B\big|^{(2,2)}_{(2,2)} - \boldsymbol{L}_2^B\big|^{(2,2)}_{(2,0)}\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^B\big|^{(2,2)}_{(0,2)}\Big) \\ &= \langle\omega_s|X_0\otimes\pi_1^{(0,1)}\Big(\boldsymbol{L}_3^B\big|^{(2,4)}_{(2,2)} - \boldsymbol{L}_2^B\big|^{(0,2)}_{(0,0)}\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^B\big|^{(2,2)}_{(2,2)} - \boldsymbol{L}_2^B\big|^{(2,2)}_{(2,0)}\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^B\big|^{(2,2)}_{(0,2)}\Big).\end{aligned} \tag{4.18}$$

In the second equality we used $[\bar{\boldsymbol{\eta}}, \boldsymbol{\lambda}_n^{(1)}] = [\bar{\boldsymbol{\eta}}, \boldsymbol{L}_n^B] = 0$ and the recursion relation in Eq. (3.15a) with $n = 0$,

$$[\boldsymbol{\eta}, \boldsymbol{\lambda}_3^{(1)}\big|^{(2,4)}_{(2,2)}] = \boldsymbol{L}_3^B\big|^{(2,4)}_{(2,2)}, \qquad [\boldsymbol{\eta}, \boldsymbol{\lambda}_2^{(1)}\big|^{(0,2)}_{(0,0)}] = \boldsymbol{L}_2^B\big|^{(0,2)}_{(0,0)}. \tag{4.19}$$

4.3. (NS–NS)-(R–NS)-(NS–R)-(R–R) amplitude

Finally, let us consider the case with four external strings coming from the four different sectors. The first-quantized amplitude is given by

$$\langle\mathcal{A}_4| = \langle\omega_s|\,X_0\bar{X}_0 \otimes \pi_1^{(0,0)}\bigg(\boldsymbol{L}_3^B\big|_{(2,2)}^{(2,2)} - \boldsymbol{L}_2^B\big|_{(2,0)}^{(2,0)}\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^B\big|_{(0,2)}^{(2,2)} - \boldsymbol{L}_2^B\big|_{(0,2)}^{(0,2)}\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^B\big|_{(2,0)}^{(2,2)} - \boldsymbol{L}_2^B\big|_{(2,2)}^{(2,2)}\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^B\big|_{(0,0)}^{(2,2)}\bigg)P; \tag{4.20}$$

as a multi-linear map,

$$\langle\mathcal{A}_4|\ :\ \mathcal{H}_Q^{\text{NS–NS}} \otimes \Big(\mathcal{H}_Q^{\text{R–NS}} \wedge \mathcal{H}_Q^{\text{NS–R}} \wedge \mathcal{H}_Q^{\text{R–R}}\Big) \longrightarrow \mathbb{C}. \tag{4.21}$$

Here, P is the projection operator onto $\mathcal{H}_Q^{\text{R–NS}} \wedge \mathcal{H}_Q^{\text{NS–R}} \wedge \mathcal{H}_Q^{\text{R–R}}$ necessary to distinguish it from $\mathcal{H}_Q^{\text{NS–NS}} \wedge \mathcal{H}_Q^{\text{R–R}} \wedge \mathcal{H}_Q^{\text{R–R}}$, both of which contain two left-moving Ramond states and two right-moving Ramond states. The string field theory amplitude in this case is calculated as

$$\langle\mathcal{A}_4| = \langle\omega_s|\,\mathbb{I} \otimes \pi_1^{(0,0)}\bigg(\boldsymbol{b}_3^{(1,1)}\big|_{(2,2)}^{(2,2)} - \boldsymbol{b}_2^{(0,1)}\big|_{(2,0)}^{(2,0)}\frac{b_0^+X}{L_0^+}\boldsymbol{L}_2^B\big|_{(0,2)}^{(2,2)} - \boldsymbol{b}_2^{(1,0)}\big|_{(0,2)}^{(0,2)}\frac{b_0^+\bar{X}}{L_0^+}\boldsymbol{L}_2^B\big|_{(2,0)}^{(2,2)} - \boldsymbol{L}_2^B\big|_{(2,2)}^{(2,2)}\frac{b_0^+X\bar{X}}{L_0^+}\boldsymbol{L}_2^B\big|_{(0,0)}^{(2,2)}\bigg)P. \tag{4.22}$$

Using $\boldsymbol{b} = \bar{\boldsymbol{b}}\hat{\bar{\boldsymbol{F}}}$ and the recursion relation in Eq. (3.35b), we have

$$\begin{aligned}
\sum_{p=0,1}\boldsymbol{b}_2^{(p,1)}\big|^{(2(1-p),0)} &= (\bar{\boldsymbol{b}}\hat{\bar{\boldsymbol{F}}})_2^{(1)}\big|^0 = \bar{\boldsymbol{b}}_2^{(1)}\big|^0\\
&= [\boldsymbol{Q}, \boldsymbol{\mu}_2^{(1)}\big|^0], && (4.23\text{a})\\
\sum_{p=0,1,2}\boldsymbol{b}_3^{(p,1)}\big|^{(2(2-p),2)} &= (\bar{\boldsymbol{b}}\hat{\bar{\boldsymbol{F}}})_3^{(1)}\big|^2 = \bar{\boldsymbol{b}}_3^{(1)}\big|^2 + \bar{\boldsymbol{b}}_2^{(0)}\big|^2\,\bar{\Xi}\pi_1^{(*,1)}\mathcal{X}\bar{\boldsymbol{b}}_2^{(0)}\big|^2\\
&= [\boldsymbol{Q}, \boldsymbol{\mu}_3^{(1)}\big|^2] + [\![\bar{\boldsymbol{b}}_2^{(0)}\big|^2, \boldsymbol{\mu}_2^{(1)}\big|^0]\!]^{\bar{1}} + \bar{\boldsymbol{b}}_2^{(0)}\big|^2\,\bar{\Xi}\pi_1^{(*,1)}\mathcal{X}\bar{\boldsymbol{b}}_2^{(0)}\big|^2, && (4.23\text{b})
\end{aligned}$$

from which we can find that

$$\begin{aligned}
\pi_1^{(0,0)}\boldsymbol{b}_2^{(0,1)}\big|_{(2,0)}^{(2,0)} &= \pi_1^{(0,0)}[\boldsymbol{Q}, \boldsymbol{\mu}_2^{(0,1)}\big|_{(2,0)}^{(2,0)}], && (4.24\text{a})\\
\pi_1^{(0,0)}\boldsymbol{b}_3^{(1,1)}\big|_{(2,2)}^{(2,2)}P &= \pi_1^{(0,0)}\big([\boldsymbol{Q}, \boldsymbol{\mu}_3^{(1,1)}\big|_{(2,2)}^{(2,2)}] - \boldsymbol{\mu}_2^{(0,1)}\big|_{(2,0)}^{(2,0)}X\boldsymbol{L}_2^B\big|_{(0,2)}^{(2,2)}\\
&\quad + \boldsymbol{L}_2^B\big|_{(2,2)}^{(2,2)}X\,\bar{\Xi}\,\boldsymbol{L}_2^B\big|_{(0,0)}^{(2,2)} + \bar{\boldsymbol{b}}_2^{(1,0)}\big|_{(0,2)}^{(0,2)}\,\bar{\Xi}\,\boldsymbol{L}_2^B\big|_{(2,0)}^{(2,2)}\big)P && (4.24\text{b})
\end{aligned}$$

by decomposing it with respect to the cyclic Ramond and Ramond numbers. Then, the string field theory amplitude in Eq. (4.22) can be rewritten as

