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Conley index theory for slow-fast systems: multi-dimensional slow manifold*

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Based on joint works with
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Abstract

A method for proving the existence of periodic and heteroclinic orbits in a singularly perturbed ODE system called a slow-fast system is given by using the Conley index theory. This is a continuation of the authors' earlier work[1] which is now extended to systems with multi-dimensional slow variables. As an application, we show, in a system of reaction-diffusion equations studied by Gardner and Smoller[2], the existence of periodic traveling waves solutions as well as the set of traveling wave solutions that are encoded by symbolic sequences of two symbols. This is based on joint works[3, 5] with M. Gameiro, T. Gedeon, H. Kokubu, and K. Mischaikow.

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1 Slow-fast system

Consider a family of differential equations on $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$ of the form

$$\dot{x} = f(x,y), \qquad \dot{y} = \epsilon g(x,y), \tag{1}$$

where $f : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k$ and $g : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^l$ are C^1 functions and $\epsilon \geq 0$. We call this kind of differential equations a *slow-fast system.*

The solutions to this system generate a flow $\varphi^\epsilon : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. When $\epsilon = 0$, (1) has a simpler form since y becomes constant, hence y can be viewed as a parameter for the flow ψ_y on $\mathbb{R}^k$ denoted for each $y \in \mathbb{R}^l$ by

$$(\psi_y(t,x), y) = \varphi^0(t,x,y). \tag{2}$$

Another way to simplify the equation (1) is to first rescale the time by $\tau = \epsilon t$ and then let $\epsilon = 0$ in the new equations. We obtain

$$0 = f(x,y), \qquad \dot{y} = g(x,y). \tag{3}$$

The set of points $(x,y) \in \mathbb{R}^{k+l}$ with $f(x,y) = 0$ is called a *slow manifold* of the problem (1). If $\frac{\partial f}{\partial x}$ is invertible for y in some bounded set Y, then by the implicit function theorem, there is a function $x = m(y)$ such that $f(m(y),y) = 0$. We call $M := \{(x,y) \in \mathbb{R}^{k+l} \mid x = m(y), y \in Y\}$ (a branch of) the slow manifold over Y. Solutions of $\dot{y} = g(m(y),y)$ generate the slow flow $\varphi_M^{\text{slow}} : \mathbb{R} \times M \to M$.

Although our theory applies to slow-fast systems of any dimension, in this paper, we mainly illustrate the results by a specific example of slow-fast system studied by Gardner and Smoller[2], which is the following system of equations describing the traveling waves in a prey-predator model:

$$\begin{aligned} \dot{u} &= w, & \dot{w} &= -\theta w - uh(u,v), \\ \dot{v} &= \epsilon z, & \dot{z} &= \epsilon(-\theta z - vk(u,v)). \end{aligned} \tag{4}$$

Here $\theta \in \mathbb{R}$ represents the wave velocity, which we fix to -0.25. For simplicity we consider the case $h(u,v) = (1-u)(u-v)$ and $k(u,v) = au - b - v$ with $a = 1.7, b = 0.25$. This is an example of the slow-fast system of the form (1) with $n = 4$, $k = l = 2$, $x = (u,w)$ and $y = (v,z)$.

There are three equilibrium points $(u,w) = (0,0), (0,v), (0,1)$ in the fast system for any θ, and we consider two slow manifold branches given by $M_0 := \{(u,w) \mid (u,w) = (0,0)\}$ and $M_1 := \{(u,w) \mid (u,w) = (0,1)\}$. When

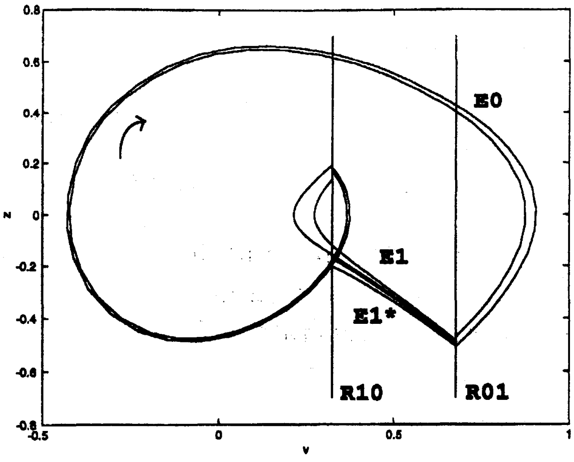

Figure 1: The slow flow for (4) with $\theta = -0.25$. The very thin strips $\mathcal{R}_{1\to 0}$ and $\mathcal{R}_{0\to 1}$ indicate the domains where the fast flow possibly has a connecting orbit.

