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Interface Vanishing for Solutions to Maxwell and Stokes Systems

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In the previous work [1], we studied interface regularity of three dimensional Maxwell system when the interface is C^2, and that of Stokes system when it is flat. In this article, we continue the study and show refined interface vanishing theorems for general interface.

Geometric situation which we are concerned in is described as follows. Namely, $\Omega\subset\mathbf{R}^3$ denotes a bounded domain with Lipschitz boundary $\partial\Omega$, and $\mathcal{M}\subset\mathbf{R}^3$ is a Lipschitz hypersurface cutting Ω transversally. Thus, it holds that

$$\begin{aligned}&\mathcal{M}\cap\Omega\neq\phi\\ &\Omega=\Omega_+\cup(\Omega\cap\mathcal{M})\cup\Omega_- \quad \text{(disjoint union)}\end{aligned} \tag{0.1}$$

with the open subsets $\Omega_\pm$ of Ω. First, we consider the Maxwell system in magnetostatics,

$$\left.\begin{aligned}\nabla\times B=J\\ \nabla\cdot B=0\end{aligned}\right\} \quad \text{in} \quad \Omega_\pm, \tag{0.2}$$

where $B={}^t(B^1(x),B^2(x),B^3(x))$ and $J={}^t(J^1(x),J^2(x),J^3(x))$ stand for three dimensional vector fields, indicating magnetic field and total current density, respectively. Furthermore, $\nabla = {}^t(\partial_1, \partial_2, \partial_3)$ denotes the gradient operator and $\times$ and $\cdot$ are outer and inner products in $\mathbf{R}^3$, so that $\nabla\times$ and $\nabla\cdot$ are the operations of rotation and divergence, respectively.

In the context of magnetoencephalography, Suzuki, Watanabe, and Shimogawara [3] introduced an interface vanishing theorem when the interface is given by the boundary ∂D of a smooth bounded domain $D \subset \mathbf{R}^3$ in use of the layer potential. More precisely, if J is continuous on $\overline{\Omega_\pm}$ and system (0.2) has a solution $B \in C(\mathbf{R}^3)^3 \cap C^1(\mathbf{R}^3 \setminus \partial D)^3$ for $\Omega_- = D$ and $\Omega_+ = \mathbf{R}^3 \setminus D$, then

$$[\nabla(n \cdot B)]_-^+ = 0 \quad \text{on} \quad \partial D$$

follows, regardless with the continuity of J across ∂D. Here, n denotes the outer unit normal vector to ∂D, $[A]_-^+ = A_+ - A_-$, and

$$A_+(\xi) = \lim_{x \to \xi,\, x \in \mathbf{R}^3 \setminus D} A(x), \qquad A_-(\xi) = \lim_{x \to \xi,\, x \in D} A(x)$$

for $\xi \in \partial D$. Then, Kobayashi, Suzuki and Watanabe [1] studied local version, the case where the bounded domain Ω is given with the interface $\mathcal{M} \cap \Omega$ as in (0.1), and showed that even if $n \times J$ has an interface on $\mathcal{M} \cap \Omega$, the normal component $n \cdot B$ of B gains the regularity in one rank. In this article, we refine the argument and reduce smoothness of the interface. This refinement is very useful to study similar problems for the Stokes system as will be described later.

To state our result for (0.2), we take some preliminaries on function spaces from Girault and Raviart [2]. Namely, if $D \subset \mathbf{R}^3$ is a bounded domain with Lipschitz boundary ∂D and n denotes the unit normal vector to ∂D, then for $p \in [1, \infty]$, $L^p(D)$ denotes the standard L^p space on D provided with the norm $\|\cdot\|_{L^p(D)}$, and the Sobolev space $W^{m,p}(D)$ is defined by

$$W^{m,p}(D) = \{u \in L^p(D) \mid \partial^\alpha u \in L^p(D) \quad \text{for} \quad |\alpha| \le m\}$$

for a positive integer m, where $\partial^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ for the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Put $H^m(D) = W^{m,2}(D)$. Given $\sigma \in (0,1)$, we say that $u \in H^{m+\sigma}(D)$ if $u \in H^m(D)$ and

$$\int_D \int_D \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x-y|^{n+2\sigma}} dxdy < +\infty$$

