The behaviour of dimension functions on unions of closed subsets I

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1 Introduction

All spaces we shall consider here are separable metrizable spaces.

It is well known that there exist (transfinite) dimension functions $d$ such that $d(X_1 \cup X_2) > \max\{dX_1, dX_2\}$ even if the subspaces $X_1$ and $X_2$ are closed in the union $X_1 \cup X_2$.

Let $\mathcal{K}$ be a class of spaces, $\beta, \alpha$ be ordinals such that $\beta < \alpha$, and $X$ be a space from $\mathcal{K}$ with $dX = \alpha$ which is the union of finitely many closed subsets with $d \leq \beta$. Define $m(X, d, \beta, \alpha) = \min\{k : X = \bigcup_{i=1}^{k} X_i, \text{ where } X_i \text{ is closed in } X \text{ and } dX_i \leq \beta\}$, $m_{\mathcal{K}}(d, \beta, \alpha) = \min\{m(X, d, \beta, \alpha) : X \in \mathcal{K} \text{ and } m(X, d, \beta, \alpha) \text{ exists}\}$ and $M_{\mathcal{K}}(d, \beta, \alpha) = \sup\{m(X, d, \beta, \alpha) : X \in \mathcal{K} \text{ and } m(X, d, \beta, \alpha) \text{ exists}\}$.

We will say that $m_{\mathcal{K}}(d, \beta, \alpha)$ and $M_{\mathcal{K}}(d, \beta, \alpha)$ do not exist if there is no space $X$ from $\mathcal{K}$ with $dX = \alpha$ which is the union of finitely many closed subsets with $d \leq \beta$. It is evident that either $m_{\mathcal{K}}(d, \beta, \alpha)$ and $M_{\mathcal{K}}(d, \beta, \alpha)$ satisfy $2 \leq m_{\mathcal{K}}(d, \beta, \alpha) \leq M_{\mathcal{K}}(d, \beta, \alpha) \leq \infty$ or they do not exist.

Two natural questions arise.

**Question 1.1** Determine the values of $m_{\mathcal{K}}(d, \beta, \alpha)$ and $M_{\mathcal{K}}(d, \beta, \alpha)$ for given $\mathcal{K}, d, \beta, \alpha$.

**Question 1.2** Find a (transfinite) dimension function $d$ having for given pair $2 \leq k \leq l \leq \infty$, $m_{\mathcal{K}}(d, \beta, \alpha) = k$ and $M_{\mathcal{K}}(d, \beta, \alpha) = l$.

Let $\mathcal{C}$ be the class of metrizable compact spaces and $\mathcal{P}$ be the class of separable completely metrizable spaces. By $\text{trind} (\text{trInd})$ we denote Hurewicz's (Smirnov's) transfinite extension of $\text{ind}$ ($\text{Ind}$) and $\text{Cmp}$ is the large inductive compactness degree introduced by de Groot. We shall recall their definitions in the next section. Let $\alpha = \lambda(\alpha) + n(\alpha)$ be the natural...
decomposition of the ordinal \( \alpha \geq 0 \) into the sum of a limit number \( \lambda(\alpha) \) (observe that \( \lambda(\text{an integer } \geq 0) = 0 \)) and a nonnegative integer \( n(\alpha) \). Let \( \beta < \alpha \) be ordinals, put 
\[
p(\beta, \alpha) = \frac{n(\alpha)+1}{n(\beta)+1} \quad \text{and} \quad q(\beta, \alpha) = \text{the smallest integer } \geq p(\beta, \alpha).
\]
We have the following theorems. The outline of the proof will be presented in section 2.

**Theorem 1.1** 1. Let \( 0 \leq \beta < \alpha \) be finite ordinals. Then we have \( m_P(Cmp, \beta, \alpha) = q(\beta, \alpha) \) and \( M_P(Cmp, \beta, \alpha) = \infty \).