$\theta = -0.25$, the fast flow has a connecting orbit from M_0 to M_1 at $v = \sigma_1 \approx 0.674$ and another from M_1 to M_0 at $v = \sigma_2 \approx 0.321$.

In Figure 1, two vertical lines (each of which is in fact a very thin strip) show the places where the connecting orbits exist in the fast flow, and the several slow flow lines on the slow manifolds M_0 and M_1 are drawn and superimposed simultaneously.

We obtain the following conclusions[5] by applying our general theory[3] to the Gardner-Smoller system (4):

Theorem 1.1 *(i) The Gardner-Smoller system (4) with appropriate choice of parameters has a periodic solution whose image under the projection* $\Pi : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^l$ *is contained in the set* $E_0 \cup E_1$ *(or* $E_0 \cup E_{1*}$*). See Figure 1.*

(ii) There is an uncountable set of bounded solutions in (4) which can be encoded by bi-infinite sequences of two symbols E_1 *and* E_{1*} *in such a way that, given a symbol sequence, the projection of the corresponding orbit under* Π *passes through* E_1 *or* E_{1*}*, and* E_0*, alternatively in the prescribed order by the symbol sequence.*

Note that Gardner and Smoller[2] obtained the existence of periodic traveling waves also using the Conley index theory. However, their method uses very concrete construction of homotopy between (4) and the van del Pol equation, and is hence limited to the specific form of the system (4). Our theory is, on the other hand, applicable to general slow-fast systems. The existence of the set of solutions encoded by symbolic sequences is new, which shows the existence of a rich variety of traveling wave solutions with complicated behaviors.

2 Conley index theory for slow-fast systems

Consider a flow γ defined on a locally compact metric space X. A compact set $N \subset X$ is called an *isolating neighborhood* if

$$\mathrm{Inv}(N,\gamma) := \{x \in X \mid \gamma(\mathbb{R},x) \subset N\} \subset \mathrm{Int}N.$$

The set $S = \mathrm{Inv}(N,\gamma)$ is called the *isolated invariant set* in N. The *Conley index*[4] is associated to isolating neighborhoods and provides a topological invariant of the isolated invariant sets with the property that if $\mathrm{Inv}(N,\gamma) = \mathrm{Inv}(N',\gamma)$ then the corresponding Conley indices are identical. Here we use the so-called cohomological Conley index denoted by $CH^*(S)$ which takes values in graded vector spaces over the finite field $\mathbb{Z}_2$.

Given an isolating neighborhood its Conley index describes the dynamics of the associated isolated invariant set. In our case we will present theorems which can be used to prove the existence of periodic orbits as well as the set of orbits encoded by symbolic sequences.

We look for solutions which consist of pieces approximating slow manifolds and of pieces approximating heteroclinic orbits between these manifolds. To prove such solutions exist, the first step is to find the appropriate isolating neighborhoods. We do this in two steps. We choose sequence of flow boxes E_i of the flows $\varphi^{\mathrm{slow}}_{M_i}$ on slow manifolds M_i, called *slow sheets*. We assume that adjacent pair of the slow sheets E_i and E_{i-1} are joined in such a way that $\mathcal{P}_i := \Pi(E_i) \cap \Pi(E_{i-1}) \cap \mathcal{R}_i \neq \emptyset$, where $\mathcal{R}_i \subset \mathbb{R}^l$ is a domain in which the fast flow possibly has a connecting orbit between the slow manifolds. Define $P_i := E_i \cap \Pi^{-1}(\mathcal{P}_i)$ and $P'_{i-1} := E_{i-1} \cap \Pi^{-1}(\mathcal{P}_i)$, which are the places where jump from E_i to E_{i-1} by the fast flow may occur. We introduce (extension of) boundary curves of P_i and P'_i which are denoted by $\tilde{W}^{\mathrm{in}}_i$ and $\tilde{W}^{\mathrm{out}}_i$, which bound a region in E_i denoted by $\tilde{W}_i$. Let $W_i := P'_i \cup \tilde{W}_i \cup P_i$. The sets $Z^{\pm}_i$ are the entrance and exit sets of W_i with respect to the slow flow on the slow sheet E_i. See Figure 2. We construct a collection of sets that consists of parts of slow sheets E_i such as W_i and P_i, together with neighborhoods $\mathbf{B}_i$ (called boxes) of connecting orbits between the slow manifolds. We call

