数理解析研究所講究録 1369 巻 2004 年 163-177

N-body Choreography on the Lemniscate

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Abstract

We show that the lemniscate curve is the simplest candidate of the orbit for the three-body figure-eight choreography. Here figure-eight choreography is the periodic motion of equal masses with equal time spacing on the eight shaped orbit. We deduce that parameterization of lemniscate $(1, \mathrm{cn}(t))\mathrm{sn}(t)/(1+\mathrm{cn}^2(t))$ using the Jacobian elliptic functions sn and cn is the three-body figure-eight choreography under an inhomogeneous interaction potential. Though we already published this analytic choreography on the lemniscate in 2003, here we especially give complementary explanation how we could find this analytic choreography. And we suggest that, if interaction potential is homogeneous or logarithmic, there are no three-body choreography on the lemniscate. We also investigate the same parameterization with different modulus for N-body system with $N = 5, 7, 9, \ldots$ on the lemniscate. We show that it conserves the center of masses, the angular momentum and the moment of inertia, but that it may not satisfy equation of motion under any interaction potential, which means unfortunately it is not a N-body choreography.

1 Introduction

On June 14, 2001, one of the author (T. F.) found a short column on a newspaper. It told us "a figure-eight three-body solution under the Newton gravity was found". He was astonished by this solution. Because, he once tried to find such a solution. In the famous paper by Chenciner and Montgomery[1], Chenciner mentions a simulator "Gravitation Ltd" by Jeff Rommereide. When T. F. was using a Macintosh SE with 8 MHz CPU and 4MB memory, long ago, he spent a lots of time to play with this nice simulator. (It still works on classic environment on Mac OS X.) Among some sample solar systems enclosed in the software, there are "Figure 8 Attractors", a small mass moving on an eight shaped orbit around two spatially fixed big masses. He tried to find figure eight solution in more realistic three-body problem. But he found nothing special. At that time, he did not know what he was looking for. The figure-eight solution is excellent. It is far beyond he had imagined. After reading this column on the newspaper, T. F. mailed the other authors H. F. and H. O. to work with on this field.

Figure-eight solution is a periodic solution to the planer three body problem with equal masses. Three bodies chase each other on an eight shaped curve. Simó named such orbits "simple choreography" – N-bodies moves on a *single* closed orbit with equal time spacing.

$$\{q_1(t), q_2(t), q_3(t)\} = \{q(t), q(t+T/3), q(t-T/3)\}. \tag{1}$$

In the above, $q_i, q \in \mathbb{R}^2$ represent the coordinate of masses $i = 1, 2, 3$ and the common orbit respectively. The period is expressed by T.

This solution is first found by Moore in 1993[2]. Existence of this orbit is proved by Chenciner and Montgomery[1]. Numerical parameters of this orbit is established by Simó[3, 4]. No one knows, however, analytic form of the orbit $q(t)$, algebraic or transcendent nature of the orbit.

We started our investigations from fitting the figure-eight orbit with known curves. It looks like to be a slightly squashed lemniscate curve. The fitting is fine but not extremely fine. The discrepancy is order 1/1000.

Obviously, we need some criteria to choose curves for the orbit. We find a criterion. Let us show this in the next section. From this criterion, we understand the lemniscate curve is the simplest candidate for the figure-eight orbit as we explain in section 3. In this lecture note, we concentrate on the motion on the lemniscate.

In sections 4.1–4.3, we summarize the story how we could find an analytic three body choreography on the lemniscate according to our study in 2002, which complements our paper[5]. Then, we consider all the other choreographic motion on the lemniscate in section 4.4. It is suggested that any homogeneous or logarithmic interaction potential may not support the three body choreography on the lemniscate.

Finally, in section 5 we discuss $N(>3)$-body system on the lemniscate. In section 5.1, we point out that a similar parameterization to three body system on the lemniscate satisfies several conservation laws even in $N(>3)$-body system. However, we almost conclude that it may not satisfy equation of motion under any interaction potential including very artificial one, in section 5.2. Brief summary is given in section 6.

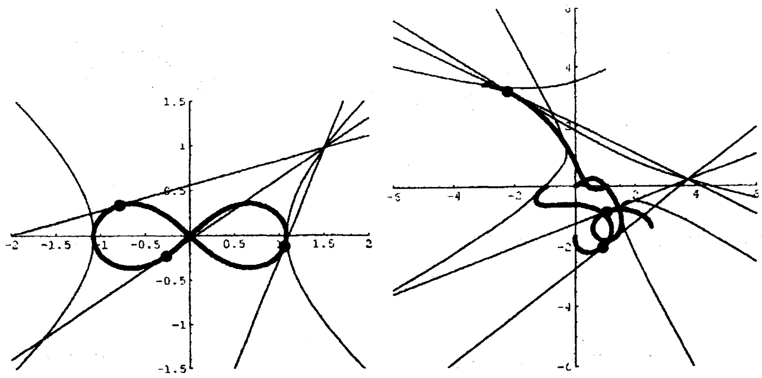

Figure 1: Three tangents to orbit with zero angular momentum meet at a point. Thick curves and thin curves represent the orbit of the masses and the orbit of the centre on tangents respectively. Three lines represent tangents. Left: Snapshot for the figure-eight solution. Right: Snapshot for an orbit with $m_1 = 1.0, m_2 = 1.1, m_3 = 1.2$.

