The behaviour of dimension functions on unions of closed subsets I

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1 Introduction

All spaces we shall consider here are separable metrizable spaces.

It is well known that there exist (transfinite) dimension functions $d$ such that $d(X_1 \cup X_2) > \max\{dX_1, dX_2\}$ even if the subspaces $X_1$ and $X_2$ are closed in the union $X_1 \cup X_2$.

Let $\mathcal{K}$ be a class of spaces, $\beta, \alpha$ be ordinals such that $\beta < \alpha$, and $X$ be a space from $\mathcal{K}$ with $dX = \alpha$ which is the union of finitely many closed subsets with $d \leq \beta$. Define $m(X, d, \beta, \alpha) = \min\{k : X = \bigcup_{i=1}^{k} X_i$, where $X_i$ is closed in $X$ and $dX_i \leq \beta\}$, $m_{\mathcal{K}}(d, \beta, \alpha) = \min\{m(X, d, \beta, \alpha) : X \in \mathcal{K} \text{ and } m(X, d, \beta, \alpha) \text{ exists}\}$ and $M_{\mathcal{K}}(d, \beta, \alpha) = \sup\{m(X, d, \beta, \alpha) : X \in \mathcal{K} \text{ and } m(X, d, \beta, \alpha) \text{ exists}\}$.

We will say that $m_{\mathcal{K}}(d, \beta, \alpha)$ and $M_{\mathcal{K}}(d, \beta, \alpha)$ do not exist if there is no space $X$ from $\mathcal{K}$ with $dX = \alpha$ which is the union of finitely many closed subsets with $d \leq \beta$. It is evident that either $m_{\mathcal{K}}(d, \beta, \alpha)$ and $M_{\mathcal{K}}(d, \beta, \alpha)$ satisfy $2 \leq m_{\mathcal{K}}(d, \beta, \alpha) \leq M_{\mathcal{K}}(d, \beta, \alpha) \leq \infty$ or they do not exist.

Two natural questions arise.

**Question 1.1** Determine the values of $m_{\mathcal{K}}(d, \beta, \alpha)$ and $M_{\mathcal{K}}(d, \beta, \alpha)$ for given $\mathcal{K}, d, \beta, \alpha$.

**Question 1.2** Find a (transfinite) dimension function $d$ having for given pair $2 \leq k \leq l \leq \infty$, $m_{\mathcal{K}}(d, \beta, \alpha) = k$ and $M_{\mathcal{K}}(d, \beta, \alpha) = l$.

Let $\mathcal{C}$ be the class of metrizable compact spaces and $\mathcal{P}$ be the class of separable completely metrizable spaces. By $\text{trind}(\text{trInd})$ we denote Hurewicz's (Smirnov's) transfinite extension of $\text{ind}$ ($\text{Ind}$) and $\text{Cmp}$ is the large inductive compactness degree introduced by de Groot. We shall recall their definitions in the next section. Let $\alpha = \lambda(\alpha) + n(\alpha)$ be the natural
decomposition of the ordinal $\alpha \geq 0$ into the sum of a limit number $\lambda(\alpha)$ (observe that $\lambda$(an integer $\geq 0$) = 0) and a nonnegative integer $n(\alpha)$. Let $\beta < \alpha$ be ordinals, put $p(\beta, \alpha) = \frac{n(\alpha)+1}{n(\beta)+1}$ and $q(\beta, \alpha) = \text{the smallest integer} \geq p(\beta, \alpha)$. We have the following theorems. The outline of the proof will be presented in section 2.

**Theorem 1.1** 1. Let $0 \leq \beta < \alpha$ be finite ordinals. Then we have $m_\mathcal{P}(\text{Cmp, } \beta, \alpha) = q(\beta, \alpha)$ and $M_\mathcal{P}(\text{Cmp, } \beta, \alpha) = \infty$.