for any α in $|\alpha| = m$ and $n = 3$. The space $H^s(\Gamma)$ is defined similarly with $n = 2$ through the local chart of Γ, where $s \in [0,1]$ and $\Gamma \subset \partial D$ is a relatively open connected set. Then, we set $H^{-s}(\Gamma) = H_0^s(\Gamma)'$, where $H_0^s(\Gamma)$ denotes the closure in $H^s(\Gamma)$ of the space composed of Lipschitz continuous functions on Γ with compact supports. Thus, we have $H_0^s(\Gamma) = H^s(\Gamma)$ if $\Gamma \subset \partial D$ is a closed surface, and in particular, it holds that $H^{1/2}(\partial D) = H_0^{1/2}(\partial D)$. In this context, let us remember the standard trace theorem that $H^1(D)|_{\partial D} \cong H^{1/2}(\partial D)$. We also put

$$H(\mathrm{div}, D) = \left\{u \in L^2(D)^3 \mid \nabla \cdot u \in L^2(D)\right\}$$

and

$$H(\mathrm{rot}, D) = \left\{u \in L^2(D)^3 \mid \nabla \times u \in L^2(D)^3\right\}.$$

Here and henceforth, $(\cdot,\cdot)_D$ and $((\cdot,\cdot))_D$ denote $L^2(D)$ and $L^2(D)^3$ inner products, respectively, and $\langle\cdot,\cdot\rangle_{\partial D}$ and $\langle\langle\cdot,\cdot\rangle\rangle_{\partial D}$ the duality pairing between $H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D) = H_0^{1/2}(\partial D)$, and $H^{-1/2}(\partial D)^3$ and $H^{1/2}(\partial D)^3$, respectively. Then we have the following.

Proposition 0.1 *Each $v \in H(div, D)$ admits the trace*

$$n \cdot v|_{\partial D} \in H^{-1/2}(\partial D),$$

and Green's formula

$$((v, \nabla\varphi))_D + (\nabla \cdot v, \varphi)_D = \langle n \cdot v, \varphi\rangle_{\partial D}$$

holds for $\varphi \in H^1(D)$.

Proposition 0.2 *Each $v \in H^1(rot, D)$ admits the trace*

$$n \times v|_{\partial D} \in H^{-1/2}(\partial D)^3,$$

and the Stokes formula

$$((\nabla \times v, w))_D - ((v, \nabla \times w))_D = \langle\langle n \times v, w\rangle\rangle_{\partial D}$$

holds for $w \in H^1(D)^3$.

To discuss the interface regularity of the solution B to the Maxwell system (0.2), we take that

$$\Gamma_\pm = \partial\Omega_\pm \cap \mathcal{M}$$

with $\partial\Omega_\pm$ being the boundary of $\Omega_\pm$. This means that Γ_+ and Γ_- coincide as sets, but are regarded as parts of the boundaries of Ω_+ and Ω_-, respectively. Henceforth, n denotes the outer unit normal vector to Γ_- so that $-n$ is the outer unit normal vector to Γ_+. Unless otherwise stated, if $\mathcal{M}$ is $C^{k,1}$, then $C^{k,1} \cap W^{k+1,\infty}_{loc}$ extension of the normal vector n defined on $\Gamma = \mathcal{M} \cap \Omega$ is always taken to Ω henceforth, where k is a non-negative integer. Furthermore, given a function $A(x)$ on $\Omega_\pm$, we set

$$[A]^+_- = A_+ - A_- \qquad \text{on} \qquad \Gamma,$$

where $A_\pm(\xi) = \lim_{x\to\xi,\, x\in\Omega_\pm} A(x)$ for $\xi \in \Gamma$ are always taken in the sense of traces to $\Gamma_\pm$.

Suppose that B and J are in $L^2(\Omega_\pm)^3$ and satisfy (0.2). This means that those relations hold piecewisely in $\Omega_\pm$ in the sense of distributions $\mathcal{D}'(\Omega_\pm)$, that is,

$$\int_{\Omega_\pm} B\cdot\nabla\times C = \int_{\Omega_\pm} J\cdot C \qquad \text{and} \qquad \int_{\Omega_\pm} B\cdot\nabla\varphi = 0$$

for any $C \in C_0^\infty(\Omega_\pm)^3$ and $\varphi \in C_0^\infty(\Omega_\pm)$. Unless otherwise stated, those vector fields $B \in L^2(\Omega_\pm)$ and $J \in L^2(\Omega_\pm)^3$ are identified with the elements in $L^2(\Omega)^3$.