$$\begin{aligned}
\langle\mathcal{A}_4| = \langle\omega_l|\xi\bar{\xi}\otimes\pi_1^{(0,0)}\Big([\boldsymbol{Q},\boldsymbol{\mu}_3^{(1,1)}\big|_{(2,2)}^{(2,2)}] - \boldsymbol{\mu}_2^{(0,1)}\big|_{(2,0)}^{(2,0)}X\boldsymbol{L}_2^B\big|_{(0,2)}^{(2,2)} \\
+ \boldsymbol{L}_2^B\big|_{(2,2)}^{(2,2)}X\bar{\Xi}\,\boldsymbol{L}_2^B\big|_{(0,0)}^{(2,2)} + \bar{\boldsymbol{b}}_2^{(1,0)}\big|_{(0,2)}^{(0,2)}\bar{\Xi}\,\boldsymbol{L}_2^B\big|_{(2,0)}^{(2,2)} - [\boldsymbol{Q},\boldsymbol{\mu}_2^{(0,1)}\big|_{(2,0)}^{(2,0)}]\frac{b_0^+X}{L_0^+}\boldsymbol{L}_2^B\big|_{(0,2)}^{(2,2)} \\
- \boldsymbol{b}_2^{(1,0)}\big|_{(0,2)}^{(0,2)}\frac{b_0^+\{\boldsymbol{Q},\bar{\Xi}\}}{L_0^+}\boldsymbol{L}_2^B\big|_{(2,0)}^{(2,2)} - \boldsymbol{L}_2^B\big|_{(2,2)}^{(2,2)}\frac{b_0^+X\{\boldsymbol{Q},\bar{\Xi}\}}{L_0^+}\boldsymbol{L}_2^B\big|_{(0,0)}^{(2,2)}\Big)P \\
= -\langle\omega_l|(\xi\bar{X}_0 - X_0\bar{\xi})\otimes\pi_1^{(0,0)}\Big(\boldsymbol{\mu}_3^{(1,1)}\big|_{(2,2)}^{(2,2)} - \boldsymbol{\mu}_2^{(0,1)}\big|_{(2,0)}^{(2,0)}\frac{b_0^+X}{L_0^+}\boldsymbol{L}_2^B\big|_{(0,2)}^{(2,2)} \\
- \boldsymbol{b}_2^{(1,0)}\big|_{(0,2)}^{(0,2)}\frac{b_0^+\bar{\Xi}}{L_0^+}\boldsymbol{L}_2^B\big|_{(2,0)}^{(2,2)} - \boldsymbol{L}_2^B\big|_{(2,2)}^{(2,2)}\frac{b_0^+X\bar{\Xi}}{L_0^+}\boldsymbol{L}_2^B\big|_{(0,0)}^{(2,2)}\Big)P,
\end{aligned} \tag{4.25}$$

except for the terms which vanish when they hit the states in $\mathcal{H}_Q$. Inserting $1 = \{\bar{\eta},\bar{\xi}\}$ or $1 = \{\eta,\xi\}$, we find that

$$\begin{aligned}
\langle\mathcal{A}_4| = \langle\omega_s|\bar{X}_0\otimes\pi_1^{(0,0)}\Big((\boldsymbol{b}_H)_3^{(1)}\big|_{(2,2)}^{(2,2)} - \boldsymbol{L}_2^B\big|_{(2,0)}^{(2,0)}\frac{b_0^+X}{L_0^+}\boldsymbol{L}_2^B\big|_{(0,2)}^{(2,2)} \\
- (\boldsymbol{b}_H)_2^{(1)}\big|_{(0,2)}^{(0,2)}\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^B\big|_{(2,0)}^{(2,2)} - \boldsymbol{L}_2^B\big|_{(2,2)}^{(2,2)}\frac{b_0^+X}{L_0^+}\boldsymbol{L}_2^B\big|_{(0,0)}^{(2,2)}\Big)P
\end{aligned} \tag{4.26}$$

by using

$$[\boldsymbol{\eta},\boldsymbol{\mu}] = [\boldsymbol{\eta},\boldsymbol{L}^B] = 0 \tag{4.27}$$

and

$$[\bar{\boldsymbol{\eta}},\boldsymbol{\mu}_3^{(1,1)}\big|_{(2,2)}^{(2,2)}] = \bar{\boldsymbol{b}}_3^{(1,0)}\big|_{(2,2)}^{(2,2)} = (\boldsymbol{b}_H)_3^{(1)}\big|_{(2,2)}^{(2,2)}, \tag{4.28a}$$

$$[\bar{\boldsymbol{\eta}},\boldsymbol{\mu}_2^{(0,1)}\big|_{(2,0)}^{(2,0)}] = \bar{\boldsymbol{b}}_2^{(0,0)}\big|_{(2,0)}^{(2,0)} = \boldsymbol{L}_2^B\big|_{(2,0)}^{(2,0)}, \tag{4.28b}$$

following from Eq. (3.31).

We can repeat a similar procedure for the left-moving sector. Using $\boldsymbol{b}_H = \boldsymbol{B}\hat{\boldsymbol{F}}$, we have, in particular,

$$\begin{aligned}
(\boldsymbol{b}_H)_2^{(1)}\big|^0 = (\boldsymbol{B}\hat{\boldsymbol{F}})_2^{(1)}\big|^0 = \boldsymbol{B}_2^{(1)}\big|^0 \\
= [\boldsymbol{Q},\boldsymbol{\lambda}_2^{(1)}\big|^0],
\end{aligned} \tag{4.29a}$$

$$\begin{aligned}
(\boldsymbol{b}_H)_3^{(1)}\big|^2 = (\boldsymbol{B}\hat{\boldsymbol{F}})_3^{(1)}\big|^2 = \boldsymbol{B}_3^{(1)}\big|^2 + \boldsymbol{B}_2^{(0)}\big|^2\,\Xi\pi_1^{(1,*)}\boldsymbol{B}_2^{(0)}\big|^2 \\
= [\boldsymbol{Q},\boldsymbol{\lambda}_3^{(1)}\big|^2] + [\boldsymbol{B}_2^{(0)}\big|^2,\boldsymbol{\lambda}_2^{(1)}\big|^0]|^2 + \boldsymbol{B}_2^{(0)}\big|^2\,\Xi\pi_1^{(1,*)}\boldsymbol{B}_2^{(0)}\big|^2,
\end{aligned} \tag{4.29b}$$

by using Eq. (3.15b). Decomposing these with respect to the cyclic Ramond and Ramond numbers, we find that

$$\pi_1^{(0,0)}(\boldsymbol{b}_H)_2^{(1)}\big|_{(0,2)}^{(0,2)} = \pi_1^{(0,0)}[\boldsymbol{Q}, \boldsymbol{\lambda}_2^{(1)}\big|_{(0,2)}^{(0,2)}], \tag{4.30a}$$

$$\begin{aligned}\pi_1^{(0,0)}(\boldsymbol{b}_H)_3^{(1)}\big|_{(2,2)}^{(2,2)}P = \pi_1^{(0,0)}\Big([\boldsymbol{Q}, \boldsymbol{\lambda}_3^{(1)}\big|_{(2,2)}^{(2,2)}] - \boldsymbol{\lambda}_2^{(1)}\big|_{(0,2)}^{(0,2)}\,\boldsymbol{L}_2^B\big|_{(2,0)}^{(2,2)} \\ + \boldsymbol{L}_2^B\big|_{(2,0)}^{(2,0)}\,\Xi\boldsymbol{L}_2^B\big|_{(0,2)}^{(2,2)} + \boldsymbol{L}_2^B\big|_{(2,2)}^{(2,2)}\,\Xi\boldsymbol{L}_2^B\big|_{(0,0)}^{(2,2)}\Big)P.\end{aligned} \tag{4.30b}$$