2. Let $\beta < \alpha$ be infinite ordinals. Then we have

$$m_\mathcal{C}(\text{trInd, } \beta, \alpha) = \begin{cases} q(\beta, \alpha), & \text{if } \lambda(\beta) = \lambda(\alpha), \\ \text{does not exist}, & \text{otherwise} \end{cases}$$

$$M_\mathcal{C}(\text{trInd, } \beta, \alpha) = \begin{cases} \infty, & \text{if } \lambda(\beta) = \lambda(\alpha), \\ \text{does not exist}, & \text{otherwise} \end{cases}$$

**Theorem 1.2** 1. For every finite $\alpha \geq 1$ there exists a space $X_\alpha \in \mathcal{P}$ such that

(a) $\text{Cmp}X_\alpha = \alpha$;

(b) $X_\alpha = \bigcup_{i=1}^{\infty}Y_i$, where each $Y_i$ is closed in $X_\alpha$ and $\text{Cmp}Y_i \leq 0$;

(c) $X_\alpha \neq \bigcup_{i=1}^{m}Z_i$, where each $Z_i$ is closed in $X_\alpha$ and $\text{Cmp}Z_i \leq \alpha - 1$ and $m$ is any integer $\geq 1$.

2. For every infinite $\alpha$ with $n(\alpha) \geq 1$ there exists a space $X_\alpha \in \mathcal{C}$ such that

(a) $\text{trInd}X_\alpha = \alpha$;

(b) $X_\alpha = \bigcup_{i=1}^{\infty}Y_i$, where each $Y_i$ is closed in $X_\alpha$ and finite-dimensional;

(c) $X_\alpha \neq \bigcup_{i=1}^{m}Z_i$, where each $Z_i$ is closed in $X_\alpha$ and $\text{trInd}Z_i \leq \alpha - 1$ and $m$ is any integer $\geq 1$.

2 Evaluations of $m_\mathcal{K}(d, \beta, \alpha)$ and $M_\mathcal{K}(d, \beta, \alpha)$

The notation $X \sim Y$ means that the spaces $X$ and $Y$ are homeomorphic. At first we consider the following construction.

**Step 1.** Let $X$ be a space without isolated points and $P$ a countable dense subset of $X$. Consider Alexandroff’s dublicate $D = X \cup X^1$ of $X$, where each point of $X^1$ is clopen in $D$. Remove from $D$ those points of $X^1$ which do not correspond to any point from $P$. Denote the obtained space by $L(X, P)$. Observe that $L(X, P)$ is the disjoint union of $X$ with the countable dense subset $P^1$ of $L(X, P)$ consisting of points from $X^1$ corresponding to the points from $P$. The space $L(X, P)$ is separable and metrizable. It will be compact if $X$ is compact. Put $L_1(X, P) = L(X, P)$. Assume that $X$ is a completely metrizable space (recall that the increment $bX \setminus X$ in any compactification $bX$ of $X$ is an $F_\sigma$-set in $bX$). Observe that $L(bX, P)$ is a compactification of $L(X, P)$ and the increment
$L(bX, P) \setminus L(X, P)$ ($\sim bX \setminus X$) is an $F_\sigma$-set in $L(bX, P)$. Hence $L(X, P)$ is also completely metrizable.

**Step 2.** Let $X$ be a space with a countable subset $R$ consisting of isolated points. Let $Y$ be a space. Substitute each point of $R$ in $X$ by a copy of $Y$. The obtained set $W$ has the natural projection $pr : W \to X$. Define the topology on $W$ as the smallest topology such that the projection $pr$ is continuous and each copy of $Y$ has its original topology as a subspace of this new space. The obtained space is denoted by $L(X, R, Y)$. It is separable and metrizable and it will be compact (completely metrizable) if $X$ and $Y$ are the same. Moreover $L(X, R, Y)$ is the disjoint union of the closed subspace $X \setminus R$ of $X$ (which we will call basic for the space $L(X, R, Y)$) and countably many clopen copies of $Y$.

**Step 3.** Let $X$ be a space without isolated points and $P$ be a countable dense subset of $X$. Define $L_n(X, P) = L(L_1(X, P), P_1, L_{n-1}(X, P))$, $n \geq 2$. Observe that for any open subset $O$ of $L_n(X, P)$ meeting the basic subset $X$ of $L_n(X, P)$ there is a copy of $L_n(X, P)$ contained in $O$. Put $L_*(X, P) = \{*\} \cup \oplus_{i=1}^{\infty}L_n(X, P)$. (Here by $\{*\} \cup \oplus_{i=1}^{\infty}X_i$ we mean the one-point extension of the free union $\oplus_{i=1}^{\infty}X_i$ such that a neighborhood base at the point $*$ consists of the sets $\{*\} \cup \oplus_{i=1}^{\infty}X_i$, $k = 1, 2, \ldots$). Observe that $L_*(X, P)$ is separable and metrizable, and it contains a copy of $L_q(X, P)$ for each $q$. $L_*(X, P)$ will be compact (completely metrizable) if $X$ is the same.