2 Three tangents theorem

The figure eight solution has zero angular momentum.

$$L = \sum_i q_i \wedge p_i = 0, \tag{2}$$

where p_i are the momentum. Because the orbit has left-right symmetry and the masses travel the orbit left lobe and right lobe with opposite orientation. Therefore, the time average of the angular momentum for each masses are zero. Then the total angular momentum, which is the constant of motion, must be zero. This is the reason why the angular moment is zero.

Then, what dose $L = 0$ mean? In other words, does $L = 0$ give any constraints for the orbit? Yes. We find the following theorem.

Theorem 1 (Three Tangents). *In the centre of mass frame of the three body problem, i.e., when $P = \sum_i p_i = 0$, the following two statements are equivalent. (1) The system has zero angular momentum. (2) Three tangents lines to the masses "almost always" ("almost everywhere" in time) meet at a point for each instant. Otherwise, for countable times, three tangent lines are allowed to be parallel.*

Proof. (1) $\Rightarrow$ (2): Assume two tangent lines to masses 1 and 2 meet at a point C_t. Since $\sum_i p_i = 0$ and $\sum_i q_i \wedge p_i = 0$, we have $\sum_i (q_i - C_t) \wedge p_i = 0$. By the assumption, we have $(q_1 - C_t) \wedge p_1 = (q_2 - C_t) \wedge p_2 = 0$. Thus we have $(q_3 - C_t) \wedge p_3 = 0$. That is, the tangent line to mass 3 also passes through the point C_t. If two tangents lines are parallel the third line must also be parallel to the other two lines since $\sum_i p_i = 0$. Now, (2) $\Rightarrow$ (1) is obvious. □

We call this theorem the "three tangents theorem" and C_t the "centre of tangents". Note that this theorem stands for the three body system with general masses and any potential energies but only for the three body system. In figure 1, the orbit of the masses and the orbit of the centre of

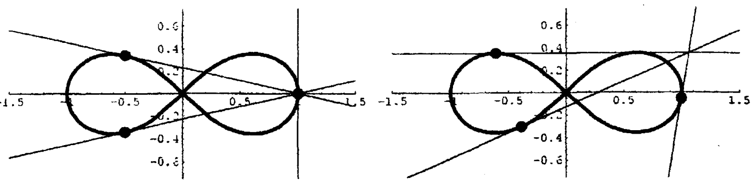

Figure 2: Left: Three tangents at isosceles configuration. Right: Three tangents at $dy/dx = 0$.

tangents are shown. The left figure is a snapshot for the figure-eight solution and the right for an orbit with general masses.

Therefore, the candidate curves for figure-eight orbit must have the following properties. (i) On this curve, we can take continuously moving three points whose centre of mass is fixed. (ii) The three tangents lines to these three points must meet at a point C_t for each instant. We would like to call the second condition the "three tangents criterion". Conditions (i) and (ii) are equivalent to $L = 0$ and $P = 0$.

3 The simplest curve

Then, we apply the "three tangents criterion" to the simplest algebraic curve, the fourth order polynomial in the Cartesian coordinate x, y. After a transformation $x \to \mu x$, $y \to \nu y$, the polynomial has the following form

$$x^4 + \alpha x^2 y^2 + \beta y^4 = x^2 - y^2. \tag{3}$$

The figure-eight solution has two isosceles configurations, the configuration

$$q_1 = (1, 0),\ q_2 = (-1/2, y_0),\ q_3 = (-1/2, -y_0) \tag{4}$$

and configuration with $x \to -x$. From the symmetry, the three tangents lines at this configuration must meet at $q_1 = (1, 0)$. See the left picture in figure 2. This criterion determines the value $\alpha = 2$.

To determine the value for β, we take a point q_1 with $dy/dx = 0$ and calculate *numerically* the other points by the condition (i). See the right picture figure 2. Then, applying the "three tangents criterion", we find $\beta = 1$ *numerically*.

Therefore, the simplest candidate is the lemniscate curve

$$(x^2 + y^2)^2 = x^2 - y^2 \tag{5}$$

or its affine transformation, $x \to x/\mu$, $y \to y/\nu$.

At this stage, we apply the "three tangents criterion" to only two configurations. Moreover, the value $\beta = 1$ is not proved. So, we need to prove that the lemniscate curve surely satisfy the conditions (i) and (ii). To do this we have to find some parametrisation for the positions q_i on the lemniscate curve.