Thanks to these relations, the amplitude in Eq. (4.26) can be further rewritten as

$$\begin{aligned}\langle\mathcal{A}_4| = \langle\omega_l|\bar{X}_0\xi\bar{\xi}\otimes\pi_1^{(0,0)}\Big(&[\boldsymbol{Q}, \boldsymbol{\lambda}_3^{(1)}\big|_{(2,2)}^{(2,2)}] - \boldsymbol{\lambda}_2^{(1)}\big|_{(0,2)}^{(0,2)}\,\boldsymbol{L}_2^B\big|_{(2,0)}^{(2,2)} \\ &+ \boldsymbol{L}_2^B\big|_{(2,0)}^{(2,0)}\,\Xi\boldsymbol{L}_2^B\big|_{(0,2)}^{(2,2)} + \boldsymbol{L}_2^B\big|_{(2,2)}^{(2,2)}\,\Xi\boldsymbol{L}_2^B\big|_{(0,0)}^{(2,2)} \\ &- \boldsymbol{L}_2^B\big|_{(2,0)}^{(2,0)}\frac{b_0^+\{Q,\Xi\}}{L_0^+}\boldsymbol{L}_2^B\big|_{(0,2)}^{(2,2)} - [\boldsymbol{Q}, \boldsymbol{\lambda}_2^{(1)}\big|_{(0,2)}^{(0,2)}]\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^B\big|_{(2,0)}^{(2,2)} \\ &- \boldsymbol{L}_2^B\big|_{(2,2)}^{(2,2)}\frac{b_0^+\{Q,\Xi\}}{L_0^+}\boldsymbol{L}_2^B\big|_{(0,0)}^{(2,2)}\Big)P \\ = -\langle\omega_l|\bar{X}_0(\xi\bar{X}_0 - X_0\bar{\xi})\otimes\pi_1^{(0,0)}\Big(&\boldsymbol{\lambda}_3^{(1)}\big|_{(2,2)}^{(2,2)} - \boldsymbol{L}_2^B\big|_{(2,0)}^{(2,0)}\frac{b_0^+\Xi}{L_0^+}\boldsymbol{L}_2^B\big|_{(0,2)}^{(2,2)} \\ &- \boldsymbol{\lambda}_2^{(1)}\big|_{(0,2)}^{(0,2)}\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^B\big|_{(2,0)}^{(2,2)} - \boldsymbol{L}_2^B\big|_{(2,2)}^{(2,2)}\frac{b_0^+\Xi}{L_0^+}\boldsymbol{L}_2^B\big|_{(0,0)}^{(2,2)}\Big)P,\end{aligned} \tag{4.31}$$

except for the terms vanishing when they hit the states in $\mathcal{H}_Q$. Again, inserting $1 = \{\eta, \xi\} = \{\bar{\eta}, \bar{\xi}\}$, the string field theory amplitude eventually becomes

$$\begin{aligned}\langle\mathcal{A}_4| = \langle\omega_s|X_0\bar{X}_0\otimes\pi_1^{(0,0)}\Big(&\boldsymbol{L}_3^B\big|_{(2,2)}^{(2,2)} - \boldsymbol{L}_2^B\big|_{(2,0)}^{(2,0)}\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^B\big|_{(0,2)}^{(2,2)} \\ &- \boldsymbol{L}_2^B\big|_{(0,2)}^{(0,2)}\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^B\big|_{(2,0)}^{(2,2)} - \boldsymbol{L}_2^B\big|_{(2,2)}^{(2,2)}\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^B\big|_{(0,0)}^{(2,2)}\Big)P,\end{aligned} \tag{4.32}$$

using $[\boldsymbol{\eta}, \boldsymbol{\lambda}^{(1)}] = \boldsymbol{L}^B$. This reproduces the first-quantized amplitude of Eq. (4.20).

5. Relation to the WZW-like formulation

So far we have constructed a complete gauge-invariant action for the type II superstring field theory in the small Hilbert space based on the cyclic L_∞ structure. In open superstring field theory [14] and heterotic string field theory [22], we can map it to a gauge-invariant action in the WZW-like formulation through a field redefinition. In this section we consider whether it is also possible to construct a complete WZW-like action in a similar way for the type II superstring field theory.

Here, let us consider the restriction of the construction to the pure NS–NS sector. If we define generating functions by

$$\boldsymbol{L}_H(s,t)|^{(0,0)} \equiv \big(\boldsymbol{Q}+\boldsymbol{B}(s,t)\big)\big|^{(0,0)} = \big(\boldsymbol{Q}+\sum_{m,n=0}^{\infty} s^m t^n \boldsymbol{B}^{(n)}_{m+n+1}\big)\big|^{(0,0)}, \tag{5.1a}$$

$$\boldsymbol{L}(s,t)\big|^{(0,0)} \equiv \big(\boldsymbol{Q}+\bar{\boldsymbol{B}}(s,t)\big)\big|^{(0,0)} = \big(\boldsymbol{Q}+\sum_{\bar{m},\bar{n}=0}^{\infty} s^{\bar{m}} t^{\bar{n}} \bar{\boldsymbol{B}}^{(\bar{n})}_{\bar{m}+\bar{n}+1}\big)\big|^{(0,0)}, \tag{5.1b}$$

they are the generating functions of $\boldsymbol{L}_H|^{(0,0)}$ and $\boldsymbol{L}|^{(0,0)}$ since the cohomomorphisms $\hat{\boldsymbol{F}}$ and $\hat{\hat{\boldsymbol{F}}}$ reduce to the identity in the NS–NS sector. The string products in the NS–NS action can be obtained by $\boldsymbol{L}|^{(0,0)} = \boldsymbol{L}(0,1)|^{(0,0)}$. These coderivations satisfy the L_∞ relations

$$[\boldsymbol{L}_H(s,t)\big|^{(0,0)}, \boldsymbol{L}_H(s,t)\big|^{(0,0)}] = 0, \tag{5.2a}$$

$$[\boldsymbol{L}(s,t)\big|^{(0,0)}, \boldsymbol{L}(s,t)\big|^{(0,0)}] = 0, \tag{5.2b}$$

and both of them are closed in the small Hilbert space:

$$[\boldsymbol{\eta}, \boldsymbol{L}_H(s,t)\big|^{(0,0)}] = [\bar{\boldsymbol{\eta}}, \boldsymbol{L}_H(s,t)\big|^{(0,0)}] = [\boldsymbol{\eta}, \boldsymbol{L}(s,t)\big|^{(0,0)}] = [\bar{\boldsymbol{\eta}}, \boldsymbol{L}(s,t)\big|^{(0,0)}] = 0. \tag{5.3}$$