All our dimension functions $d$ are assumed to be monotone with respect to closed subsets and $d(\text{a point}) \leq 0$.

**Lemma 2.1** Let $d$ be a dimension function and $X$ be a space without isolated points which cannot be written as the union of $k \geq 1$ closed subsets with $d \leq \alpha$, where $\alpha$ is an ordinal. Let also $P$ be a countable dense subset of $X$. Then

(a) for every $q$ we have $L_q(X, P) \neq \bigcup_{i=1}^{k}L_i$, where each $X_i$ is closed in $L_q(X, P)$ and $dX_i \leq \alpha$;

(b) $L_*(X, P) \neq \bigcup_{i=1}^{m}L_i$, where each $X_i$ is closed in $L_*(X, P)$ and $dX_i \leq \alpha$, and $m$ is any integer $\geq 1$.

All our classes $\mathcal{K}$ of topological spaces are assumed to be monotone with respect to closed subsets and closed under operations $L(,)$ and $L(,,)$.

**Lemma 2.2** Let $\mathcal{K}$ be a class of topological spaces, $\alpha$ be an ordinal $\geq 0$ and $d$ be a dimension function such that $dL(L(S, P), P_1, T) \leq \alpha$ for any $S, T$ from $\mathcal{K}$ with $dS \leq \alpha$, $dT \leq \alpha$ and any $P$. Let $X \in \mathcal{K}$ such that $X = \bigcup_{i=1}^{k}X_i$, where each $X_i$ is closed in $X$, without isolated points and $dX_i \leq \alpha$. Let also $P_i$ be a countable dense subset of $X_i$ for each $i$. Then for each $q$ the space $L_q(X, \bigcup_{i=1}^{k}P_i)$ exists and is the union of $k^{q}$ closed subsets with $d \leq \alpha$. 
We will say that a dimension function $d$ satisfies the sum theorem of type $A$ if for any $X$ being the union of two closed subspaces $X_1$ and $X_2$ with $dX_i \leq \alpha_i$, where each $\alpha_i$ is finite and $\geq 0$, we have $dX \leq \alpha_1 + \alpha_2 + 1$. A space $X$ is completely decomposable in the sense of the dimension function $d$ if $dX = \alpha$, where $\alpha$ is an integer $\geq 1$, and $X = \bigcup_{i=1}^{n+1} X_i$, where each $X_i$ is closed in $X$ and $dX_i = 0$. Observe that if this space $X$ belongs to a class $\mathcal{K}$ of topological spaces then $m_{\mathcal{K}}(d, \beta, \alpha) \leq m(X, d, \beta, \alpha) \leq \alpha + 1$ for each $\beta$ with $0 \leq \beta < \alpha$.

We will say that a transfinite dimension function $d$ satisfies the sum theorem of type $A_{tr}$ if for any $X$ being the union of two closed subspaces $X_1$ and $X_2$ with $dX_i \leq \alpha_i$ and $\alpha_2 \geq \alpha_1$ we have $dX \leq \alpha_2$, if $\lambda(\alpha_1) < \lambda(\alpha_2)$, and $dX \leq \alpha_2 + n(\alpha_1) + 1$, if $\lambda(\alpha_1) = \lambda(\alpha_2)$. A space $X$ is completely decomposable in the sense of the transfinite dimension function $d$ if $dX = \alpha$, where $\alpha$ is an infinite ordinal with $n(\alpha) \geq 1$, and $X = \bigcup_{i=1}^{n(\alpha)+1} X_i$, where each $X_i$ is closed in $X$ and $dX_i = \lambda(\alpha)$. Observe that if this space $X$ belongs to a class $\mathcal{K}$ of topological spaces then $m_{\mathcal{K}}(d, \beta, \alpha) \leq m(X, d, \beta, \alpha) \leq n(\alpha) + 1$ for each $\beta$ with $\lambda(\alpha) \leq \beta < \alpha$.

To every space $X$ one assigns the large inductive compactness degree $\operatorname{Cmp}$ as follows.