2. Let \( \beta < \alpha \) be infinite ordinals. Then we have

\[
m_C(\text{trInd}, \beta, \alpha) = \begin{cases} 
q(\beta, \alpha), & \text{if } \lambda(\beta) = \lambda(\alpha), \\
does \text{not exist}, & \text{otherwise}
\end{cases}
\]

\[
M_C(\text{trInd}, \beta, \alpha) = \begin{cases} 
\infty, & \text{if } \lambda(\beta) = \lambda(\alpha), \\
does \text{not exist}, & \text{otherwise}
\end{cases}
\]

**Theorem 1.2** 1. For every finite \( \alpha \geq 1 \) there exists a space \( X_\alpha \in \mathcal{P} \) such that

(a) \( CmpX_\alpha = \alpha \);

(b) \( X_\alpha = \bigcup_{i=1}^{\infty} Y_i \), where each \( Y_i \) is closed in \( X_\alpha \) and \( CmpY_i \leq 0 \);

(c) \( X_\alpha \neq \bigcup_{i=1}^{m} Z_i \), where each \( Z_i \) is closed in \( X_\alpha \) and \( CmpZ_i \leq \alpha - 1 \) and \( m \) is any integer \( \geq 1 \).

2. For every infinite \( \alpha \) with \( n(\alpha) \geq 1 \) there exists a space \( X_\alpha \in \mathcal{C} \) such that

(a) \( \text{trInd}X_\alpha = \alpha \);

(b) \( X_\alpha = \bigcup_{i=1}^{\infty} Y_i \), where each \( Y_i \) is closed in \( X_\alpha \) and finite-dimensional;

(c) \( X_\alpha \neq \bigcup_{i=1}^{m} Z_i \), where each \( Z_i \) is closed in \( X_\alpha \) and \( \text{trInd}Z_i \leq \alpha - 1 \) and \( m \) is any integer \( \geq 1 \).

2 Evaluations of \( m_K(d, \beta, \alpha) \) and \( M_K(d, \beta, \alpha) \)

The notation \( X \sim Y \) means that the spaces \( X \) and \( Y \) are homeomorphic. At first we consider the following construction.

Step 1. Let \( X \) be a space without isolated points and \( P \) a countable dense subset of \( X \). Consider Alexandroff's dublicate \( D = X \cup X^1 \) of \( X \), where each point of \( X^1 \) is clopen in \( D \). Remove from \( D \) those points of \( X^1 \) which do not correspond to any point from \( P \). Denote the obtained space by \( L(X, P) \). Observe that \( L(X, P) \) is the disjoint union of \( X \) with the countable dense subset \( P^1 \) of \( L(X, P) \) consisting of points from \( X^1 \) corresponding to the points from \( P \). The space \( L(X, P) \) is separable and metrizable. It will be compact if \( X \) is compact. Put \( L_1(X, P) = L(X, P) \). Assume that \( X \) is a completely metrizable space (recall that the increment \( bX \setminus X \) in any compactification \( bX \) of \( X \) is an \( F_\sigma \)-set in \( bX \)). Observe that \( L(bX, P) \) is a compactification of \( L(X, P) \) and the increment
$L(bX, P) \setminus L(X, P) \sim bX \setminus X$ is an $F_\sigma$-set in $L(bX, P)$. Hence $L(X, P)$ is also completely metrizable.

**Step 2.** Let $X$ be a space with a countable subset $R$ consisting of isolated points. Let $Y$ be a space. Substitute each point of $R$ in $X$ by a copy of $Y$. The obtained set $W$ has the natural projection $pr : W \to X$. Define the topology on $W$ as the smallest topology such that the projection $pr$ is continuous and each copy of $Y$ has its original topology as a subspace of this new space. The obtained space is denoted by $L(X, R, Y)$. It is separable and metrizable and it will be compact (completely metrizable) if $X$ and $Y$ are the same. Moreover $L(X, R, Y)$ is the disjoint union of the closed subspace $X \setminus R$ of $X$ (which we will call basic for the space $L(X, R, Y)$) and countably many clopen copies of $Y$.