The L_∞ relations in Eq. (5.2) follow from the differential equations

$$\partial_t \boldsymbol{L}_H(s,t)\big|^{(0,0)} = [\boldsymbol{L}_H(s,t)\big|^{(0,0)}, \boldsymbol{\lambda}(s,t)\big|^{(0,0)}], \tag{5.4a}$$

$$\partial_s \boldsymbol{L}_H(s,t)\big|^{(0,0)} = [\boldsymbol{\eta}, \boldsymbol{\lambda}(s,t)\big|^{(0,0)}], \tag{5.4b}$$

and

$$\partial_t \boldsymbol{L}(s,t)\big|^{(0,0)} = [\boldsymbol{L}(s,t)\big|^{(0,0)}, \bar{\boldsymbol{\lambda}}(s,t)\big|^{(0,0)}], \tag{5.5a}$$

$$\partial_s \boldsymbol{L}(s,t)\big|^{(0,0)} = [\bar{\boldsymbol{\eta}}, \bar{\boldsymbol{\lambda}}(s,t)\big|^{(0,0)}], \tag{5.5b}$$

derived from Eqs. (3.13) and (3.28), respectively, where $\boldsymbol{\lambda}(s,t)|^{(0,0)}$ and $\bar{\boldsymbol{\lambda}}(s,t)|^{(0,0)}$ are the similar restrictions of the gauge products in Eqs. (3.14) and (3.29) to the NS–NS sector. The required string products are obtained in two steps: first, we recursively solve Eq. (5.4) starting from the initial condition $\boldsymbol{L}_H(s,0)|^{(0,0)} = \boldsymbol{L}_B(s)|^{(0,0)}$, and then solve Eq. (5.5) with the initial condition $\boldsymbol{L}(s,0)|^{(0,0)} = \boldsymbol{L}_H(0,s)|^{(0,0)}$. This is nothing but the asymmetric construction proposed in Ref. [3], and thus the string and gauge products we constructed reduce in the NS–NS sector to those obtained by their asymmetric construction [3].

Using the fact that $\boldsymbol{L}(s,t)|^{(0,0)}$ satisfies the differential equation in Eq. (5.5), we can show that the string products restricted in the NS–NS sector $\boldsymbol{L}|^{(0,0)} = \boldsymbol{L}(0,1)|^{(0,0)}$ can be written in the form of the similarity transformation [14,22] as

$$\boldsymbol{L}|^{(0,0)} = \boldsymbol{Q} + \bar{\boldsymbol{B}}(0,1)\big|^{(0,0)} = \hat{\boldsymbol{g}}^{-1}\boldsymbol{Q}\hat{\boldsymbol{g}} \tag{5.6}$$

by the cohomomorphism

$$\hat{\boldsymbol{g}} = \vec{\mathcal{P}} \exp\Big(\int_0^1 dt \bar{\boldsymbol{\lambda}}(0,t)|^{(0,0)}\Big). \tag{5.7}$$

Due to the commutativity $[\boldsymbol{\eta}, \bar{\boldsymbol{\lambda}}(0,t)] = 0$ that holds by construction, however, it transforms $\boldsymbol{\eta}$ and $\bar{\boldsymbol{\eta}}$ asymmetrically. The constraints $\eta\Phi_{\text{NS–NS}} = \bar{\eta}\Phi_{\text{NS–NS}} = 0$, restricting $\Phi_{\text{NS–NS}}$ to the small Hilbert space, are mapped to

$$\eta\, \pi_1 \hat{\boldsymbol{g}}(e^{\wedge \Phi_{\text{NS—NS}}}) = \pi_1 \boldsymbol{L}^{\bar{\eta}}(e^{\wedge \pi_1 \hat{\boldsymbol{g}}(e^{\wedge \Phi_{\text{NS—NS}}})}) = 0, \tag{5.8}$$

with $\boldsymbol{L}^{\bar{\boldsymbol{\eta}}} = \hat{\boldsymbol{g}}\bar{\boldsymbol{\eta}}\hat{\boldsymbol{g}}^{-1}$. Using this map, therefore, we can only obtain a *half*-WZW-like formulation, in which the NS–NS string field V has the ghost and picture numbers $(1, 0)$ and $(-1, 0)$, respectively, and takes value in the medium Hilbert space introduced in Eq. (2.22): $V \in \mathcal{H}_{\bar{m}}$. If we identify the string field through the map

$$\pi_1 \hat{\boldsymbol{g}}(e^{\wedge \Phi_{\text{NS—NS}}}) = G_{\bar{\eta}}(V), \tag{5.9}$$

the pure-gauge string field $G_{\bar{\eta}}(V)$ satisfies the asymmetric Maurer–Cartan equations

$$\eta G_{\bar{\eta}}(V) = 0, \qquad \boldsymbol{L}^{\bar{\eta}}(e^{\wedge G_{\bar{\eta}}(V)}) = 0, \tag{5.10}$$

and thus is given by the one used in heterotic string field theory [19,22].[6] The string fields in the other sectors are simply identified in two formulations. We denote the string fields of the R–NS, NS–R, and R–R sectors in the half-WZW-like formulation as Ψ, $\bar{\Psi}$, and Σ, respectively, to distinguish which formulation they belong to:

$$\Phi_{\text{R–NS}} = \Psi, \qquad \Phi_{\text{NS–R}} = \bar{\Psi}, \qquad \Phi_{\text{R–R}} = \Sigma. \tag{5.11}$$

The half-WZW-like formulation obtained in this way is the dual (in the sense that the roles of Q and $\bar{\eta}$ are exchanged) to that given in Ref. [12]. It is not the completely WZW-like formulation defined using the (whole) large Hilbert space $\mathcal{H}_l$, but we also construct here an action and a gauge transformation to complete the story. First of all, we rewrite the action in Eq. (2.47) in the WZW-like form by extending the NS–NS string field $\Phi_{\text{NS–NS}}$ to $\Phi_{\text{NS–NS}}(t)$ with $t \in [0, 1]$ satisfying $\Phi_{\text{NS–NS}}(0) = 0$ and $\Phi_{\text{NS–NS}}(1) = \Phi_{\text{NS–NS}}$. Using the cyclicity, we find that

$$\begin{aligned}
S = &\int_0^1 dt\, \omega_m(\bar{\xi}\,\partial_t \Phi_{\text{NS–NS}}(t), \pi_1^{(0,0)} \boldsymbol{L}(e^{\wedge \Phi_{\text{NS—NS}}(t)}))\\
&+ \frac{1}{2}\omega_s(\Phi_{\text{R–NS}}, YQ\Phi_{\text{R–NS}}) + \frac{1}{2}\omega_s(\Phi_{\text{NS–R}}, \bar{Y}Q\Phi_{\text{NS–R}}) + \frac{1}{2}\omega_s(\Phi_{\text{R–R}}, \mathcal{Y}\bar{\mathcal{Y}}Q\Phi_{\text{R–R}})\\
&+ \sum_{r=0}^{\infty} \frac{1}{(2r+2)!}\Big(\omega_s\big(\Phi_{\text{R–NS}}, \pi_1^{(1,0)} \boldsymbol{b}(e^{\wedge \Phi_{\text{NS—NS}}} \wedge {\Phi_{\text{R–NS}}}^{\wedge 2r+1})\big)\\
&\qquad\qquad + \omega_s\big(\Phi_{\text{NS–R}}, \pi_1^{(0,1)} \boldsymbol{b}(e^{\wedge \Phi_{\text{NS—NS}}} \wedge {\Phi_{\text{NS–R}}}^{\wedge 2r+1})\big)\\
&\qquad\qquad\quad + \omega_s\big(\Phi_{\text{R–R}}, \pi_1^{(1,1)} \boldsymbol{b}(e^{\wedge \Phi_{\text{NS—NS}}} \wedge {\Phi_{\text{R–R}}}^{\wedge 2r+1})\big)\Big)\\
&+ \sum_{r_1, r_2=0}^{\infty} \frac{1}{(2r_1+2)!(2r_2+2)!}\\
&\qquad \times \bigg[\frac{1}{2}\Big(\omega_s\big(\Phi_{\text{R–NS}}, \pi_1^{(1,0)} \boldsymbol{b}(e^{\wedge \Phi_{\text{NS—NS}}} \wedge {\Phi_{\text{R–NS}}}^{\wedge 2r_1+1} \wedge {\Phi_{\text{NS–R}}}^{2r_2+2})\big)
\end{aligned}$$