(i) $\operatorname{Cmp} X = -1$ iff $X$ is compact.
(ii) $\operatorname{Cmp} X = 0$ iff there is a base $B$ for the open sets of $X$ such that the boundary $\operatorname{Bd} U$ is compact for each $U \in B$.
(iii) $\operatorname{Cmp} X \leq \alpha$, where $\alpha$ is an integer $\geq 1$, if for each pair of disjoint closed subsets $A$ and $B$ of $X$ there exists a partition $\mathcal{C}$ between $A$ and $B$ in $X$ such that $\operatorname{Cmp} \mathcal{C} \leq \alpha - 1$.
(iv) $\operatorname{Cmp} X = \alpha$ if $\operatorname{Cmp} X \leq \alpha$ and $\operatorname{Cmp} X > \alpha - 1$.
(v) $\operatorname{Cmp} X = \infty$ if $\operatorname{Cmp} X > \alpha$ for every positive integer $\alpha$.

Recall also the definitions of the transfinite inductive dimensions $\operatorname{trInd}$ and $\operatorname{trInd}$.

(i) $\operatorname{trInd} X = -1$ iff $X = \emptyset$.
(ii) $\operatorname{trInd} X \leq \alpha$, where $\alpha$ is an ordinal $\geq 0$, if for each pair of disjoint closed subsets $A$ and $B$ of $X$ there exists a partition $\mathcal{C}$ between $A$ and $B$ in $X$ such that $\operatorname{trInd} \mathcal{C} < \alpha$.
(iii) $\operatorname{trInd} X = \alpha$ if $\operatorname{trInd} X \leq \alpha$ and $\operatorname{trInd} X \leq \beta$ holds for no $\beta < \alpha$.
(iv) $\operatorname{trInd} X = \infty$ if $\operatorname{trInd} X \leq \alpha$ holds for no ordinal $\alpha$.

The definition of $\operatorname{trInd}$ is obtained by replacing the set $A$ in (ii) with a point of $X$.

**Remark 2.1**
(i) Note that $\operatorname{Cmp}$ satisfies the sum theorem of type $A$ ([ChH, Theorem 2.2]) and for each integer $\alpha \geq 1$ there exists a separable completely metrizable space $C_{\alpha}$ with $\operatorname{Cmp} C_{\alpha} = \alpha$ which is completely decomposable in the sense of $\operatorname{Cmp}$ ([ChH, Theorem 3.1]).

For the convenience of the reader, we recall that $C_{\alpha} = \{0\} \times ([0,1]^\alpha \setminus (0,1)^\alpha) \cup \bigcup_{i=1}^{\infty} \{x_i\} \times [0,1]^\alpha \subset I_{\alpha+1}$, where $\{x_i\}_{i=1}^{\infty}$ is a sequence of real numbers such that $0 < x_{i+1} < x_{i} \leq 1$ for all $i$ and $\lim_{i \to \infty} x_{i-1} = 0$. Note that the closed subsets in the decomposition of $C_{\alpha}$ can be assumed without isolated points.

(ii) Note also that $\operatorname{trInd}$ satisfies the sum theorem of type $A_{tr}$ ([E, Theorem 7.2.7]) and for each infinite ordinal $\alpha$ with $n(\alpha) \geq 1$ there exists a metrizable compact space $S_{\alpha}$ (Smirnov's
compactum) with \( \text{trInd}S^\alpha = \alpha \) which is completely decomposable in the sense of \( \text{trInd} \) ([Ch, Lemma 3.5]). Recall that Smirnov's compacta \( S^0, S^1, \ldots, S^\alpha, \ldots \), \( \alpha < \omega_1 \), are defined by transfinite induction: \( S^0 \) is the one-point space, \( S^\alpha = S^\beta \times [0, 1] \) for \( \alpha = \beta + 1 \), and if \( \alpha \) is a limit ordinal, then \( S^\alpha = \{ \ast \alpha \} \cup \bigcup_{\beta < \alpha} S^\beta \) is the one-point compactification of the free union of all the previously defined \( S^\beta \)'s, where \( \ast \alpha \) is the compactifying point. Note that the closed subsets in the decomposition of \( S^\alpha \) can be assumed without isolated points.