**Step 3.** Let $X$ be a space without isolated points and $P$ be a countable dense subset of $X$. Define $L_n(X, P) = L(L_1(X, P), P^1, L_{n-1}(X, P))$, $n \geq 2$. Observe that for any open subset $O$ of $L_n(X, P)$ meeting the basic subset $X$ of $L_n(X, P)$ there is a copy of $L_{n-1}(X, P)$ contained in $O$. Put $L_*(X, P) = \{\ast\} \cup \oplus_{n=1}^\infty L_n(X, P)$. (Here by $\{\ast\} \cup \oplus_{n=1}^\infty X_i$ we mean the one-point extension of the free union $\oplus_{n=1}^\infty X_i$ such that a neighborhood base at the point * consists of the sets $\{\ast\} \cup \oplus_{i=k}^\infty X_i$, $k = 1, 2, \ldots$). Observe that $L_*(X, P)$ is separable and metrizable, and it contains a copy of $L_q(X, P)$ for each $q$. $L_*(X, P)$ will be compact (completely metrizable) if $X$ is the same.

All our dimension functions $d$ are assumed to be monotone with respect to closed subsets and $d(\ast)$ \leq 0.

**Lemma 2.1** Let $d$ be a dimension function and $X$ be a space without isolated points which cannot be written as the union of $k \geq 1$ closed subsets with $d \leq \alpha$, where $\alpha$ is an ordinal. Let also $P$ be a countable dense subset of $X$. Then

(a) for every $q$ we have $L_q(X, P) \neq \bigcup_{i=1}^k X_i$, where each $X_i$ is closed in $L_q(X, P)$ and $dX_i \leq \alpha$;

(b) $L_*(X, P) \neq \bigcup_{i=1}^m X_i$, where each $X_i$ is closed in $L_*(X, P)$ and $dX_i \leq \alpha$, and $m$ is any integer \geq 1.

All our classes $\mathcal{K}$ of topological spaces are assumed to be monotone with respect to closed subsets and closed under operations $L(,)$ and $L(,)$.

**Lemma 2.2** Let $\mathcal{K}$ be a class of topological spaces, $\alpha$ be an ordinal \geq 0 and $d$ be a dimension function such that $dL(L(S, P), P^1, T) \leq \alpha$ for any $S, T$ from $\mathcal{K}$ with $dS \leq \alpha$, $dT \leq \alpha$ and any $P$. Let $X \in \mathcal{K}$ such that $X = \bigcup_{i=1}^k X_i$, where each $X_i$ is closed in $X$, without isolated points and $dX_i \leq \alpha$. Let also $P_i$ be a countable dense subset of $X_i$ for each $i$. Then for each $q$ the space $L_q(X, \bigcup_{i=1}^k P_i)$ exists and is the union of $k^q$ closed subsets with $d \leq \alpha$. 


We will say that a dimension function $d$ satisfies the sum theorem of type $A$ if for any $X$ being the union of two closed subspaces $X_1$ and $X_2$ with $dX_i \leq \alpha_i$, where each $\alpha_i$ is finite and $\geq 0$, we have $dX \leq \alpha_1 + \alpha_2 + 1$. A space $X$ is completely decomposable in the sense of the dimension function $d$ if $dX = \alpha$, where $\alpha$ is an integer $\geq 1$, and $X = \bigcup_{i=1}^{\alpha+1} X_i$, where each $X_i$ is closed in $X$ and $dX_i = 0$. Observe that if this space $X$ belongs to a class $\mathcal{K}$ of topological spaces then $m_{\mathcal{K}}(d, \beta, \alpha) \leq m(X, d, \beta, \alpha) \leq \alpha + 1$ for each $\beta$ with $0 \leq \beta < \alpha$.

We will say that a transfinite dimension function $d$ satisfies the sum theorem of type $A_{tr}$ if for any $X$ being the union of two closed subspaces $X_1$ and $X_2$ with $dX_i \leq \alpha_i$ and $\alpha_2 \geq \alpha_1$ we have $dX \leq \alpha_2$, if $\lambda(\alpha_1) < \lambda(\alpha_2)$, and $dX \leq \alpha_2 + n(\alpha_1) + 1$, if $\lambda(\alpha_1) = \lambda(\alpha_2)$. A space $X$ is completely decomposable in the sense of the transfinite dimension function $d$ if $dX = \alpha$, where $\alpha$ is an infinite ordinal with $n(\alpha) \geq 1$, and $X = \bigcup_{i=1}^{n(\alpha)+1} X_i$, where each $X_i$ is closed in $X$ and $dX_i = \lambda(\alpha)$. Observe that if this space $X$ belongs to a class $\mathcal{K}$ of topological spaces then $m_{\mathcal{K}}(d, \beta, \alpha) \leq m(X, d, \beta, \alpha) \leq n(\alpha) + 1$ for each $\beta$ with $\lambda(\alpha) \leq \beta < \alpha$.