[6] The role of η in heterotic string field theory is played here by $\bar{\eta}$.

$$
\begin{aligned}
&\qquad\qquad + \omega_s\big(\Phi_{\text{NS--R}}, \pi_1^{(0,1)}\boldsymbol{b}(e^{\wedge\Phi_{\text{NS--NS}}} \wedge \Phi_{\text{R--NS}}{}^{\wedge 2r_1+2} \wedge \Phi_{\text{NS--R}}{}^{2r_2+1})\big)\Big)\\
&\qquad\qquad + \omega_s\big(\Phi_{\text{R--R}}, \pi_1^{(1,1)}\boldsymbol{b}(e^{\wedge\Phi_{\text{NS--NS}}} \wedge \Phi_{\text{R--NS}}{}^{\wedge 2r_1+2} \wedge \Phi_{\text{R--R}}{}^{\wedge 2r_2+1})\big)\\
&\qquad\qquad + \omega_s\big(\Phi_{\text{R--R}}, \pi_1^{(1,1)}\boldsymbol{b}(e^{\wedge\Phi_{\text{NS--NS}}} \wedge \Phi_{\text{NS--R}}{}^{\wedge 2r_1+2} \wedge \Phi_{\text{R--R}}{}^{\wedge 2r_2+1})\big)\Big]\\
&+ \sum_{r_1,r_2,r_3=0}^{\infty}\bigg[\frac{1}{(2r_1+1)!(2r_2+1)!(2r_3+1)!}\\
&\times \omega_s\big(\Phi_{\text{R--R}}\,, \pi_1^{(1.1)}\boldsymbol{b}(e^{\wedge\Phi_{\text{NS--NS}}} \wedge \Phi_{\text{R--NS}}{}^{\wedge 2r_1+1} \wedge \Phi_{\text{NS--R}}{}^{\wedge 2r_2+1} \wedge \Phi_{\text{R--R}}{}^{\wedge 2r_3})\big)\\
&\qquad + \frac{1}{(2r_1+2)!(2r_2+2)!(2r_3+2)!}\\
&\times \omega_s\big(\Phi_{\text{R--R}}, \pi_1^{(1,1)}\boldsymbol{b}(e^{\wedge\Phi_{\text{NS--NS}}} \wedge \Phi_{\text{R--NS}}{}^{\wedge 2r_1+2} \wedge \Phi_{\text{NS--R}}{}^{\wedge 2r_2+2} \wedge \Phi_{\text{R--R}}{}^{\wedge 2r_3+1})\big)\bigg].
\end{aligned}
\tag{5.12}
$$

It is mapped to the half-WZW-like action through the identification in Eqs. (5.9) and (5.11) as

$$
\begin{aligned}
S = &\int_0^1 dt\, \omega_m(B_t(V(t)), QG_{\tilde\eta}(V(t)))\\
&+ \frac{1}{2}\omega_s(\Psi, YQ\Psi) + \frac{1}{2}\omega_s(\bar\Psi, \bar YQ\bar\Psi) + \frac{1}{2}\omega_s(\Sigma, \mathcal{Y}\bar{\mathcal{Y}}Q\Sigma)\\
&+ \sum_{r=0}^{\infty}\frac{1}{(2r+2)!}\Big(\omega_s\big(\Psi, \pi_1^{(1,0)}\tilde{\boldsymbol{b}}(e^{\wedge G_{\tilde\eta}(V)} \wedge \Psi^{\wedge 2r+1})\big)\\
&\qquad\qquad + \omega_s\big(\bar\Psi, \pi_1^{(0,1)}\tilde{\boldsymbol{b}}(e^{\wedge G_{\tilde\eta}(V)} \wedge \bar\Psi^{\wedge 2r+1})\big)\\
&\qquad\qquad\quad + \omega_s\big(\Sigma, \pi_1^{(1,1)}\tilde{\boldsymbol{b}}(e^{\wedge G_{\tilde\eta}(V)} \wedge \Sigma^{\wedge 2r+1})\big)\Big)\\
&+ \sum_{r_1,r_2=0}^{\infty}\frac{1}{(2r_1+2)!(2r_2+2)!}\\
&\qquad\times\bigg[\frac{1}{2}\Big(\omega_s\big(\Psi, \pi_1^{(1,0)}\tilde{\boldsymbol{b}}(e^{\wedge G_{\tilde\eta}(V)} \wedge \Psi^{\wedge 2r_1+1} \wedge \bar\Psi^{2r_2+2})\big)\\
&\qquad\qquad + \omega_s\big(\bar\Psi, \pi_1^{(0,1)}\boldsymbol{b}(e^{\wedge G_{\tilde\eta}(V)} \wedge \Psi^{\wedge 2r_1+2} \wedge \bar\Psi^{2r_2+1})\big)\Big)\\
&\qquad + \omega_s\big(\Sigma, \pi_1^{(1,1)}\tilde{\boldsymbol{b}}(e^{\wedge G_{\tilde\eta}(V)} \wedge \Psi^{\wedge 2r_1+2} \wedge \Sigma^{\wedge 2r_2+1})\big)\\
&\qquad + \omega_s\big(\Sigma, \pi_1^{(1,1)}\tilde{\boldsymbol{b}}(e^{\wedge G_{\tilde\eta}(V)} \wedge \bar\Psi^{\wedge 2r_1+2} \wedge \Sigma^{\wedge 2r_2+1})\big)\bigg]\\
&+ \sum_{r_1,r_2,r_3=0}^{\infty}\bigg[\frac{1}{(2r_1+1)!(2r_2+1)!(2r_3+1)!}\\
&\times \omega_s\big(\Sigma, \pi_1^{(1.1)}\tilde{\boldsymbol{b}}(e^{\wedge G_{\tilde\eta}(V)} \wedge \Psi^{\wedge 2r_1+1} \wedge \bar\Psi^{\wedge 2r_2+1} \wedge \Sigma^{\wedge 2r_3})\big)
\end{aligned}
$$

$$+\frac{1}{(2r_1+2)!(2r_2+2)!(2r_3+2)!}$$
$$\times\, \omega_s\big(\Sigma, \pi_1^{(1,1)}\tilde{\boldsymbol{b}}(e^{\wedge G_{\bar{\eta}}(V)} \wedge \Psi^{\wedge 2R_1+2} \wedge \bar{\Psi}^{\wedge 2R_2+2} \wedge \Sigma^{\wedge 2R_3+1})\big)\Bigg], \tag{5.13}$$

where $\tilde{\boldsymbol{b}} = \hat{\boldsymbol{g}}(\boldsymbol{b} - \bar{\boldsymbol{B}}(0,1)|^{(0,0)})\hat{\boldsymbol{g}}^{-1}$. Here, we defined the associated fields as

$$B_d(V(t)) = \pi_1^{(0,0)}\hat{\boldsymbol{g}}\bar{\boldsymbol{\xi}}_{\boldsymbol{d}}(e^{\wedge \Phi_{\mathrm{NS-NS}}(t)}), \tag{5.14}$$

with $d = \partial_t$ or δ, by introducing one-coderivations $\bar{\boldsymbol{\xi}}_{\boldsymbol{d}}$ derived from $\bar{\xi}d$. We can show that they satisfy the characteristic identities of the associated fields,

$$dG_{\bar{\eta}}(V(t)) = \pi_1^{(0,0)}\boldsymbol{L}^{\bar{\eta}}(e^{\wedge G_{\bar{\eta}}(V(t))} \wedge B_d(V(t))), \tag{5.15a}$$

$$D_{\bar{\eta}}(t)\big(\partial_t B_\delta(V(t)) - \delta B_t(V(t)) - \pi_1^{(0,0)}\boldsymbol{L}^{\bar{\eta}}(e^{\wedge G_{\bar{\eta}}(V(t))} \wedge B_t(V(t)) \wedge B_\delta(V(t)))\big) = 0. \tag{5.15b}$$

The nilpotent linear operator $D_{\bar{\eta}}(t)$ was introduced as

$$D_{\bar{\eta}}(t)\varphi = \pi_1^{(0,0)}\boldsymbol{L}^{\bar{\eta}}\big(e^{\wedge G_{\bar{\eta}}(V(t))} \wedge \varphi\big), \tag{5.16}$$

for a general string field $\varphi \in \mathcal{H}_{\tilde{m}}$.