(iii) Observe that \( \text{trind} \) satisfies another sum theorem. Namely, for any \( X \) being the union of two closed subspaces \( X_1 \) and \( X_2 \) with \( \text{trind}X_1 \leq \alpha_1 \) and \( \alpha_2 \geq \alpha_1 \) we have \( \text{trind}X \leq \alpha_2 \), if \( \lambda(\alpha_1) < \lambda(\alpha_2) \), and \( \text{trind}X \leq \alpha_2 + 1 \), if \( \lambda(\alpha_1) = \lambda(\alpha_2) \) [Ch, Theorem 3.9].

**Proposition 2.1** (i) Let \( \mathcal{K} \) be a class of topological spaces, \( d \) be a dimension function satisfying the sum theorem of type \( A \), \( \alpha \) be an integer \( \geq 1 \) and \( X \) be a space from \( \mathcal{K} \) with \( dX = \alpha \) which is completely decomposable in the sense of \( d \). Then for any integer \( 0 \leq \beta < \alpha \) we have \( m_{\mathcal{K}}(d, \beta, \alpha) = m(X, d, \beta, \alpha) = q(\beta, \alpha) \).

(ii) Let \( \mathcal{K} \) be a class of topological spaces, \( d \) be a transfinite dimension function satisfying the sum theorem of type \( A_{\text{tr}} \), \( \alpha \) be an infinite ordinal with \( n(\alpha) \geq 1 \) and \( X \) be a space from \( \mathcal{K} \) with \( dX = \alpha \) which is completely decomposable in the sense of \( d \). Then for any infinite ordinal \( \beta < \alpha \) we have \( m_{\mathcal{K}}(d, \beta, \alpha) = m(X, d, \beta, \alpha) = q(\beta, \alpha) \) if \( \lambda(\beta) = \lambda(\alpha) \) and \( m_{\mathcal{K}}(d, \beta, \alpha) \) does not exist otherwise.

The deficiency \( \text{def} \) is defined in the following way: For a space \( X \),

\[
\text{def} X = \min\{\dim(Y \setminus X) : Y \text{ is a metrizable compactification of } X\}
\]

Recall that \( \text{Cmp} X \leq \text{def} X \) and \( \text{def} X = 0 \) iff \( \text{Cmp} X = 0 \).

**Lemma 2.3** (i) \( \text{def} L(L(X, P), P^1, Y) = \max\{\text{def} X, \text{def} Y\} \) for any \( X, P, Y \). In particular, we have \( \text{Cmp} L(L(X, P), P^1, Y) \leq 0 \) if \( \text{Cmp} X \leq 0 \) and \( \text{Cmp} Y \leq 0 \).

(ii) \( \text{trInd} L(L(X, P), P^1, Y) = \max\{\text{trInd} X, \text{trInd} Y\} \) for any compacta \( X, Y \) and any \( P \).

Proof. (i) Let \( bX \) and \( bY \) be metrizable compactifications of \( X \) and \( Y \) respectively such that \( \dim(bX \setminus X) = \text{def} X \) and \( \dim(bY \setminus Y) = \text{def} Y \). Observe that the space \( L(L(bX, P), P^1, bY) \) is a compactification of \( L(L(X, P), P^1, Y) \) and the increment \( Z = L(L(bX, P), P^1, bY) \setminus L(L(X, P), P^1, Y) \) is the union of countably many closed subsets, one of which is homeomorphic to \( bX \setminus X \) and the others are homeomorphic to \( bY \setminus Y \). So by the countable sum theorem for \( \dim \) we get that \( \dim Z = \max\{\dim(bX \setminus X), \dim(bY \setminus Y)\} = \max\{\text{def} X, \text{def} Y\} \). Hence \( \text{def} L(L(X, P), P^1, Y) \leq \max\{\text{def} X, \text{def} Y\} \), thereby \( \text{def} L(L(X, P), P^1, Y) = \max\{\text{def} X, \text{def} Y\} \).

(ii) At first let us prove the statement when \( Y \) is a singleton. Observe that in this case \( L(L(X, P), P^1, Y) = L(X, P) \). Consider two disjoint closed subsets \( A \) and \( B \) of \( L(X, P) \).
Recall that $L(X, P)$ contains a copy of $X$. Choose a partition $C$ between $A \cap X$ and $B \cap X$ in $X$. Extend the partition to a partition $C_1$ between $A$ and $B$ in $L(X, P)$. Consider another partition $C_2$ between $A$ and $B$ in $L(X, P)$ which is "thin" (i.e. $\text{Int}_{L(X, P)}C_2 = \emptyset$) and is in $C_1$. Observe that $C_2 \subset C$. Hence $\text{trInd}L(X, P) = \text{trInd}X$.