For the moment, we assume that $\mathcal{M}$ is Lipschitz continuous and relation (0.2) holds for $B \in L^2(\Omega_\pm)^3$ and $J \in L^2(\Omega_\pm)^3$. This implies that $B \in H(\text{rot}, \Omega_\pm) \cap H(\text{div}, \Omega_\pm)$, which assures the well-definedness of

$$n \times B|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)^3 \qquad \text{and} \qquad n\cdot B|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm),$$

and hence $B|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)^3$ follows. Furthermore,

$$[n\times B]^+_- = 0 \qquad \text{and} \qquad [n\cdot B]^+_- = 0 \tag{0.3}$$

if and only if

$$\nabla\times B = J \in L^2(\Omega)^3 \qquad \text{and} \qquad \nabla\cdot B = 0 \in L^2(\Omega) \tag{0.4}$$

as distributions in Ω, respectively. If both relations of (0.4) are satisfied, then $B \in H^1(\Omega)^3$ follows, because $B \in H^1(\Omega)^3$ is equivalent to $[B]^+_- = 0$ on Γ for $B \in H^1(\Omega_\pm)^3$. A slightly weaker fact $B \in H^1_{loc}(\Omega)^3$ is also obtained by Corollary I.2.10 of [2],

$$H(\text{rot}, \Omega) \cap H(\text{div}, \Omega) \subset H^1_{loc}(\Omega)^3, \tag{0.5}$$

as (0.4) implies $B \in H^1_{loc}(\Omega)^3$.

Our first result is stated as follows. There, $\mathcal{M}$ is supposed to be $C^{1,1}$ and hence $C^{1,1} \cap W^{2,\infty}_{loc}(\Omega)$ extension is taken to n.

Theorem 0.1 *If $\mathcal{M}$ is $C^{1,1}$, and $B \in H^1(\Omega)^3$ and $J \in H(rot, \Omega_\pm)$ satisfy (0.2), then it holds that $n \cdot B \in H^2_{loc}(\Omega)$.*

Above theorem is a slight improvement of a theorem of [1], but new argument for the proof is presented. Similarly to that case, $B \in H^1(\Omega)^3$ solves (0.2) in Ω as a distribution, so that

$$\int_\Omega B \cdot \nabla \times C = \int_\Omega J \cdot C \qquad \text{and} \qquad \int_\Omega B \cdot \nabla\varphi = 0$$

hold for any $C \in C_0^\infty(\Omega)^3$ and $\varphi \in C_0^\infty(\Omega)$. We note that $J \in H(\mathrm{rot}, \Omega_\pm)$ belongs to $J \in H(\mathrm{rot}, \Omega)$ if and only if $[n \times J]_-^+ = 0$ on Γ, and if this condition is satisfied furthermore, then we have

$$-\Delta B = \nabla \times J \in L^2(\Omega)^3$$

(as distributions in Ω), because $\nabla \times B = J \in H(\mathrm{rot}, \Omega)$ and $\nabla \cdot B = 0 \in L^2(\Omega)^3$ are valid similarly in Ω. Then, $B \in H^2_{loc}(\Omega)^3$ is obtained from the standard elliptic regularity. Theorem 0.1 says, in contrast, that even if $n \times J$ has an interface on $\Gamma = \mathcal{M} \cap \Omega$, the normal component $n \cdot B$ of B gains the regularity in one rank. It is not difficult to suspect that the solenoidal condition $\nabla \cdot B = 0$ plays an essential role in such a regularity. In this connection, it must be noted that in Theorem 0.1, interface to $n \cdot J$ is not permitted. In fact, the first equation of (0.2) holds in Ω, and therefore,

$$\nabla \cdot J = \nabla \cdot (\nabla \times B) = 0$$

follows there. This implies $J \in H(\mathrm{div}, \Omega)$, and hence $[n \cdot J]_-^+ = 0$ holds on Γ.

The second theme of this article is the stationary Stokes system;

$$\left.\begin{array}{r} -\Delta v = \nabla p + f \\ \nabla \cdot v = 0 \end{array}\right\} \quad \text{in} \quad \Omega_\pm \tag{0.6}$$

where $v = {}^t(v^1(x), v^2(x), v^3(x))$ denotes the velocity of fluid, $p = p(x)$ the pressure, and $f(x) = {}^t(f^1(x), f^2(x), f^3(x))$ the external force. The following theorem is proven by Theorem 0.1 and the vorticity formulation. There, $\mathcal{M}$ is supposed to be $C^{2,1}$ so that $C^{2,1} \cap W^{3,\infty}_{loc}(\Omega)$ extension is taken to n. Actually, [1] showed the theorem when $\mathcal{M}$ is flat.