4 Three body choreography on the lemniscate

4.1 Parametrisation

From the definition of the lemniscate (5) we get,

$$1 = \left(\frac{x}{x^2+y^2}\right)^2 - \left(\frac{y}{x^2+y^2}\right)^2. \tag{6}$$

That is, the inversion of the lemniscate $(x/(x^2+y^2), y/(x^2+y^2))$ is the rectangular hyperbola. Let us parameterize the rectangular hyperbola

$$\frac{x}{x^2+y^2} = \frac{1}{s}, \ \frac{y}{x^2+y^2} = \frac{c}{s} \tag{7}$$

with functions $s(t)$ and $c(t)$ which satisfy

$$s(t)^2 + c(t)^2 = 1. \tag{8}$$

Take the inversion again, we get a parameterization of the lemniscate

$$(x, y) = \left(\frac{s(t)}{1+c(t)^2}, \frac{s(t)c(t)}{1+c(t)^2}\right). \tag{9}$$

We take, as a first trial, $s(t) = \sin(t)$, $c(t) = \cos(t)$ and $T = 2\pi$. But the centre of mass does not stay at the origin. This is clearly seen at the isosceles configuration (4). Since $q_1(t) = q(t) = (1, 0)$, t is $\pi/2$. Then the x-component of both $q_2(t)$ and $q_3(t)$ are $\sin(\pi/6)/(1+\cos^2(\pi/6)) = -2/7 \neq -1/2$ and the center of mass is not zero.

We take, of course, as the next trial, $s(t) = \mathrm{sn}(t,k)$ and $c(t) = \mathrm{cn}(t,k)$.

$$(x, y) = \left(\frac{\mathrm{sn}(t,k)}{1+\mathrm{cn}(t,k)^2}, \frac{\mathrm{sn}(t,k)\,\mathrm{cn}(t,k)}{1+\mathrm{cn}(t,k)^2}\right). \tag{10}$$

Here sn and cn are the Jacobian elliptic functions. This is natural, because the lemniscate is the birth place of the elliptic functions. These functions have one parameter, the modulus k. The period T is $4K(k)$ where $K(k)$ is the complete elliptic integral of the first kind,

$$K(k) = \int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}. \tag{11}$$

If we take $k \to 0$, then $\mathrm{sn}(t,k) \to \sin(t)$ and $\mathrm{cn}(t,k) \to \cos(t)$.

To find appropriate k, consider again the isosceles configuration (4). In this case, $T = 4K$ and $t = K$. Then the x-component of $q_2(t) = q(7K/3)$ must be $-1/2$. This gives the equation for k

$$\mathrm{sn}\left(\frac{K(k)}{3}, k\right) = \sqrt{3} - 1. \tag{12}$$

Using the Mathematica®, we can solve this equation *numerically*. The value is

$$k^2 = 0.933\,012\,701\,892\,219\,323\,381\,861... \tag{13}$$

Using this *numerical* value, we have checked that this parameterization surely satisfies the conditions (i) and (ii), that is, the centre of mass *numerically* stays at the origin and the angular momentum is *numerically* zero for the whole range of t. Moreover, we find that the kinetic energy and the moment of inertia is *numerically* constant

$$I = \sum_i q_i^2 = \text{ constant.} \tag{14}$$

On the other hand, this orbit does not satisfy the equation of motion under the Newton potential. We noticed that the smashed orbit by the value of k^2 given by (13), i.e.,

$$(x, y) = \left(\frac{\mathrm{sn}(t,k)}{1+\mathrm{cn}(t,k)^2},\ k^2\, \frac{\mathrm{sn}(t,k)\,\mathrm{cn}(t,k)}{1+\mathrm{cn}(t,k)^2} \right) \tag{15}$$

is close to the figure-eight solution under the Newton potential. The discrepancy is order 1/1000. So, we take a lot of time to find more "natural" time-depending smashing function instead of the constant k^2. After these trial failed, we proceeded to study the motion on the lemniscate (10) itself.

4.2 Proof of conservation of the centre of mass, angular momentum and the moment of inertia

Proofs are straightforward, because we already know the equations to be proved.

We first determine the exact value of the modulus k^2, which is defined by the equation (12). To do this, evaluate $\mathrm{sn}(10K/3)$ in two ways,

$$\mathrm{sn}\left(\frac{10K}{3}\right) = \mathrm{sn}\left(3K + \frac{K}{3}\right) = -\frac{\mathrm{cn}(K/3)}{\mathrm{dn}(K/3)}$$

and using the periodicity

$$\mathrm{sn}\left(\frac{10K}{3}\right) = \mathrm{sn}\left(4K - \frac{2K}{3}\right) = -\,\mathrm{sn}\left(\frac{2K}{3}\right) = -\frac{2\,\mathrm{sn}(K/3)\,\mathrm{cn}(K/3)\,\mathrm{dn}(K/3)}{1-k^2\,\mathrm{sn}^4(K/3)},$$

where

$$\mathrm{dn}(t) = (1 - k^2\,\mathrm{sn}^2(t))^{1/2}.$$

Equating the right hand side of the above two equations, we get

$$k^2 = \frac{1 - 2\,\mathrm{sn}(K/3)}{\mathrm{sn}^4(K/3) - 2\,\mathrm{sn}^3(K/3)} = \frac{2+\sqrt{3}}{4}. \tag{16}$$

This is the exact value of (13).