Now let us consider the general case. Assume that $A$ and $B$ are disjoint closed subsets in $L(L(X, P), P^1, Y)$. Recall that there is the natural continuous projection $pr : L(L(X, P), P^1, Y) \rightarrow L(X, P)$. Consider the closed subsets $prA$ and $prB$ of $L(X, P)$. If they are disjoint, choose a partition $C$ between $prA$ and $prB$ in $L(X, P)$ like in the previous part. Observe that $pr^{-1}C$ is a partition between $A$ and $B$ in $L(L(X, P), P^1, Y)$ such that $pr^{-1}C$ is homeomorphic to a closed subset of $C$. Assume now that $prA \cap prB \neq \emptyset$. Note that $Q^1 = prA \cap prB$ is finite and $L(L(X, P), P^1, Y)$ is the free union of $L(L(X, (P \setminus Q)), P^1 \setminus Q^1, Y)$, where $Q$ is the finite subset of $P$ corresponding to $Q^1$ and finitely many copies of $Y$. Choose a partition between $A$ and $B$ in $X$ and a partition between $A$ and $B$ in each of the copies of $Y$ corresponding to points of $Q$. It follows from the foregoing discussion that the free union of these partitions constitutes a partition in $L(L(X, P), P^1, Y)$ between $A$ and $B$. We conclude that $\text{trInd}L(L(X, P), P^1, Y) = \max\{\text{trInd}X, \text{trInd}Y\}$. □

Proof of Theorem 1.1.

(i) Because of Remark 2.1 and Proposition 2.1, we need only establish that $M_p(\text{Cmp}, \beta, \alpha) = \infty$. Consider the space $C_\alpha = \bigcup_{i=1}^{\alpha+1} X_i$, where each $X_i$ is closed in $X$, without isolated points and Cmp $X_i = 0$, from Remark 2.1. Let $P_i$ be a countable dense subset of $X_i$. Put $P = \bigcup_{i=1}^{\alpha+1} P_i$. Recall that def $C_\alpha = \alpha$ ([ChH, Theorem 3.1]). So by Lemma 2.3 for any integer $q$ we have def $L_q(C_\alpha, P) = \alpha$ and hence Cmp $L_q(C_\alpha, P) = \alpha$. Observe that by Lemmas 2.2 and 2.3, we get that the completely metrizable space $L_q(C_\alpha, P)$ is the union of $(\alpha+1)^q$ many closed subspaces with Cmp $\leq 0$. Hence $m(L_q(C_\alpha, P), \text{Cmp}, \beta, \alpha) \leq (\alpha+1)^q$. Since Cmp satisfies the sum theorem of type A, $C_\alpha$ cannot be represented as $\alpha$-many closed subspaces with Cmp $\leq 0$. By Lemma 2.1, we have $m(L_q(C_\alpha, P), \text{Cmp}, \beta, \alpha) \geq \alpha q \geq q$. Since $\lim_{q \rightarrow \infty} q = \infty$ we get $M_p(\text{Cmp}, \beta, \alpha) = \infty$.

(ii) By similar arguments as in the proof of (i) one can prove $M_c(\text{trInd}, \beta, \alpha) = \infty$, if $\lambda(\beta) = \lambda(\alpha)$; and does not exist otherwise. □

Proof of Theorem 1.2.

(i) Put $X_\alpha = \{\ast\} \cup \bigoplus_{i=1}^{\infty} L_i(C_\alpha, P)$. Observe that $X_\alpha$ is completely metrizable and is the union of countably many closed subspaces with Cmp $\leq 0$. Since def $X_\alpha = \alpha$, we have Cmp $X_\alpha = \alpha$. Now observe that $\lim_{i \rightarrow \infty} m(L_i(C_\alpha, P), \text{Cmp} , \alpha - 1, \alpha) = \infty$. Hence $X_\alpha$ cannot be written as the finite union of closed subspaces with Cmp $\leq \alpha - 1$.