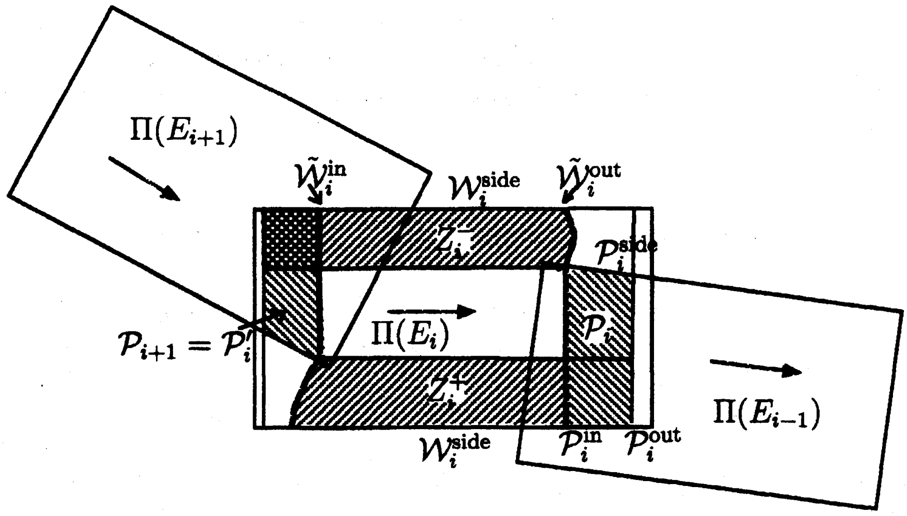

Figure 2: Explanation of various sets like $W_i, P_i, P_i' \subset E_i$. Note that by script letters we denote the projection under Π of the unscripted objects.

the collection of sets $E_i \subset M_i$ and boxes $\mathbf{B}_i$ a *periodic corridor*, which is also defined more precisely later. Finally, we construct the set N by fattening relevant pieces of slow manifolds in the corridor transversally into the fast direction. This set N is a candidate for an isolating neighborhood for φ^ϵ.

Definition 2.1 A set $\mathbf{B} \subset \mathbb{R}^k \times \mathbb{R}^l$ is called a *box*, if:

(1) $\mathbf{B}$ is an isolating neighborhood for $\psi_{\mathcal{P}}$ where $\mathcal{P} := \Pi(\mathbf{B})$, and the *parametrized flow* $\psi_{\mathcal{P}} : \mathbb{R} \times \mathbb{R}^k \times Y \to \mathbb{R}^k \times Y$ is given by $\psi_{\mathcal{P}}(t,x,y) := (\psi_y(t,x), y)$ for $y \in \mathcal{P}$.

(2) Let $S(\mathbf{B}) := \mathrm{Inv}(\mathbf{B}, \psi_{\mathcal{P}})$. Then there exists an attractor-repeller decomposition[4] $\mathcal{M}(S(\mathbf{B})) := \{M(p,\mathbf{B}) \mid p = 1,2 \ (2 > 1)\}$.

(3) There are isolating neighborhoods $V(p,\mathbf{B})$ for $M(p,\mathbf{B})$, $p = 1,2$, such that $V(p,\mathbf{B}) \subset \mathrm{Int}\,\mathbf{B}$ and $V(1,\mathbf{B}) \cap V(2,\mathbf{B}) = \emptyset$.

(4) For each $y \in \mathcal{P}$, the set $\mathbf{B}_y = \mathbf{B} \cap \Pi^{-1}(y)$ is a k-dimensional disc.

(5) Let $S_y(\mathbf{B}) := \mathrm{Inv}(\mathbf{B}_y, \psi_y)$ and let $\{M_y(p,\mathbf{B}) \mid p = 1,2\}$ be the corresponding attractor-repeller decomposition of $S_y(\mathbf{B})$. Then there are

relatively open subsets $\mathcal{P}^0, \mathcal{P}^1 \subset \mathcal{P}$, such that for every $y \in \mathcal{P}^0 \cup \mathcal{P}^1$, $S_y(\mathbf{B}) = \cup_{p=1,2} M_y(p, \mathbf{B})$, namely ψ_y has no connecting orbit in $\mathbf{B}_y$.