Next, we prove the conservation of the centre of mass. It is convenient to use complex variable

$$x^{(\pm)}(t) = x(t) \pm \mathrm{i}\,y(t) = \frac{\mathrm{sn}(t)}{1 \mp \mathrm{i}\,\mathrm{cn}(t)} \tag{17}$$

and consider the complex plane of t. The equation to be proved is

$$f(t) = x^{(+)}(t) + x^{(+)}\left(t + \frac{4K}{3}\right) + x^{(+)}\left(t - \frac{4K}{3}\right) = 0. \tag{18}$$

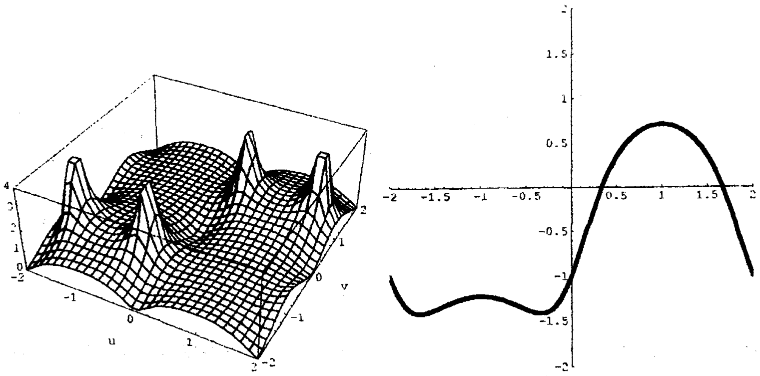

Figure 3: Left: Absolute value of $x^{(+)}(uK + \mathrm{i}\,vK')$ in its fundamental domain $-2 \le u \le 2$ and $-2 \le v \le 2$. There are two poles on the line $\Im t = K'$ and other two poles on the line $\Im t = -K'$. Right: Value of $1/x^{(+)}(uK + \mathrm{i}\,K')$ for $-2 \le u \le 2$. Two zeros correspond to the two poles on the line $\Im t = K'$ in the left picture.

Since the function $f(t)$ is an elliptic function and $f(0) = 0$, showing the absence of poles in its fundamental domain is enough to show $f(t) = 0$. For the fundamental domain of the function $x^{(\pm)}(t)$, we take $-2K \le \Re t \le 2K$, $-2K' \le \Im t \le 2K'$ where K' is the complementary elliptic integral of the first kind

$$K'(k) = K(\sqrt{1-k^2}).$$

The left picture of figure 3, the absolute value of $x^{(+)}(uK+\mathrm{i}\,vK')$ for $-2 \le u \le 2$ and $-2 \le v \le 2$ shows the places of the poles of $x^{(+)}(t)$ in its fundamental domain. It suggests the poles are on the line $t = u \pm \mathrm{i}\,K'$ with $u \in \mathbb{R}$. Indeed,

$$x^{(+)}(u + \mathrm{i}\,K') = \frac{1}{k\,\mathrm{sn}(u) - \mathrm{dn}(u)}. \tag{19}$$

The right picture of figure 3 shows the behavior of the function $1/x^{(+)}(uK + iK') = k\,\mathrm{sn}(uK) - \mathrm{dn}(uK)$ for $-2 \le u \le 2$. The two zeros in the right picture correspond to the two poles in the left picture. Note the "w" shaped structure in the third quadrant. Since it can take a value 4 times, the function $x^{(+)}(t)$ is at least degree 4. Actually this function is the elliptic function of degree 4.

We can easily show that the poles are at

$$t = \frac{K}{3} + \mathrm{i}\,K',\ \frac{5K}{3} + \mathrm{i}\,K' \tag{20}$$

and

$$t = -\frac{5K}{3} - \mathrm{i}\,K',\ -\frac{K}{3} - \mathrm{i}\,K'. \tag{21}$$

Since degree of the elliptic function $x^{(+)}$ is 4, these four poles are simple pole. As shown in the two zeros in the right picture of figure 3, the residues for the poles with the same imaginary part have the opposite sign and the same magnitude.

Note that the difference of the poles with the same imaginary part is exactly one third of the period $T = 4K$. Therefore, all poles in the right hand side of the equation (18) cancel each other. Thus, the equation (18) is proved. See FFO[5] for details.

Proofs for conservation of the angular momentum ($= 0$) and the moment of inertia are similar to that for the conservation of the centre of mass. These quantities have a form

$$g(t) + g\left(t + \frac{4K}{3}\right) + g\left(t - \frac{4K}{3}\right) = \text{constant}$$

and the function $g(t)$ has a similar structure of poles as $x^{(+)}$. It has simple poles. The distance of the poles are $4K/3$ and the residue is opposite. See FFO[5] for details.

4.3 Equation of motion for the three body choreography on the lemniscate

It was quite difficult to show the motion on the lemniscate (10) satisfies an equation of motion with some force vector F,

$$\frac{d^2q(t)}{dt^2} = F(q(t)) \tag{22}$$

or its complex equivalent

$$\frac{d^2x^{(+)}(t)}{dt^2} = F^{(+)}(x^{(+)}(t), x^{(-)}(t)). \tag{23}$$

Because, we did not know the function F. In other words, we did not know the equation to be proved. Indeed, we did not know whether such functions F exist or not. No numerical calculations can tell the answer. So, it takes long time to find F.