(ii) Put $X_\alpha = \{\ast\} \cup \bigoplus_{i=1}^{\infty} L_i(S^\alpha, P)$. Observe that $X_\alpha$ is compact and is the union of countably many finite-dimensional closed subspaces (recall that $S^\alpha$ and therefore $L_i(S^\alpha, P)$ have the same property). Since for each $i$, $\text{trInd}L_i(S^\alpha, P) = \alpha$, we have $\text{trInd}X_\alpha = \alpha$. Now
observe that \( \lim_{i \to \infty} m(L_i(S^\alpha, P), \text{trInd}, \alpha - 1, \alpha) = \infty \). Hence \( X_\alpha \) cannot be written as the finite union of closed subsets with \( \text{trInd} \leq \alpha - 1 \). 

**Remark 2.2** Let \( Q \) be the set of rational numbers of the closed interval \([0, 1]\). Recall that for the spaces \( X = Q \times [0, 1]^n \) and \( Y = ([0, 1] \setminus Q) \times I^n \) we have \( \text{Cmp} X = \text{def} X = \text{Cmp} Y = \text{def} Y = n \) ([AN, p. 18 and 56]). It is easy to observe that \( X \) satisfies points (a)-(c) of Theorem 1.2 (i). However, \( X \) is not completely metrizable. Note that \( Y \) is completely metrizable and satisfies points (a) and (c) of Theorem 1.2 (i) but not (b). Observe that Smirnov’s compactum \( S^\alpha \) with \( n(\alpha) \geq 1 \) satisfies points (a) and (b) of Theorem 1.2 (ii) but not (c). Note also that any Cantor manifold \( Z \) with \( \text{trInd}Z = \alpha \), where \( \alpha \) is infinite ordinal with \( n(\alpha) \geq 1 \), (see for such spaces for example in [O]) satisfies points (a) and (c) of Theorem 1.2 (ii) but not (b).

Let \( d \) be a (transfinite) dimension function. A space \( X \) with \( dX \neq \infty \) is said to have property \( (*)_d \) if for every open nonempty subset \( O \) of the space \( X \) there exists a closed in \( X \) subset \( F \subset O \) with \( dF = dX \).

Observe that the spaces \( X, Y \) from Remark 2.2 have property \( (*)_{\text{Cmp}} \) and \( Z \) has property \( (*)_{\text{trInd}} \).

**Proposition 2.2** Let \( X \) be a completely metrizable space with \( dX \neq \infty \). Then \( X \neq \bigcup_{i=1}^{\infty} X_i \), where each \( X_i \) is closed in \( X \) and \( dX_i < dX \) iff there exists a closed subspace \( Y \) of \( X \) such that

(i) \( dY = dX \) and

(ii) \( Y \) has the property \( (*)_d \).

**Remark 2.3** This remark concerns non-metrizable compact spaces. Using the construction of Lokucievskij’s example ([E, p. 140]), Chatyrko, Kozlov and Pasyukov [ChKP, Remark 3.15 (b)] presented for each \( n = 3, 4, \ldots \) a compact Hausdorff space \( X_n \) such that \( \text{ind} X_n = 2 \) and \( m(X_n, \text{ind}, 1, 2) = n \). Hence it is clear that \( m_N(\text{ind}, 1, 2) = 2 \) and \( M_N(\text{ind}, 1, 2) = \infty \), where \( N \) is the class of compact Hausdorff spaces. In [K] Kotkin constructed a compact Hausdorff space \( X \) with \( \text{ind} X = 3 \) which is the union of three one-dimensional in the sense of \( \text{ind} \) closed subspaces. Hence, \( m_N(\text{ind}, 1, 3) = 3 \) and \( M_N(\text{ind}, 2, 3) = 2 \). Filippov in [F] presented for every \( n \) a compact Hausdorff space \( F_n \) with \( \text{ind} F_n = n \), which is the union of finitely many one-dimensional in the sense of \( \text{ind} \) closed subspaces, thereby \( m_N(\text{ind}, k, n) < \infty \) for each \( 1 \leq k < n \). By the sum theorem from Remark 2.1 (iii) for \( \text{ind} \) which is valid in fact for all regular spaces, one can get that \( m_N(\text{ind}, 1, n) \geq 2^{n-2} + 1 \) for each \( n \).
参考文献