To every space $X$ one assigns the large inductive compactness degree Cmp as follows.

(i) Cmp $X = -1$ iff $X$ is compact;

(ii) Cmp $X = 0$ iff there is a base $B$ for the open sets of $X$ such that the boundary $\text{Bd} U$ is compact for each $U$ in $B$;

(iii) Cmp $X \leq \alpha$, where $\alpha$ is an integer $\geq 1$, if for each pair of disjoint closed subsets $A$ and $B$ of $X$ there exists a part $C$ between $A$ and $B$ in $X$ such that Cmp $C \leq \alpha - 1$;

(iv) Cmp $X = \alpha$ if Cmp $X \leq \alpha$ and Cmp $X > \alpha - 1$;

(v) Cmp $X = \infty$ if Cmp $X > \alpha$ for every positive integer $\alpha$.

Recall also the definitions of the transfinite inductive dimensions trind and trInd.

(i) trInd$X = -1$ iff $X = \emptyset$;

(ii) trInd$X \leq \alpha$, where $\alpha$ is an ordinal $\geq 0$, if for each pair of disjoint closed subsets $A$ and $B$ of $X$ there exists a part $C$ between $A$ and $B$ in $X$ such that trInd$C < \alpha$;

(iii) trInd$X = \alpha$ if trInd$X \leq \alpha$ and trInd$X \leq \beta$ holds for no $\beta < \alpha$;

(iv) trInd$X = \infty$ if trInd$X \leq \alpha$ holds for no ordinal $\alpha$.

The definition of trind is obtained by replacing the set $A$ in (ii) with a point of $X$.

**Remark 2.1** (i) Note that Cmp satisfies the sum theorem of type $A$ ([ChH, Theorem 2.2]) and for each integer $\alpha \geq 1$ there exists a separable completely metrizable space $C_\alpha$ with Cmp $C_\alpha = \alpha$ which is completely decomposable in the sense of Cmp ([ChH, Theorem 3.1]).

For the convenience of the reader, we recall that $C_\alpha = \{0\} \times ([0,1]^\alpha \setminus (0,1)^\alpha) \cup \bigcup_{i=1}^{\infty} \{x_i\} \times [0,1]^\alpha \subset I^{\alpha+1}$, where $\{x_i\}_{i=1}^{\infty}$ is a sequence of real numbers such that $0 < x_{i+1} < x_i \leq 1$ for all $i$ and $\lim_{i \to \infty} x_i = 0$. Note that the closed subsets in the decomposition of $C_\alpha$ can be assumed without isolated points.

(ii) Note also that trInd satisfies the sum theorem of type $A_{tr}$ ([E, Theorem 7.2.7]) and for each infinite ordinal $\alpha$ with $n(\alpha) \geq 1$ there exists a metrizable compact space $S^\alpha$ (Smirnov's
compactum) with $\text{trInd} S^\alpha = \alpha$ which is completely decomposable in the sense of $\text{trInd}$ ([Ch, Lemma 3.5]). Recall that Smirnov's compacta $S^0, S^1, \ldots, S^\alpha, \ldots, \alpha < \omega_1$, are defined by transfinite induction: $S^0$ is the one-point space, $S^\alpha = S^\beta \times [0,1]$ for $\alpha = \beta + 1$, and if $\alpha$ is a limit ordinal, then $S^\alpha = \{ *_\alpha \} \cup \cup_{\beta<\alpha} S^\beta$ is the one-point compactification of the free union of all the previously defined $S^\beta$'s, where $*_\alpha$ is the compactifying point. Note that the closed subsets in the decomposition of $S^\alpha$ can be assumed without isolated points.