One important observation is the following. Since $x^{(+)}(t)$ has four simple poles, the left hand side of the equation (23) has four triple poles. Therefore, $F^{(+)}(x^{(+)}(t), x^{(-)}(t))$ must have four triple poles at the same place in the complex plane of t. Let z_0 be one of the position of poles given by equations (20) and (21). Then, the right hand side must behave like

$$\frac{a_{-3}}{(t - z_0)^3} + a_0 + a_1(t - z_0) + ..., \tag{24}$$

where the coefficients a_n are known value because the left hand side is the known function.

The other important observation, which leads us to an equation of motion, is the following. Since $(x(t), y(t))$ is on the lemniscate, it satisfies the equation (5). Or equivalently, in the complex variables

$$\left(x^{(+)}(t)^2 - \frac{1}{2}\right)\left(x^{(-)}(t)^2 - \frac{1}{2}\right) = \frac{1}{4}. \tag{25}$$

Therefore, if $x^{(+)}(t)$ has a simple pole at z_0

$$x^{(+)}(z_0 + \epsilon) = \frac{1}{\epsilon} + O(1)$$

then $x^{(-)}(t)$ must behave as

$$x^{(-)}(z_0 + \epsilon)^2 = \frac{1}{2} + \frac{\epsilon^2}{4} + O(\epsilon^3).$$

In figure 4, the value of $1/x^{(+)}(t)$ and the value of $1/x^{(-)}(t)$ for $t = uK + \mathrm{i}K'$ and $-2 \le u \le 2$ are shown. The $1/x^{(-)}(t)$ has local maximum when $1/x^{(+)}(t)$ has zeros, i.e., $x^{(+)}(t)$ has poles. The exact value of the maximum for the $1/x^{(-)}(t)$ is $\sqrt{2}$ and the curvatures at the local maximum are the same according to the above equation.

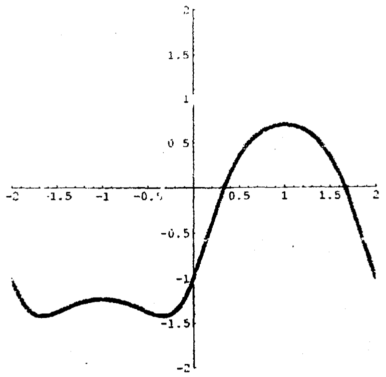

Figure 4: Lower curve: The value of $1/x^{(+)}(uK + \mathrm{i}\,K')$ for $-2 \leq u \leq 2$. Upper curve: The value of $1/x^{(-)}(uK + \mathrm{i}\,K')$.

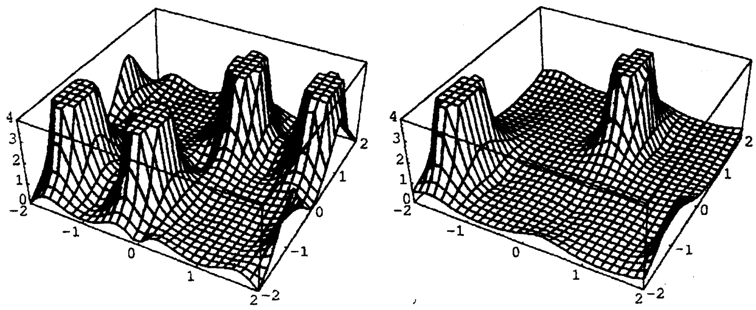

Figure 5: Left: Absolute value of $d^2x^{(+)}(t)/dt^2$. This function has four triple poles. Right: Absolute value of $[2\,(x^{(-)}(t+4K/3) - x^{(-)}(t))]^{-1}$. This function has two triple poles.

Let z_1 and z_2 be the poles in equation (20). Remember that the residues of these poles are opposite. Therefore, we get

$$x^{(-)}(z_1+\epsilon) - x^{(-)}(z_2+\epsilon) = a\,\epsilon^3\left(1 + b\,\epsilon^2 + O(\epsilon^4)\right). \tag{26}$$

The absence of even power of ϵ is due to the symmetry around $t = K + \mathrm{i}\,K'$. See figure 4. Therefore, we get

$$\frac{1}{x^{(-)}(z_1+\epsilon) - x^{(-)}(z_2+\epsilon)} = \frac{a'_{-3}}{\epsilon^3} + \frac{a'_{-1}}{\epsilon} + a'_1\,\epsilon + O(\epsilon^3). \tag{27}$$

This pole structure is similar to the structure what we are looking for. Compare the equations (24) and (27). The equation (27) has triple pole term we need. But this also has unwanted $1/\epsilon$ term. Therefore, comparing the coefficient of the triple poles, we find the following equation.

$$\frac{d^2x^{(+)}(t)}{dt^2} = \frac{1}{2}\left\{\frac{1}{x^{(-)}(t+4K/3) - x^{(-)}(t)} + \frac{1}{x^{(-)}(t-4K/3) - x^{(-)}(t)}\right\} + \text{term with 4 simple poles.} \tag{28}$$

In figure 5, the pole structure of the both sides of the equation are shown. The left picture is the absolute value of $d^2x^{(+)}(t)/dt^2$. This function has four triple poles. The right picture is the absolute value of $[2\,(x^{(-)}(t+4K/3) - x^{(-)}(t))]^{-1}$. This function has two triple poles. The other two poles in the left picture is produced by $[2\,(x^{(-)}(t-4K/3) - x^{(-)}(t))]^{-1}$.

