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Characterizations of Tree Maps Having Positive Entropy

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1. INTORDUCTION

考える空間はすべてコンパクト距離空間とし、写像はすべて連続とする。また $f: X \to X$ を X からそれ自身への写像とし、f の位相的エントロピーを $h(f)$ で表す。X がグラフあるいは、tree、閉区間の場合はそれぞれ graph map、tree map、interval map と呼ぶことにする。

この小論は graph map あるいは、tree map における位相的エントロピーを考える。interval map が正の位相的エントロピーを持つための特徴付けは、色々な研究者が調べており、知り尽くされている感がある。しかし、graph map に関しては J. Llibre と M. Misiurewicz([LM, Theorem B]) の結果しか知られていないし、あるいは、tree map においては X. Ye 達の結果以外ほとんど知られていない。また、1994 年に B. Schweizer and J. Smítal[SS] は distributionally chaos を導入した。彼らは interval map について調べて、interval map が正の位相的エントロピーを持つ必要条件はその写像が distributionally chaos であることであることを示した。すなわち、

Theorem 1.1. [SS] *Let $f : [0,1] \to [0,1]$ be a map. Then f has positive topological entropy if and only if f is distributionally chaotic.*

2001 年に J. Cánovas [C] が n-star $\mathbb{X}_n = \{z \in \mathbb{C} | z^n \in [0,1]\}$ $(n \in \mathbb{N})$ 上の写像 f で以下のことを示した。

Theorem 1.2. [C] *Let $f : \mathbb{X}_n \to \mathbb{X}_n$ be a map with periodic point* 0. *Then f has positive topological entropy if and only if f is distributionally chaotic.*

この論文からもわかるが、閉区間から tree に拡張するにあたっても、かなりギャップがあることがわかる。そこで私達は空間を tree に拡張し、写像の制限なしで証明した。すなわち、tree map が正の位相的エントロピーを持つ必要条件はその写像が distributionally chaos であることであることを示した。

2. A SURVEY OF SOME DEFINITIONS OF "CHAOS"

カオスが論文に初めて登場したのは Li と Yorke の論文 [LY] と言われている。

Definition 2.1. Let $\varepsilon > 0$. The subset $D \subset X$ with Card(D) ≥ 2 is an *scramble set* if for each $x, y \in D$ with $x \neq y$,

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \text{ and}$$

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0.$$

A map f is *chaotic in the sense of Li-Yorke* if f has an uncountable scramble set. (this is an extension of an inessentially modified version of the original definition of Li and Yorke [LY]) A map f is *weakly chaotic in the sense of Li-Yorke* if f has a scramble set.

定義から、Li-Yorke のカオスならば Li-Yorke の弱カオスであることがわかる。interval map ならば逆も成立する。

Theorem 2.2. [KS] *Every weakly chaotic in the sense of Li-Yorke is chaotic in the sense of Li-Yorke.*

しかし、一般には成立しない。

Example 2.3. There exists a weakly chaotic map $f : [0,1]^2 \to [0,1]^2$ in the sense of Li-Yorke which is non-chaotic in the sense of Li-Yorke.

つぎに、よく知られているカオスは Devaney によって定義されたカオスである。

Definition 2.4. This map f is *chaotic in the sense of Devaney* if

(1) f is *topologically transitive*, that is, for any non-empty open sets U and V in X, there exists some non-negative integer k such that $f^k(U) \cap V \neq \emptyset$,

(2) the set of all periodic points of f is dense in X, and

(3) f has *sensitive dependence on initial conditions*, i.e., there exists a number $\delta > 0$ such that for every point x of X and every neighborhood V of x, there exists a point y of V and a non-negative integer n such that $d(f^n(x), f^n(y)) > \delta$.