(iii) Observe that trind satisfies another sum theorem. Namely, for any $X$ being the union of two closed subspaces $X_1$ and $X_2$ with $\text{trind} X_1 \leq \alpha_1$ and $\alpha_2 \geq \alpha_1$ we have $\text{trind} X \leq \alpha_2$, if $\lambda(\alpha_1) < \lambda(\alpha_2)$, and $\text{trind} X \leq \alpha_2 + 1$, if $\lambda(\alpha_1) = \lambda(\alpha_2)$ [Ch, Theorem 3.9].

Proposition 2.1 (i) Let $\mathcal{K}$ be a class of topological spaces, $d$ be a dimension function satisfying the sum theorem of type A, $\alpha$ be an integer $\geq 1$ and $X$ be a space from $\mathcal{K}$ with $dX = \alpha$ which is completely decomposable in the sense of $d$. Then for any integer $0 \leq \beta < \alpha$ we have $m_{\mathcal{K}}(d, \beta, \alpha) = m(X, d, \beta, \alpha) = q(\beta, \alpha)$.

(ii) Let $\mathcal{K}$ be a class of topological spaces, $d$ be a transfinite dimension function satisfying the sum theorem of type $A_{tr}$, $\alpha$ be an infinite ordinal with $n(\alpha) \geq 1$ and $X$ be a space from $\mathcal{K}$ with $dX = \alpha$ which is completely decomposable in the sense of $d$. Then for any infinite ordinal $\beta < \alpha$ we have $m_{\mathcal{K}}(d, \beta, \alpha) = m(X, d, \beta, \alpha) = q(\beta, \alpha)$ if $\lambda(\beta) = \lambda(\alpha)$ and $m_{\mathcal{K}}(d, \beta, \alpha)$ does not exist otherwise.

The deficiency $\text{def}$ is defined in the following way: For a space $X$,

$$\text{def} X = \min\{\dim(Y \setminus X) : Y \text{ is a metrizable compactification of } X\}.$$  

Recall that $\text{Cmp} X \leq \text{def} X$ and $\text{def} X = 0$ iff $\text{Cmp} X = 0$.

Lemma 2.3 (i) def $L(L(X, P), P^1, Y) = \max\{\text{def } X, \text{def } Y\}$ for any $X$, $P$, $Y$. In particular, we have $\text{Cmp} L(L(X, P), P^1, Y) \leq 0$ if $\text{Cmp} X \leq 0$ and $\text{Cmp} Y \leq 0$.

(ii) $\text{trInd} L(L(X, P), P^1, Y) = \max\{\text{trInd} X, \text{trInd} Y\}$ for any compacta $X$, $Y$ and any $P$.

Proof. (i) Let $bX$ and $bY$ be metrizable compactifications of $X$ and $Y$ respectively such that $\dim(bX \setminus X) = \text{def } X$ and $\dim(bY \setminus Y) = \text{def } Y$. Observe that the space $L(bX, P, P^1, bY)$ is a compactification of $L(L(X, P), P^1, Y)$ and the increment $Z = L(bX, P, P^1, bY) \setminus L(L(X, P), P^1, Y)$ is the union of countably many closed subsets, one of which is homeomorphic to $bX \setminus X$ and the others are homeomorphic to $bY \setminus Y$. So by the countable sum theorem for $\dim$ we get that $\dim Z = \max\{\dim(bX \setminus X), \dim(bY \setminus Y)\} = \max\{\text{def } X, \text{def } Y\}$. Hence $\text{def} L(L(X, P), P^1, Y) \leq \max\{\text{def } X, \text{def } Y\}$, thereby $\text{def} L(L(X, P), P^1, Y) = \max\{\text{def } X, \text{def } Y\}$.

(ii) At first let us prove the statement when $Y$ is a singleton. Observe that in this case $L(L(X, P), P^1, Y) = L(X, P)$. Consider two disjoint closed subsets $A$ and $B$ of $L(X, P)$.
Recall that $L(X, P)$ contains a copy of $X$. Choose a partition $C$ between $A \cap X$ and $B \cap X$ in $X$. Extend the partition to a partition $C_1$ between $A$ and $B$ in $L(X, P)$. Consider another partition $C_2$ between $A$ and $B$ in $L(X, P)$ which is "thin" (i.e. $\text{Int}_{L(X, P)}C_2 = \emptyset$) and is in $C_1$. Observe that $C_2 \subset C$. Hence $\text{trInd}L(X, P) = \text{trInd}X$.