How can we do with the four simple poles? We find that the structure of the four simple poles are the same as that of $x^{(+)}(t)$. That is

$$\text{the term with 4 simple poles} = \frac{\sqrt{3}}{4}x^{(+)}(t). \tag{29}$$

Thus, we find an equation of motion for the choreographic motion on the lemniscate

$$\frac{d^2q_i(t)}{dt^2} = \frac{1}{2}\left\{\frac{q_j - q_i}{(q_j - q_i)^2} + \frac{q_k - q_i}{(q_k - q_i)^2}\right\} + \frac{\sqrt{3}}{4}q_i(t) \tag{30}$$

where (i, j, k) is a permutation of $(1, 2, 3)$. Or equivalently, using $\sum_i q_i = 0$, we get

$$\frac{d^2q_i(t)}{dt^2} = \frac{1}{2}\left\{\frac{q_j - q_i}{(q_j - q_i)^2} + \frac{q_k - q_i}{(q_k - q_i)^2}\right\} - \frac{\sqrt{3}}{12}\left\{(q_j - q_i) + (q_k - q_i)\right\}. \tag{31}$$

The corresponding potential energy is

$$U = \sum_{i<j}\left(\frac{1}{2}\log r_{ij} - \frac{\sqrt{3}}{24}r_{ij}^2\right). \tag{32}$$

Where $r_{ij} = |q_i - q_j|$ represent mutual distances of the masses. The negative sign for the second term produces a repulsive force.

4.4 Other parameterization

We can change the speed of three bodies on the orbit simultaneously by the following transformation of time

$$t \to \tau(t). \tag{33}$$

As far as the function $\tau(t)$ satisfies

$$\tau\left(t + \frac{4K}{3}\right) = \tau(t) + \frac{4K}{3}, \tag{34}$$

conservation of the centre of mass and the angular momentum hold. The equation of motion will be changed by this transformation. But, these transformed motion defined by such function of $\tau(t)$ will not be a motion under a homogeneous or log potential. This is because these transformed motions still have constant moment of inertia.

It has been shown that the figure-eight solution with constant moment of inertia does not exist under homogeneous or log potential with only one exceptional potential, $-1/r^2$ potential[6]. And, as shown in figure 6, the figure-eight orbit under the $-1/r^2$ potential is not the lemniscate.

Actually, we can show that the figure-eight solution under the potential energy $-1/r^2$ cannot be expressed by a fourth order polynomial of the Cartesian co-ordinate x and y. Let us show a brief proof. Details will be published elsewhere.

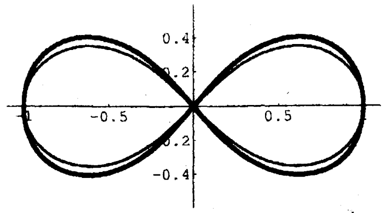

Figure 6: The figure-eight orbit under the $-1/r^2$ potential (bold curve) and the lemniscate (thin curve).

Let us take the isosceles configuration for the initial condition $q_1(0) = (1, 0)$, $q_2(0) = (-1/2, y_0)$ and $q_3(0) = (-1/2, -y_0)$. It is known that the total energy for the figure-eight solution under this potential must be zero. Therefore, with the three tangents theorem, the initial velocities of the masses are determined uniquely up to time reversal. Choosing one direction of the motion, the initial condition is completely determined with only one unknown parameter y_0. Then, we can calculate the power series expansion of t for the orbit $q_i(t)$,

$$q_i(t) = \sum_{n=0}^{N} (x_{in}(y_0), y_{in}(y_0))\, t^n + O(t^{N+1}), \tag{35}$$

where $x_{in}(y_0)$ and $y_{in}(y_0)$ are the coefficient for the Cartesian co-ordinate. They are functions of y_0. Indeed, we determined these coefficients to $N = 18$, although such higher order terms are not needed for our present purpose.

Now, let us assume the orbits $q_i(t) = (x_i(t), y_i(t))$ are on a fourth order polynomial

$$x_i(t)^4 + \alpha x_i(t)^2 y_i(t)^2 + \beta y_i(t)^4 = x_i(t)^2 - \gamma y_i(t)^2. \tag{36}$$

Here, we have used the fact that the figure-eight orbit must passes through the point $(1, 0)$ by the initial condition. The power series given by the equation (35) must satisfy the equation (36) to the order $N = 18$.

Then, we have $N + 1 = 19$ equations for only 4 parameters α, β, γ and y_0. It is not difficult to show that there are no such parameters. Thus, the figure-eight orbit under the potential energy $-1/r^2$ cannot be expressed by a fourth order polynomial.