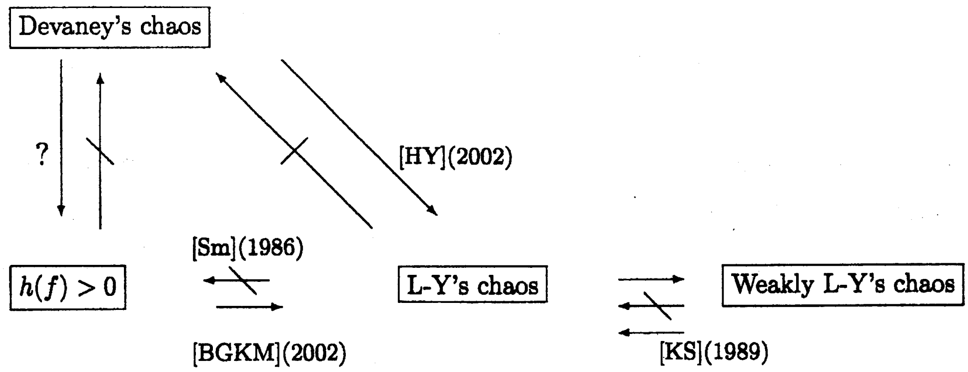
Remark 2.5. [BBCDS] The above conditions (1) and (2) imply the condition (3).

Example 2.6. (1) The tent map f_1 is chotic in the sense of Devaney and Li-Yorke with $h(f_1) = \log 2 > 0$.

(2) There exists an interval map f_2 with $h(f_2) > 0$ which is chotic in the sense of Li-Yorke and is non-chotic in the sense of Devaney.

(3) [Sm] There exists an interval map f_3 with $h(f_3) = 0$ which is chotic in the sense of Li-Yorke and is non-chotic in the sense of Devaney.

カオスと位相的エントロピーのことを図でまとめたのが以下の通りである。



3. Distributinal Chaos

つぎに Distributinal Chaos を定義しよう。

Definition 3.1. For $x, y \in X$, let us define the functions $F_{x,y}^{(n)} : \mathbb{R} \to \mathbb{R}$ by

$$F_{x,y}^{(n)}(t) = \frac{1}{n}\mathrm{Card}(\{\mathrm{m} | 0 \leq \mathrm{m} \leq \mathrm{n}-1 \text{ and } d(f^m(x), f^m(y)) < t\}),$$

where Card(P) denote the cardinality of a set P.

Remark 3.2. 以下簡単な性質を挙げておく。

(1) $0 \leq F_{x,y}^{(n)}(t) \leq 1$.

(2) $F_{x,y}^{(n)}(t) = 0$ if for all $t \leq 0$.

(3) $F_{x,y}^{(n)}(t) = 1$ if for all $\mathrm{diam(X)} \leq \mathrm{t}$.

(4) If let $\epsilon_a : \mathbb{R} \to \mathbb{R}$ be the distribution function given by

$$\epsilon_a(t) = \begin{cases} 0 & \text{if } t \leq a \\ 1 & \text{if } t > a, \end{cases} \tag{3.1}$$

$F_{x,x}^{(n)} = \epsilon_0$ if for all $x \in X$ and $n \in \mathbb{N}$.

Definition 3.3. Let us define the *upper and lower distribution functions* as :

$$F_{x,y}^{*}(t) = \limsup_{n\to\infty} F_{x,y}^{(n)}(t) \quad \text{and} \quad F_{x,y}(t) = \liminf_{n\to\infty} F_{x,y}^{(n)}(t)$$

Remark 3.4. 以下簡単な性質を挙げておく。

(1) If $\lim_{n\to\infty} d(f^n(x), f^n(y)) = a$,
then $F_{x,y} = F^*_{x,y} = \epsilon_a$.

(2) $\epsilon_{\mathrm{diam(X)}} \le F_{x,y} \le F^*_{x,y} \le \epsilon_0$.

Definition 3.5. [SS] The map f is said to be *distributionally chaotic* if there exist $x, y \in X$ and $t > 0$ such that $F^*_{x,y}(t) > F_{x,y}(t)$

Theorem 1.1 により、interval map は位相的エントロピーが正であることと distributionally chaos であることは同値であるが、彼らは A. N. Sharkovsky の結果「最大 ω-limit set」のタイプを分類して証明した。

しかし、2次元ではこの定理は成り立たない。

Theorem 3.6. [FP] *There exists a distributionally chaotic map* $f : [0,1]^2 \to [0,1]^2$ *with* $h(f) = 0$.