The gauge transformation

$$\pi_1\delta(e^{\wedge\Phi}) = \pi_1\boldsymbol{L}(e^{\wedge\Phi} \wedge \Lambda) \tag{5.17}$$

generated by the parameter $\Lambda = \Lambda_{\mathrm{NS-NS}} + \Lambda_{\mathrm{R-NS}} + \Lambda_{\mathrm{NS-R}} + \Lambda_{\mathrm{R-R}}$ is also mapped to that in the half-WZW-like formulation with the gauge parameters

$$\Lambda = -\pi_1^{(0,0)}\hat{\boldsymbol{g}}(e^{\wedge\Phi_{\mathrm{NS-NS}}} \wedge \bar{\xi}\Lambda_{\mathrm{NS-NS}}), \quad \lambda = \Lambda_{\mathrm{R-NS}}, \quad \bar{\lambda} = \Lambda_{\mathrm{NS-R}}, \quad \rho = \Lambda_{\mathrm{R-R}}, \tag{5.18}$$

as

$$B_\delta(V) = \pi_1^{(0,0)}\tilde{\boldsymbol{L}}\big(e^{\wedge G_{\bar{\eta}}+\Psi+\bar{\Psi}\Sigma} \wedge (\Lambda - \bar{\xi}\lambda - \bar{\xi}\bar{\lambda} - \bar{\xi}\rho)\big), \tag{5.19a}$$

$$\delta\Psi = -\pi_1^{(1,0)}\tilde{\boldsymbol{L}}\big(e^{\wedge G_{\bar{\eta}}+\Psi+\bar{\Psi}\Sigma} \wedge (\bar{\eta}\Lambda - \lambda - \bar{\lambda} - \rho)\big), \tag{5.19b}$$

$$\delta\bar{\Psi} = -\pi_1^{(0,1)}\tilde{\boldsymbol{L}}\big(e^{\wedge G_{\bar{\eta}}+\Psi+\bar{\Psi}\Sigma} \wedge (\bar{\eta}\Lambda - \lambda - \bar{\lambda} - \rho)\big), \tag{5.19c}$$

$$\delta\Sigma = -\pi_1^{(1,1)}\tilde{\boldsymbol{L}}\big(e^{\wedge G_{\bar{\eta}}+\Psi+\bar{\Psi}\Sigma} \wedge (\bar{\eta}\Lambda - \lambda - \bar{\lambda} - \rho)\big), \tag{5.19d}$$

where

$$\pi_1\tilde{\boldsymbol{L}} = \pi_1\hat{\boldsymbol{g}}\boldsymbol{L}\hat{\boldsymbol{g}}^{-1} = \pi_1\boldsymbol{Q} + \mathcal{G}\tilde{\boldsymbol{b}}. \tag{5.20}$$

There is also an extra gauge invariance in the half-WZW-like formulation under the transformation

$$B_\delta(V) = D_{\bar{\eta}}\Omega, \tag{5.21}$$

because the identification in Eq. (5.9) is not one-to-one but

$$\delta G_{\bar{\eta}}(V) = D_{\bar{\eta}}B_\delta(V) = D_{\bar{\eta}}D_{\bar{\eta}}\Omega = 0. \tag{5.22}$$

The identities in Eq. (5.15) are enough to guarantee that the action in Eq. (5.13) is invariant under the gauge transformations in Eqs. (5.19) and (5.21) independently with the gauge invariance in the homotopy algebraic formulation.

6. Summary and discussion

Extending the procedure for constructing the heterotic string field theory, we have constructed the type II superstring field theory with a cyclic L_∞ structure based on the homotopy algebraic formulation. In addition to the closed string constraints, we impose extra constraints on the string fields in the R–NS, NS–R, and R–R sectors. These constraints restrict the dependence of these string fields on the bosonic ghost zero modes, and also make the field–anti-field decomposition in the BV quantization obvious. Although the kinetic term of the R–R string field is non-local, it provides the same propagator in the Siegel gauge as that naturally obtained in the first-quantized formulation. Repeating the procedure used in the construction of the heterotic string products, we have constructed the string products for the type II superstring with the cyclic L_∞ structure acting across all the NS–NS, R–NS, NS–R, and R–R sectors. We can map the action and the gauge transformation to those in the half-WZW-like formulation defined using the medium Hilbert space, although not in the completely WZW-like formulation in the large Hilbert space.

A remaining interesting task is to construct a completely WZW-like action in the large Hilbert space. In the language introduced in Ref. [12], the similarity transformation generated by the cohomomorphism in Eq. (5.7) maps the small Hilbert space L_∞ triplet $(\boldsymbol{\eta}, \bar{\boldsymbol{\eta}}; \boldsymbol{L}^{\mathrm{NS,NS}})$ to the asymmetric (heterotic) one $(\boldsymbol{\eta}, \boldsymbol{L}^{\bar{\eta}}; \boldsymbol{Q})$, while the symmetric triplet $(\boldsymbol{L}^{\eta}, \boldsymbol{L}^{\bar{\eta}}; \boldsymbol{Q})$ is necessary to realize the complete WZW-like formulation. In order to realize it and couple the NS–NS action in Ref. [33] to the string fields in the other sectors, it seems to be necessary to find a construction which is an extension of the symmetric construction proposed in Ref. [3]. Our method used in this paper, which was developed in Ref. [22] for constructing the heterotic string field theory, cannot be extended that way so some completely different approach seems to be needed.

Finally, it should be emphasized that the type II superstring field theory has the possibility of providing a solid basis to the AdS/CFT correspondence which is still mysterious and must be proved. We hope that the gauge-invariant action we have constructed will help us explore such an interesting possibility.

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Appendix A. Expanding with respect to ghost zero modes

In this appendix we expand the string field with respect to the ghost zero modes. After summarizing the Fock representation, an explicit expression is given for each sector.