5 $N = 5, 7, \cdots, \infty$ body choreography on the lemniscate

We can put more than three bodies on the lemniscate which satisfy the centre of mass is at the origin, the angular momentum is zero and the moment of inertia is constant. Choreography with even bodies inevitably collide at the origin. So, here, we consider $5, 7, \cdots, \infty$ bodies. These orbit, however, do not seem to satisfy meaningful equation of motion. Or at least, it is very difficult to find such equation of motion.

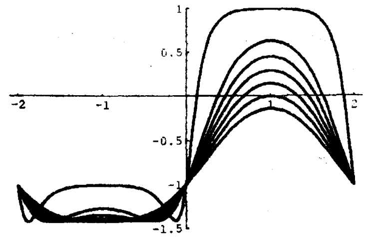

Figure 7: Value of $k\,\mathrm{sn}(uK) - \mathrm{dn}(uK)$, $-2 \le u \le 2$ for $k^2 = 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1 - 10^{-5}$. The curves with larger k^2 have larger values in the right-hand side region.

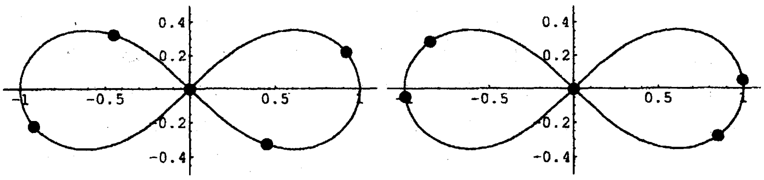

Figure 8: Five bodies on the lemniscate at $t = 0$ for $k^2 = 0.653...$ (left) and for $k^2 = 0.997...$ (right).

5.1 Conservation laws

As shown in section 4.3, interval between two zeros of the function

$$\frac{1}{x^{(+)}(u + \mathrm{i}\,K')} = k\,\mathrm{sn}(u) - \mathrm{dn}(u)$$

has an important role. As shown in figure 7, this function has no zeros for $k^2 < 1/2$, double zero at $k^2 = 1/2$ and two zeros for $k^2 > 1/2$. The interval between two zeros become large with k^2 become large continuously.

Let δz_0 be the interval of two zeros. We can take any interval between $0 \le \delta z_0 < 2K$ by taking some value $1/2 \le k^2 < 1$. To put three bodies on the lemniscate, we have taken $\delta z_0 = 4K/3$. And this structure of zeros ensure the conservation of the centre of mass, the angular momentum and the moment of inertia.

To put five bodies, we can take two values of δz_0,

$$\delta z_0 = 4K \times \frac{1}{5} \text{ and } 4K \times \frac{2}{5}.$$

Both of two δz_0 give five bodies on the lemniscate. This structure of zeros ensure the conservations stated above for five bodies. Corresponding value of modulus are

$$k^2 = \begin{cases} 0.653\,660\,413\,954\,773\,213\,440\ldots, \\ 0.997\,643\,736\,031\,613\,235\,083\ldots. \end{cases}$$

We would find exact value of k^2 if we do similar calculations in section 4.2. Note that, since the function dn is defined by $\mathrm{dn}(t)^2 = 1 - k^2\,\mathrm{sn}(t)^2$, the equation to determine zeros $k\,\mathrm{sn}(z_0) = \mathrm{dn}(z_0)$

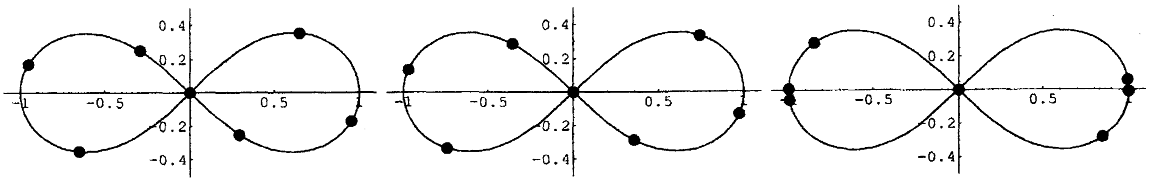

Figure 9: Seven bodies on the lemniscate at $t = 0$ for $k^2 = 0.574...$ (left), for $k^2 = 0.830...$ (middle) and for $k^2 = 0.999...$ (right).

is equivalent to

$$\mathrm{dn}(z_0) = \frac{1}{\sqrt{2}}. \tag{37}$$

Using the symmetry around K of the function $k\,\mathrm{sn}(t) - \mathrm{dn}(t)$, we know that $z_0 = K \pm \delta z_0/2$. Therefore, the modulus for five bodies are given by $\mathrm{dn}(7K(k)/5, k) = 1/\sqrt{2}$ or $\mathrm{dn}(9K(k)/5, k) = 1/\sqrt{2}$.