Question 3.7. *Is every map* f *with* $h(f) > 0$ *distributionally chaotic ?*

Theorem 3.8. [Ba] *There exists a distributionally chaotic map* $f : [0,1]^2 \to [0,1]^2$ *which is chotic in the sense of Li-Yorke.*

上の結果から distributionally chaos と Li-Yorke のカオスとは関係ないことがわかる。

4. Definition

Definition 4.1. The map f is said to be *turbulent* if there exist two arcs J, K with at most one common point such that $J \cup K \subset f(J) \cap f(K)$.

Definition 4.2. Let z be a periodic point of f. The *unstable set of z is defined to be the set*

$$W(z, f) = \{x \in X \mid \text{for any neighborhood } V \text{ of } z,\ x \in f^k(V) \text{ for some } k > 0\}.$$

A point y is homoclinic if there exists a point $z \neq y$ such that $f^n(z) = z$ for some $n > 0$, $y \in W(z, f^n)$ and $f^{kn}(y) = z$ for some $k > 0$. We say such a point y a homoclinic point.

This definition of homoclinic points first appeared in [B1].

Definition 4.3. A point $x \in X$ is a *nonwandering point for f if for any open set U containing x there exists $n > 0$ such that $f^n(U) \cap U \neq \emptyset$.*

Remark 4.4. If X is a closed interval, then f has positive topological entropy if and only if f has a (nonwandering) homoclinic point.

A circle map has positive topological entropy if and only if it has a nonwandering homoclinic point. There exists a circle map with homoclinic point and zero topological entropy.

Example 4.5 (cf.[B2, Example D, p.229]). We introduce an easy graph map $f : X \to X$. Let $\mathbb{S}^1_+ = \{e^{2\pi it} \in \mathbb{C} | 0 \le t \le 1/2\}$, $\mathbb{S}^1_- = \{e^{2\pi it} \in \mathbb{C} | 1/2 \le t \le 1\}$, $X_1 = \{re^{\pi i} \in \mathbb{C} | 1 \le r \le 3/2\}$ and $X = \mathbb{S}^1_+ \cup \mathbb{S}^1_- \cup X_1$.

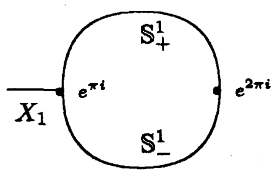


Figure 3.7.1.

Let us define a continuous map $g : X \to X$:

$$g(re^{2\pi it}) = \begin{cases} e^{2\pi i} & \text{if } re^{2\pi it} \in \mathbb{S}^1_- \\ e^{4\pi it} & \text{if } re^{2\pi it} \in \mathbb{S}^1_+ \\ e^{\pi i(r-1)} & \text{if } re^{2\pi it} \in X_1 \end{cases}$$

We see that f has only two nonwandering points $e^{\pi i}, e^{2\pi i}$ and only one periodic point $e^{2\pi i}$, and that $f(e^{\pi i}) = e^{2\pi i}$ and $e^{\pi i} \in W(e^{2\pi i}, f)$. Hence, $e^{\pi i}$ is a nonwandering homoclinic point for f. But [BC, CorollaryVIII7] implies that f has zero topological entropy.

Definition 4.6. We define the *ω-limit set of a point $x \in X$ to be the set*

$$\omega(x, f) = \bigcap_{m \ge 0} \mathrm{Cl}(\bigcup_{n \ge m} f^n(x)).$$

$y \in \omega(x, f)$ if and only if $f^{n_k}(x) \to y$ for some subsequence of positive integers $n_k \to \infty$.

5. Main Theorem

Theorem 5.1. *Let f be a continuous map from a tree X to itself. The following statements are equivalent:*

(1) *f has positive topological entropy.*
(2) *f^n is turbulent for some $n \in \mathbb{N}$.*
(3) *f has a (nonwandering) homoclinic point.*
(4) *f has an infinite ω-limit set which contains a periodic orbit.*
(5) *f is distributionally chaotic.*

J. Llibre と M. Misiurewicz([LM, Theorem B]) の結果から、主定理 (1) から主定理 (2)〜(5) を証明することは簡単であるが、逆に関しては明らかではない。特に主定理 (4) から主定理 (1) を証明することが長年ネックになっていた。interval map に関しては主定理はよく知られているが、主定理 (4) から主定理 (1) の証明には A. N. Sharkovsky の色々な結果が使われており、なかなかうまくいかなかったが、今回それを解消することに成功した。

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