Now let us consider the general case. Assume that $A$ and $B$ are disjoint closed subsets in $L(L(X, P), P^1, Y)$. Recall that there is the natural continuous projection $pr : L(L(X, P), P^1, Y) \rightarrow L(X, P)$. Consider the closed subsets $prA$ and $prB$ of $L(X, P)$. If they are disjoint, choose a partition $C_2$ between $prA$ and $prB$ in $L(X, P)$ like in the previous part. Observe that $pr^{-1}C_2$ is a partition between $A$ and $B$ in $L(L(X, P), P^1, Y)$ such that $pr^{-1}C_2$ is homeomorphic to a closed subset of $C$. Assume now that $prA \cap prB \neq \emptyset$. Note that $Q^1 = prA \cap prB$ is finite and $L(L(X, P), P^1, Y)$ is the free union of $L(L(X, P \setminus Q)), P^1 \setminus Q^1, Y)$, where $Q$ is the finite subset of $P$ corresponding to $Q^1$ and finitely many copies of $Y$. Choose a partition between $A$ and $B$ in $X$ and a partition between $A$ and $B$ in each of the copies of $Y$ corresponding to points of $Q$. It follows from the foregoing discussion that the free union of these partitions constitutes a partition in $L(L(X, P), P^1, Y)$ between $A$ and $B$. We conclude that $\text{trInd}L(L(X, P), P^1, Y) = \max\{\text{trInd}X, \text{trInd}Y\}$. □

**Proof of Theorem 1.1.**

(i) Because of Remark 2.1 and Proposition 2.1, we need only establish that $M_{\mathcal{P}}(\text{Cmp}, \beta, \alpha) = \infty$. Consider the space $C_{\alpha} = \bigcup_{i=1}^{\alpha+1} X_i$, where each $X_i$ is closed in $X$, without isolated points and $\text{Cmp} X_i = 0$, from Remark 2.1. Let $P_i$ be a countable dense subset of $X_i$. Put $P = \bigcup_{i=1}^{\alpha+1} P_i$. Recall that def $C_{\alpha} = \alpha$ ([ChH, Theorem 3.1]). So by Lemma 2.3 for any integer $q$ we have def $L_q(C_{\alpha}, P) = \alpha$ and hence $\text{Cmp} L_q(C_{\alpha}, P) = \alpha$. Observe that by Lemmas 2.2 and 2.3, we get that the completely metrizable space $L_q(C_{\alpha}, P)$ is the union of $(\alpha+1)^q$ many closed subspaces with $\text{Cmp} \leq 0$. Hence $m(L_q(C_{\alpha}, P), \text{Cmp}, \beta, \alpha) \leq (\alpha+1)^q$. Since $\text{Cmp}$ satisfies the sum theorem of type A, $C_{\alpha}$ cannot be represented as $\alpha$-many closed subsets with $\text{Cmp} \leq 0$. By Lemma 2.1, we have $m(L_q(C_{\alpha}, P), \text{Cmp}, \beta, \alpha) \geq qa \geq q$. Since $\lim_{q \rightarrow \infty} q = \infty$ we get $M_{\mathcal{P}}(\text{Cmp}, \beta, \alpha) = \infty$.

(ii) By similar arguments as in the proof of (i) one can prove $M_{\mathcal{C}}(\text{trInd}, \beta, \alpha) = \infty$, if $\lambda(\beta) = \lambda(\alpha)$; and does not exist otherwise. □

**Proof of Theorem 1.2.**

(i) Put $X_\alpha = \{\ast\} \cup \bigoplus_{i=1}^{\alpha} L_i(C_{\alpha}, P)$. Observe that $X_\alpha$ is completely metrizable and is the union of countably many closed subspaces with $\text{Cmp} \leq 0$. Since def $X_\alpha = \alpha$, we have $\text{Cmp} X_\alpha = \alpha$. Now observe that $\lim_{t \rightarrow \infty} m(L_t(C_{\alpha}, P), \text{Cmp}, \alpha - 1, \alpha) = \infty$. Hence $X_\alpha$ cannot be written as the finite union of closed subsets with $\text{Cmp} \leq \alpha - 1$.