Appendix A.1. Fock representation of ghost zero modes

Appendix A.1.1. fermionic ghost

The fermionic ghost zero modes (b_0, c_0) satisfy the anti-commutation relation

$$\{b_0, c_0\} = 1, \tag{A.1}$$

and the Hermite and BPZ conjugate relations

$$(b_0)^\dagger = b_0, \qquad (c_0)^\dagger = c_0, \tag{A.2}$$

$$\mathrm{bpz}(b_0) = b_0, \qquad \mathrm{bpz}(c_0) = -c_0. \tag{A.3}$$

Its Fock representation is two-dimensional space spanned by two states,

$$\{|\downarrow\rangle, |\uparrow\rangle\}, \tag{A.4}$$

where

$$b_0|\downarrow\rangle = 0, \qquad |\uparrow\rangle = c_0|\downarrow\rangle. \tag{A.5}$$

Its dual space is spanned by their BPZ conjugates

$$\{\langle\downarrow|, \langle\uparrow|\}, \tag{A.6}$$

with

$$\langle\downarrow|b_0 = 0, \qquad \langle\uparrow| = \langle\downarrow|c_0. \tag{A.7}$$

The inner-product matrix between two spaces is off-diagonal:

$$\langle\uparrow|\downarrow\rangle = \langle\downarrow|c_0|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = 1, \qquad \langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 0. \tag{A.8}$$

Since there are two pairs of fermionic ghost zero modes, (b_0, c_0) and $(\bar{b}_0, \bar{c}_0)$, in the closed string theory, their Fock representation is four-dimensional,

$$\{|\downarrow\downarrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\uparrow\rangle\}, \tag{A.9}$$

where $|\downarrow\downarrow\rangle \propto |\downarrow\rangle \otimes |\downarrow\rangle$. One of the closed string constraints $b_0^-|\downarrow\downarrow\rangle = 0$ restricts it to the two-dimensional space,

$$\{|\downarrow\downarrow\rangle, c_0^+|\downarrow\downarrow\rangle\}. \tag{A.10}$$

We normalize the states so that

$$\langle\downarrow\downarrow|c_0^+ c_0^-|\downarrow\downarrow\rangle = 1. \tag{A.11}$$

Appendix A.1.2. Bosonic ghost

There are also bosonic ghosts (β_n, γ_n) in the R sector. They satisfy the commutation relation

$$[\gamma_n, \beta_m] = \delta_{n+m,0}, \tag{A.12}$$

and the Hermite and BPZ conjugate relations[7]

$$(\gamma_n)^\dagger = \gamma_{-n}, \qquad (\beta_n)^\dagger = -\beta_{-n}, \tag{A.13}$$

$$\mathrm{bpz}(\gamma_n) = e^{-i\pi(n+\frac{1}{2})}\gamma_{-n}, \qquad \mathrm{bpz}(\beta_n) = e^{-i\pi(n-\frac{3}{2})}\beta_{-n}. \tag{A.14}$$

In general it is known that they have infinitely many Fock representations defined on the ground states with the picture number p as [34]

$$\beta_n|p\rangle = 0 \qquad \text{for } n > -p - \frac{3}{2}, \tag{A.15}$$

$$\gamma_n|p\rangle = 0 \qquad \text{for } n \geq p + \frac{3}{2}. \tag{A.16}$$

In string field theory we use two natural representations with $p = -1/2$ and $-3/2$, whose non-zero mode parts are common Fock space obtained by acting $(\beta_{-n}, \gamma_{-n})$ with $n > 0$ on the ground state $|0\rangle\rangle$ satisfying

$$\beta_n|0\rangle\rangle = \gamma_n|0\rangle\rangle = 0, \qquad n > 0. \tag{A.17}$$

Two representations with $p = -1/2$ and $-3/2$ are the direct product of this common non-zero mode part and the zero mode parts obtained by acting γ_0 and β_0 on the ground states $|0\rangle$ and $|\tilde{0}\rangle$ defined by

$$\beta_0|0\rangle = 0, \qquad \gamma_0|\tilde{0}\rangle = 0, \tag{A.18}$$

respectively. The representations of the zero modes,

$$\{|0\rangle, \gamma_0|0\rangle, (\gamma_0)^2|0\rangle, \ldots\} \quad \text{and} \quad \{|\tilde{0}\rangle, \beta_0|\tilde{0}\rangle, (\beta_0)^2|\tilde{0}\rangle, \ldots\}, \tag{A.19}$$

are infinite-dimensional and dual to each other with respect to the BPZ inner product induced by

$$\langle\tilde{0}|0\rangle = \langle 0|\tilde{0}\rangle = 1. \tag{A.20}$$

In order to intertwine two representations, we can introduce the *delta functions* $\delta(\gamma_0)$ and $\delta(\beta_0)$ as the Grassmann odd operators which satisfy

$$\gamma_0\delta(\gamma_0) = \delta(\gamma_0)\gamma_0 = 0, \qquad \beta_0\delta(\beta_0) = \delta(\beta_0)\beta_0 = 0, \tag{A.21}$$

$$\delta(\beta_0)\delta(\gamma_0)\delta(\beta_0) = \delta(\beta_0), \qquad \delta(\gamma_0)\delta(\beta_0)\delta(\gamma_0) = \delta(\gamma_0), \tag{A.22}$$

[7] Fractional phases in Eq. (A.14) become a simple sign factor on the Fock states restricted by the GSO projection.

and are (graded) commutative with the operators other than (β_0, γ_0). Then, two ground states $|0\rangle$ and $|\tilde{0}\rangle$ can be related as

$$|0\rangle = \delta(\beta_0)|\tilde{0}\rangle, \qquad |\tilde{0}\rangle = \delta(\gamma_0)|0\rangle. \tag{A.23}$$

The inner products in Eq. (A.20) provide

$$\langle 0|\delta(\gamma_0)|0\rangle = \langle\tilde{0}|\delta(\beta_0)|\tilde{0}\rangle = 1. \tag{A.24}$$

These apparently strange operators can be defined by

$$\delta(\gamma_0) = |\tilde{0}\rangle\langle\tilde{0}|, \qquad \delta(\beta_0) = |0\rangle\langle 0|, \tag{A.25}$$

if necessary, and are closely related to a geometric object, the integral form on the super-moduli space of the super Riemann manifold [35,36].[8]

Appendix A.2. Zero-mode expansion of string fields

Appendix A.2.1. NS–NS sector

In the NS–NS sector, only the fermionic ghosts have zero modes (b_0, c_0) and $(\bar{b}_0, \bar{c}_0)$. The NS–NS string field $\Phi_{\text{NS–NS}}$ constrained by Eq. (2.4) can be expanded with respect to these ghost zero modes as

$$\Phi_{\text{NS–NS}} = \phi_{\text{NS–NS}} - c_0^+ \psi_{\text{NS–NS}}. \tag{A.26}$$

From its Fock representation, we can separate the ghost zero-mode dependence as

$$|\phi_{\text{NS–NS}}\rangle = |\downarrow\downarrow\rangle \otimes |\phi_{\text{NS–NS}}\rangle\!\rangle, \qquad |\psi_{\text{NS–NS}}\rangle = |\downarrow\downarrow\rangle \otimes |\psi_{\text{NS–NS}}\rangle\!\rangle, \tag{A.27}$$

where the state denoted as $|\phi_{\text{NS–NS}}\rangle\!\rangle$ or $|\psi_{\text{NS–NS}}\rangle\!\rangle$ represents the non-zero mode part of the Fock representation of the string field. This expansion holds independent of whether the ghost number of $\Phi_{\text{NS–NS}}$ is restricted or not.

Appendix A.2.2. R–NS sector

In the R–NS sector there are also the bosonic ghost zero modes (β_0, γ_0) in the left-moving sector. The R–NS string field $\Phi_{\text{R–NS}}$ restricted by the constraints in Eqs. (2.4) and (2.6) can be expanded as

$$\Phi_{\text{R–NS}} = \phi_{\text{R–NS}} - \frac{1}{2}(\gamma_0 + 2c_0^+ G)\psi_{\text{R–NS}}, \tag{A.28}$$

in which the ghost zero-mode dependence can be separated as

$$|\phi_{\text{R–NS}}\rangle = |\downarrow\downarrow\rangle \otimes |0\rangle \otimes |\phi_{\text{R–NS}}\rangle\!\rangle, \qquad |\psi_{\text{R–NS}}\rangle = |\downarrow\downarrow\rangle \otimes |0\rangle \otimes |\psi_{\text{R–NS}}\rangle\!\rangle. \tag{A.29}$$

The states denoted as $|\phi_{\text{R–NS}}\rangle\!\rangle$ and $|\psi_{\text{R–NS}}\rangle\!\rangle$ are the string field after separating the ghost-zero modes. The zero modes can be integrated out by using the inner product

$$\langle 0| \otimes \langle\downarrow\downarrow| c_0^+ c_0^- \delta(\gamma_0)|\downarrow\downarrow\rangle \otimes |0\rangle = 1. \tag{A.30}$$

[8] The authors would like to thank Pietro Antonio Grassi, from whom they learned much about the integral form [37–39].