Theorem 0.2 *If $\mathcal{M}$ is $C^{2,1}$, $v \in H^2(\Omega)^3$, $p \in H^1(\Omega)$, and $f \in H^1(\Omega_\pm)^3$ satisfy (0.6), and $[n\cdot\nabla p]_-^+ = 0$ holds on Γ, then the condition $(n\cdot\nabla)^2(n\times v) \in H^1(\Omega)^3$ implies that $v \in H^3_{loc}(\Omega)^3$ and $p \in H^2_{loc}(\Omega)$.*

Standard regularity associated with the above theorem is obvious, so that $v \in H^2(\Omega)^3$, $p \in H^1(\Omega)$, and $f \in H^1(\Omega)^3$ imply $v \in H^3_{loc}(\Omega)$ and $p \in H^2_{loc}(\Omega)$ in (0.6). On the other hand, $f \in H(\mathrm{div}, \Omega)$ can take place of the assumption

$$[n \cdot \nabla p]_-^+ = \left[\frac{\partial p}{\partial n}\right]_-^+ = 0 \qquad \text{on} \qquad \Gamma \tag{0.7}$$

in Theorem 0.2, because (0.6) holds in Ω and therefore, $f \in H(\mathrm{div}, \Omega)$ gives that

$$-\Delta p = -\nabla \cdot \nabla p = \nabla \cdot f \in L^2(\Omega),$$

in Ω, or $\nabla p \in H(\mathrm{div}, \Omega)$. This implies (0.7) and also $p \in H^2_{loc}(\Omega)$ from the standard elliptic regularity. In other words, if (0.6) holds in a natural L^2 setting in Ω, then $f \in H^1(\Omega_\pm)^3 \cap H(\mathrm{div}, \Omega)$ implies H^2 interface vanishing of the pressure and H^3 interface of the velocity only in the second normal derivative of the tangential component. Namely, we have the following.

Theorem 0.3 *If $\mathcal{M}$ is $C^{2,1}$ and $v \in H^2(\Omega)^3$, $p \in H^1(\Omega)$, and $f \in L^2(\Omega)^3$ satisfy (0.6), then $f \in H(div, \Omega)$ implies $p \in H^2_{loc}(\Omega)$, and $f \in H^1(\Omega_\pm)^3$ with $(n \cdot \nabla)^2(n \times v) \in H^1(\Omega)^3$, furthermore, gives that $v \in H^3_{loc}(\Omega)^3$.*

Those interface vanishing theorems are optimal in the sense that there is $v \in H^2(\Omega)^3$, $p \in H^2(\Omega)$, and $f \in H^1(\Omega_\pm)^3 \cap H(\mathrm{div}, \Omega)$ satisfying (0.6), (0.7), and $[(n \cdot \nabla)^2(n \times v)]_-^+ \neq 0$ on Γ. Among them is the case that

$$\mathcal{M} = \{x = (x_1, x_2, x_3) \mid x_3 = 0\}$$

with $n = {}^t(0, 0, 1)$, $\Omega = \{x = (x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 < 1\}$,

$$v = \begin{pmatrix} \chi(x_1 - x_2)x_3\,|x_3| \\ \chi(x_1 - x_2)x_3\,|x_3| \\ 0 \end{pmatrix} \in H^2(\Omega)^3,$$

and $p = x_3\,|x_3| \in H^2(\Omega)$, where χ is a smooth function on $\mathbf{R}$ with the support containing 0. In fact, we have

$$f = -\begin{pmatrix} 2\chi(x_1 - x_2)H(x_3) + \chi''(x_1 - x_2)x_3\,|x_3| \\ 2\chi(x_1 - x_2)H(x_3) + \chi''(x_1 - x_2)x_3\,|x_3| \\ 2|x_3| \end{pmatrix},$$

where $H = H(s)$ is the Heaviside function:

$$H(s) = \begin{cases} 1 & (x_3 > 0) \\ -1 & (x_3 < 0) \end{cases}$$

and this f is in $H^2(\Omega_\pm)^3 \cap H(\mathrm{div}, \Omega)$. Thus, here actual interface arises in the second normal derivative of the tangential component of the velocity, in spite that any other assumption in Theorems 0.2 and 0.3 is satisfied.

References

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