(ii) Put $X_\alpha = \{\ast\} \cup \bigoplus_{i=1}^{\infty} L_i(S_\alpha, P)$. Observe that $X_\alpha$ is compact and is the union of countably many finite-dimensional closed subspaces (recall that $S_\alpha$ and therefore $L_i(S_\alpha, P)$ have the same property). Since for each $i$, $\text{trInd}L_i(S_\alpha, P) = \alpha$, we have $\text{trInd}X_\alpha = \alpha$. Now
observe that \( \lim_{i \to \infty} m(L_i(S^\alpha, P), \trInd, \alpha - 1, \alpha) = \infty. \) Hence \( X_\alpha \) cannot be written as the finite union of closed subsets with \( \trInd \leq \alpha - 1. \)

**Remark 2.2** Let \( Q \) be the set of rational numbers of the closed interval \([0, 1]\). Recall that for the spaces \( X = Q \times [0, 1]^n \) and \( Y = ([0, 1] \setminus Q) \times I^n \) we have \( \cmp X = \def X = \cmp Y = \def Y = n \). It is easy to observe that \( X \) satisfies points (a)-(c) of Theorem 1.2 (i). However, \( X \) is not completely metrizable. Note that \( Y \) is completely metrizable and satisfies points (a) and (c) of Theorem 1.2 (i) but not (b). Observe that Smirnov's compactum \( S^\alpha \) with \( n(\alpha) \geq 1 \) satisfies points (a) and (b) of Theorem 1.2 (ii) but not (c). Note also that any Cantor manifold \( Z \) with \( \trInd Z = \alpha \), where \( \alpha \) is an infinite ordinal with \( n(\alpha) \geq 1 \), (see for such spaces for example in [G]) satisfies points (a) and (c) of Theorem 1.2 (ii) but not (b).

Let \( d \) be a (transfinite) dimension function. A space \( X \) with \( dX \neq \infty \) is said to have property \((*)_d\) if for every open nonempty subset \( O \) of the space \( X \) there exists a closed in \( X \) subset \( F \subset O \) with \( dF = dX \).

Observe that the spaces \( X, Y \) from Remark 2.2 have property \((*)_{\cmp} \) and \( Z \) has property \((*)_{\trInd} \).

**Proposition 2.2** Let \( X \) be a completely metrizable space with \( dX \neq \infty \). Then \( X \neq \bigcup_{i=1}^\infty X_i \), where each \( X_i \) is closed in \( X \) and \( dX_i < dX \) iff there exists a closed subspace \( Y \) of \( X \) such that

(i) \( dY = dX \) and

(ii) \( Y \) has the property \((*)_d\).

**Remark 2.3** This remark concerns non-metrizable compact spaces. Using the construction of Lokucievskij's example ([L, p. 140]), Chatyrko, Kozlov and Pasynkov [ChKP, Remark 3.15 (b)] presented for each \( n = 3, 4, \ldots \) a compact Hausdorff space \( X_n \) such that \( \ind X_n = 2 \) and \( m(X_n, \ind, 1, 2) = n \). Hence it is clear that \( m_N(\ind, 1, 2) = 2 \) and \( M_N(\ind, 1, 2) = \infty \), where \( N \) is the class of compact Hausdorff spaces. In [K] Kotkin constructed a compact Hausdorff space \( X \) with \( \ind X = 3 \) which is the union of three one-dimensional in the sense of \( \ind \) closed subspaces. Hence, \( m_N(\ind, 1, 3) = 3 \) and \( m_N(\ind, 2, 3) = 2 \). Filippov in [F] presented for every \( n \) a compact Hausdorff space \( F_n \) with \( \ind F_n = n \), which is the union of finitely many one-dimensional in the sense of \( \ind \) closed subspaces, thereby \( m_N(\ind, k, n) < \infty \) for each \( 1 \leq k < n \). By the sum theorem from Remark 2.1 (iii) for \( \ind \) which is valid in fact for all regular spaces, one can get that \( m_N(\ind, 1, n) \geq 2^{n-2} + 1 \) for each \( n \).
参考文献