Given a box $\mathbf{B}$, one can define a 2×2 matrix $T_{\mathbf{B}}$, called the *transition matrix*[4], which carry the algebraic information of connecting orbits inside the box $\mathbf{B}$ for the fast flow, in the sense that the non-zero $(2,1)$-entry of $T_{\mathbf{B}}$ implies the existence of some $y \in \mathcal{P}$ for which the corresponding ψ_y possesses a connecting orbit from $M_y(2, \mathbf{B})$ to $M_y(1, \mathbf{B})$.

Definition 2.2 A collection $\{E_i\}_{i=0}^I$ of slow sheets with $E_0 = E_I$, together with subsets $W_i, P_i, P_i' \subset E_i$ and sets $Z_i^{\pm} \subset W_i$, and a collection of boxes $\{\mathbf{B}_i \mid i = 1, \ldots, I\}$ form a *periodic corridor*, if

(1) $\mathcal{P}_i = \Pi(\mathbf{B}_i)$ for all i;

(2) $\mathcal{P}_i^{\text{side}} \setminus \mathcal{W}_i^{\text{side}} \subset \mathcal{Z}_{i-1}^-$, and $W_i^{\text{side}} \subset \text{Int}_{W_i} Z_i^+ \cup \text{Int}_{W_i} Z_i^-$ for each i.

(3) Let $\mathcal{P}_i^{\text{in}} := \Pi(P_i^{\text{in}})$ and $\mathcal{P}_i^{\text{out}} := \Pi(P_{i-1}'^{\text{out}})$. For each $i = 1, \ldots, I$ there exist homotopy equivalences of pairs

$$\begin{aligned} h_0 : (\mathcal{P}_i^{\text{in}}, \mathcal{P}_i^{\text{in}} \cap \mathcal{Z}_{i-1}^-) &\hookrightarrow (\tilde{\mathcal{W}}_i^{\text{out}}, \tilde{\mathcal{W}}_i^{\text{out}} \cap \mathcal{Z}_i^-) \\ h_1 : (\mathcal{P}_i^{\text{out}}, \mathcal{P}_i^{\text{out}} \cap \mathcal{Z}_{i-1}^-) &\hookrightarrow (\tilde{\mathcal{W}}_{i-1}^{\text{in}}, \tilde{\mathcal{W}}_{i-1}^{\text{in}} \cap \mathcal{Z}_{i-1}^-). \end{aligned}$$

For more precise definition, see our forthcoming paper[3].

We furthermore define $\Theta := T_I \circ \cdots \circ T_1$ where T_i $(i = 1, \ldots, I)$ is the $(2,1)$-entry of the transition matrix $T_{\mathbf{B}_i}$.

After constructing a periodic corridor $\{E_i\}_{i=0,\ldots,I}$ and $\{\mathbf{B}_i\}_{i=1,\ldots,I}$, let

$$N := \bigcup_{i=1}^{I} \mathbf{B}_i \ \cup \ \bigcup_{i=0}^{I} ([-q, q]^k \times W_i).$$

Then for $q > 0$ chosen sufficiently small, N is an isolating neighborhood for φ^ϵ for sufficiently small $\epsilon > 0$.

Theorem 2.3 *Given a periodic corridor as above, assume that*

(1) for each $i = 1, \ldots, I$, the index of $M(1, \mathbf{B}_i)$ has the same Conley index as that of a hyperbolic fixed point;

(2) $H^*(\mathcal{W}, \mathcal{Z}; \mathbb{Z}_2)$ *is the same as the Conley index of a hyperbolic periodic orbit, where*

$$\mathcal{W} := \bigcup_{i=1}^{I} \Pi(W_i), \quad \mathcal{Z} := \bigcup_{i=1}^{I} \Pi(\mathcal{Z}_i^-);$$

(3) Θ *is an isomorphism.*

Then, for sufficiently small $\epsilon > 0$, $\mathrm{Inv}(N, \varphi^\epsilon)$ *contains a periodic orbit.*

This general result can be applied to the Gardner-Smoller system (4) by setting $I = 2$ and taking two kinds of periodic corridors with the slow sheets $\{E_0 = E_2, E_1\}$ or $\{E_0 = E_2, E_{1*}\}$ as in Figure 1. Therefore Theorem 1.1 in §1 follows from the above theorem and a similar construction of symbolic coding in our previous paper[6].

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