Similarly, to put seven bodies, we can take three values of δz_0,

$$\delta z_0 = 4K \times \frac{1}{7},\ 4K \times \frac{2}{7} \text{ and } 4K \times \frac{3}{7}.$$

Corresponding value of modulus are

$$k^2 = \begin{cases} 0.574\,569\,280\,934\,588\,654\,057\ldots, \\ 0.830\,609\,000\,670\,624\,071\,081\ldots, \\ 0.999\,930\,000\,538\,037\,277\,285\ldots. \end{cases}$$

For N bodies, we have $(N-1)/2$ values of δz_0 and k^2 from nearly $1/2$ to nearly 1. If k^2 approaches $1/2$, the velocity of mass or equivalently mass distribution on the lemniscate curve approaches to uniform. On the other hand, if k^2 approaches to 1 the velocity of mass or equivalently mass distribution on the lemniscate varies place to place. See figures 8 and 9. This is because the following equation holds for any k^2, which can be easily verified by a direct calculation.

$$v(t)^2 + \left(k^2 - \frac{1}{2}\right) q(t)^2 = \frac{1}{2}. \tag{38}$$

For infinitely many bodies, therefore, we can take any value of $0 \le \delta z_0 < 2K$ and $1/2 \le k^2 < 1$. In the infinite bodies choreography, noting will be changed with time goes. So any choreographic motion of infinite bodies on the lemniscate conserve any physical quantities.

5.2 Equation of motion

As shown above, it is easy to put more than three bodies on the lemniscate with some conservation laws. But, these choreography do not seem to satisfy any equation of motion.

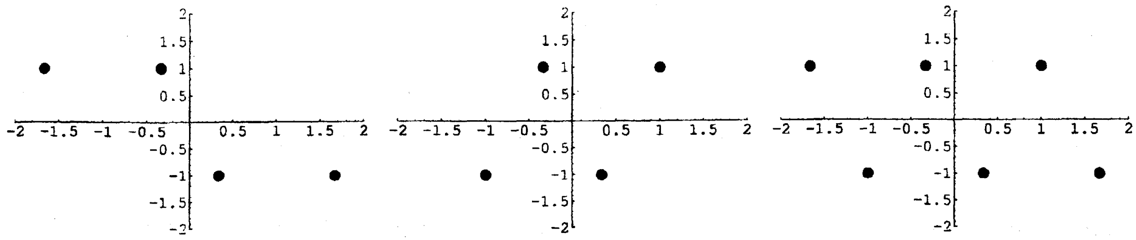

Figure 10: Poles of $x^{(-)}(t)$ (left), of $x^{(-)}(t+4K/3)$ (middle) and of $x^{(-)}(t+4K/3)-x^{(-)}(t)$ (right).

To see why the three bodies are simple whereas many bodies are complex, count the degrees of the elliptic function in the left hand side and the right hand side of the equation of motion for three bodies,

$$\frac{d^2x^{(+)}(t)}{dt^2} \sim \frac{1}{x^{(-)}\left(t+\frac{4K}{3}\right)-x^{(-)}(t)} + \frac{1}{x^{(-)}\left(t-\frac{4K}{3}\right)-x^{(-)}(t)}. \tag{39}$$

The function $x^{(+)}(t)$ is a elliptic function with 4 simple poles. Thus the left hand side of the above equation has 4 triple poles. Therefore, degree of the left hand side is 12.

In figure 10, pole structure for $x^{(-)}(t)$ and $x^{(-)}(t+4K/3)$ are shown. These poles are simple pole and 2 poles are common, therefore $x^{(-)}(t+4K/3)-x^{(-)}(t)$ has 6 simple poles as shown in the right picture. Then degree of this elliptic function is 6. Thus, inverse of this function is also degree 6. (Actually, as shown in section 4.3, the function $\left(x^{(-)}(t+4K/3)-x^{(-)}(t)\right)^{-1}$ has two triple poles.) This is the first term of the right hand side. The second term is also degree 6. Thus, in the right hand side, sum of two degrees 6 functions make degree 12 function, which balance the left hand side. Nothing remarkable is not needed in this point of view.

For a trial, let us consider the following equation of motion for five bodies,

$$\frac{d^2x^{(+)}(t)}{dt^2} \sim \sum_{j=1}^{4} \frac{1}{x^{(-)}\left(t+\frac{4jK}{5}\right)-x^{(-)}(t)}. \tag{40}$$

The left hand side has degree 12. In the right hand side, two terms would have degree 6 and two terms would have degree 8. Sum of these four terms must make degree 12 to balance the left hand side. Something remarkable would be needed here. Of course, this is not impossible. Because the lemniscate is so beautiful curve. But we don't think this is hopeful.

6 Summary

In this lecture note, we have given complementary explanation how we could find an analytic three body choreography on the lemniscate[5] according to our study in 2002. Especially we have given the reason why we are interested in the lemniscate as the figure-eight orbit by using the three tangents theorem.

Then, we discussed the possible interaction potential for any other three body choreography on the lemniscate. We could conclude that the potential would not be homogeneous or logarithmic. In order to derive this conclusion, we have shown the orbit of the figure-eight solution under $-1/r^2$ potential can not be expressed by the fourth order polynomial.

For N-body system with $N = 5, 7, 9, \ldots$ on the lemniscate, we have shown there are several parameterizations similar to the three body system which conserve the center of mass, the angular momentum and the moment of inertia. However, unfortunately, we are almost sure that they do not satisfy equation of motion under any interaction potential.

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