Appendix A.2.3. NS–R sector

In the NS–R sector the bosonic ghost zero modes $(\bar{\beta}_0, \bar{\gamma}_0)$ are in the right-moving sector. The NS–R string field $\Phi_{\text{NS–R}}$ restricted by the constraints in Eqs. (2.4) and (2.6) can be expanded as

$$\Phi_{\text{NS–R}} = \phi_{\text{NS–R}} - \frac{1}{2}(\bar{\gamma}_0 + 2c_0^+ \bar{G})\psi_{\text{NS–R}}, \tag{A.31}$$

and we can separate the zero-mode part as

$$|\phi_{\text{NS–R}}\rangle = |\downarrow\downarrow\rangle \otimes |0\rangle \otimes |\phi_{\text{NS–R}}\rangle\!\rangle, \qquad |\psi_{\text{NS–R}}\rangle = |\downarrow\downarrow\rangle \otimes |0\rangle \otimes |\psi_{\text{NS–R}}\rangle\!\rangle. \tag{A.32}$$

The ghost zero modes can be integrated out using

$$\langle 0| \otimes \langle\downarrow\downarrow| c_0^+ c_0^- \delta(\bar{\gamma}_0) |\downarrow\downarrow\rangle \otimes |0\rangle = 1. \tag{A.33}$$

Appendix A.2.4. R–R sector

In the R–R sector, there are the bosonic ghost zero modes (β_0, γ_0) and $(\bar{\beta}_0, \bar{\gamma}_0)$ in both the left-moving and right-moving sectors. The R–R string field restricted by the constraints in Eqs. (2.4) and (2.14) can be expanded as

$$\Phi_{\text{R–R}} = \phi_{\text{R–R}} - \frac{1}{2}(\gamma_0 \bar{G} - \bar{\gamma}_0 G + 2c_0^+ G\bar{G})\psi_{\text{R–R}}, \tag{A.34}$$

in which we can further separate the ghost zero modes as

$$|\phi_{\text{R–R}}\rangle = |\downarrow\downarrow\rangle \otimes |0\rangle \otimes |0\rangle \otimes |\phi_{\text{R–R}}\rangle\!\rangle, \qquad |\psi_{\text{R–R}}\rangle = |\downarrow\downarrow\rangle \otimes |0\rangle \otimes |0\rangle \otimes |\psi_{\text{R–R}}\rangle\!\rangle, \tag{A.35}$$

with the string fields $|\phi_{\text{R–R}}\rangle\!\rangle$ and $|\psi_{\text{R–R}}\rangle\!\rangle$ constructed only on the non-zero-mode Fock space. The zero modes can be integrated out by using the non-trivial inner product

$$\langle 0| \otimes \langle 0| \otimes \langle\downarrow\downarrow| c_0^+ c_0^- \delta(\gamma_0)\delta(\bar{\gamma}_0) |\downarrow\downarrow\rangle \otimes |0\rangle \otimes |0\rangle = 1. \tag{A.36}$$

In order to construct the Sen-type action, we have to introduce an extra string field with picture number $(-3/2, -3/2)$, whose zero-mode ground state $|\downarrow\downarrow\rangle \otimes |\tilde{0}\rangle \otimes |\tilde{0}\rangle$ is related to $|\downarrow\downarrow\rangle \otimes |0\rangle \otimes |0\rangle$ through the relations

$$|\downarrow\downarrow\rangle \otimes |0\rangle \otimes |0\rangle = \delta(\bar{\beta}_0)\delta(\beta_0) |\downarrow\downarrow\rangle \otimes |\tilde{0}\rangle \otimes |\tilde{0}\rangle, \tag{A.37}$$

$$|\downarrow\downarrow\rangle \otimes |\tilde{0}\rangle \otimes |\tilde{0}\rangle = \delta(\gamma_0)\delta(\bar{\gamma}_0) |\downarrow\downarrow\rangle \otimes |0\rangle \otimes |0\rangle. \tag{A.38}$$

Their BPZ conjugates are

$$\langle 0| \otimes \langle 0| \otimes \langle\downarrow\downarrow| = \langle\tilde{0}| \otimes \langle\tilde{0}| \otimes \langle\downarrow\downarrow| \delta(\bar{\beta}_0)\delta(\beta_0), \tag{A.39}$$

$$\langle\tilde{0}| \otimes \langle\tilde{0}| \otimes \langle\downarrow\downarrow| = \langle 0| \otimes \langle 0| \otimes \langle\downarrow\downarrow| \delta(\gamma_0)\delta(\bar{\gamma}_0). \tag{A.40}$$

The zero-mode integration can be performed by using the inner product

$$\langle\tilde{0}| \otimes \langle\tilde{0}| \otimes \langle\downarrow\downarrow| c_0^+ c_0^- \delta(\bar{\beta}_0)\delta(\beta_0) |\downarrow\downarrow\rangle \otimes |\tilde{0}\rangle \otimes |\tilde{0}\rangle = 1. \tag{A.41}$$

The extra string field $\tilde{\Phi}_{\text{R–R}}$ restricted by the constraint in Eq. (2.42) can be expanded with respect to the ghost zero modes as

$$\tilde{\Phi}_{\text{R–R}} = \tilde{\phi}_{\text{R–R}} - c_0^+ \tilde{\psi}_{\text{R–R}}, \tag{A.42}$$

from which we can separate the ghost zero modes as

$$\tilde{\phi}_{\mathrm{R-R}} = |\downarrow\downarrow\rangle \otimes |\tilde{0}\rangle \otimes |\tilde{0}\rangle \otimes |\tilde{\phi}_{\mathrm{R-R}}\rangle\!\rangle, \tag{A.43a}$$

$$\tilde{\psi}_{\mathrm{R-R}} = |\downarrow\downarrow\rangle \otimes |\tilde{0}\rangle \otimes |\tilde{0}\rangle \otimes |\tilde{\psi}_{\mathrm{R-R}}\rangle\!\rangle. \tag{A.43b}$$

It can be shown that $X\bar{X}\tilde{\Phi}_{\mathrm{R-R}}$ obtained by changing the picture actually has the form of Eq. (2.15):

$$X\bar{X}\tilde{\Phi}_{\mathrm{R-R}} = G\bar{G}\tilde{\tilde{\phi}}_{\mathrm{R-R}} - \frac{1}{2}(\gamma_0\bar{G} - \bar{\gamma}_0 G + 2c_0^+ G\bar{G})\tilde{\tilde{\psi}}_{\mathrm{R-R}}, \tag{A.44}$$

where

$$\tilde{\tilde{\phi}}_{\mathrm{R-R}} = |\downarrow\downarrow\rangle \otimes |0\rangle \otimes |0\rangle \otimes |\tilde{\phi}_{\mathrm{R-R}}\rangle\!\rangle, \tag{A.45a}$$

$$\tilde{\tilde{\psi}}_{\mathrm{R-R}} = |\downarrow\downarrow\rangle \otimes |0\rangle \otimes |0\rangle \otimes |\tilde{\psi}_{\mathrm{R-R}}\rangle\!\rangle. \tag{A.45b}$$

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