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On dense subsets of the boundary of a Coxeter system

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The purpose of this note is to introduce main results of my recent paper [10] about dense subsets of the boundary of a Coxeter system.

A Coxeter group is a group \( W \) having a presentation

\[
\langle S \mid (st)^{m(s,t)} = 1 \text{ for } s, t \in S \rangle,
\]

where \( S \) is a finite set and \( m : S \times S \to \mathbb{N} \cup \{\infty\} \) is a function satisfying the following conditions:

1. \( m(s,t) = m(t,s) \) for each \( s, t \in S \),
2. \( m(s,s) = 1 \) for each \( s \in S \), and
3. \( m(s,t) \geq 2 \) for each \( s, t \in S \) such that \( s \neq t \).

The pair \((W, S)\) is called a Coxeter system. Let \((W, S)\) be a Coxeter system. For a subset \( T \subset S \), \( W_T \) is defined as the subgroup of \( W \) generated by \( T \), and called a parabolic subgroup. If \( T \) is the empty set, then \( W_T \) is the trivial group. A subset \( T \subset S \) is called a spherical subset of \( S \), if the parabolic subgroup \( W_T \) is finite. For each \( w \in W \), we define \( S(w) = \{s \in S \mid \ell(ws) < \ell(w)\} \), where \( \ell(w) \) is the minimum length of word in \( S \) which represents \( w \). For a subset \( T \subset S \), we also define \( W^T = \{w \in W \mid S(w) = T\} \).

Let \((W, S)\) be a Coxeter system and let \( S' \) be the family of spherical subsets of \( S \). We denote \( WS' \) as the set of all cosets of the form \( wW_T \), with \( w \in W \) and \( T \in S' \). The sets \( S' \) and \( WS' \) are partially ordered by inclusion. Contractible simplicial complexes \( K(W, S) \) and \( \Sigma(W, S) \) are
defined as the geometric realizations of the partially ordered sets $S'$ and $WS'$, respectively ([7, §3], [5]). The natural embedding $S' \rightarrow WS'$ defined by $T \mapsto WT$ induces an embedding $K(W, S) \rightarrow \Sigma(W, S)$ which we regard as an inclusion. The group $W$ acts on $\Sigma(W, S)$ via simplicial automorphism. Then $\Sigma(W, S) = WK(W, S)$ ([5], [7]). For each $w \in W$, $wK(W, S)$ is called a chamber of $\Sigma(W, S)$. If $W$ is infinite, then $\Sigma(W, S)$ is noncompact. In [12], G. Moussong proved that a natural metric on $\Sigma(W, S)$ satisfies the CAT(0) condition. Hence, if $W$ is infinite, $\Sigma(W, S)$ can be compactified by adding its ideal boundary $\partial\Sigma(W, S)$ ([6, §4], [8]). This boundary $\partial\Sigma(W, S)$ is called the boundary of $(W, S)$. We note that the natural action of $W$ on $\Sigma(W, S)$ is properly discontinuous and cocompact ([5], [6]), and this action induces an action of $W$ on $\partial\Sigma(W, S)$.

A subset $A$ of a space $X$ is said to be dense in $X$, if $\overline{A} = X$. A subset $A$ of a metric space $X$ is said to be quasi-dense, if there exists $N > 0$ such that each point of $X$ is $N$-close to some point of $A$.

Let $(W, S)$ be a Coxeter system. Then $W$ has the word metric $d_\ell$ defined by $d_\ell(w, w') = \ell(w^{-1}w')$ for each $w, w' \in W$.

Here we obtained the following theorems in [10].

**Theorem 1.** Let $(W, S)$ be a Coxeter system. Suppose that $W^{\{s_0\}}$ is quasi-dense in $W$ with respect to the word metric and $m(s_0, t_0) = \infty$ for some $s_0, t_0 \in S$. Then there exists $\alpha \in \partial\Sigma(W, S)$ such that the orbit $W\alpha$ is dense in $\partial\Sigma(W, S)$.

Suppose that a group $\Gamma$ acts properly and cocompactly by isometries on a CAT(0) space $X$. Every element $\gamma \in \Gamma$ such that the order $o(\gamma) = \infty$ is a hyperbolic transformation of $X$, i.e., there exists a geodesic axis $c : \mathbb{R} \rightarrow X$ and a real number $a > 0$ such that $\gamma \cdot c(t) = c(t + a)$ for each $t \in \mathbb{R}$ ([3]). Then, for all $x \in X$, the sequence $\{\gamma^i x\}$ converges to $c(\infty)$ in $X \cup \partial X$. We denote $\gamma^\infty = c(\infty)$.

**Theorem 2.** Let $(W, S)$ be a Coxeter system. If the set

$$\bigcup\{W^{\{s\}} \mid s \in S \text{ such that } m(s, t) = \infty \text{ for some } t \in S\}$$
is quasi-dense in $W$, then $\{w^\infty \mid w \in W \text{ such that } o(w) = \infty\}$ is dense in $\partial \Sigma(W, S)$.

Remark. For a negatively curved group $G$ and the boundary $\partial G$ of $G$,
(1) we can show that $G\alpha$ is dense in $\partial G$ for each $\alpha \in \partial G$ by an easy argument, and
(2) it is known that $\{g^\infty \mid g \in G \text{ such that } o(g) = \infty\}$ is dense in $\partial G$ ([2]).

Example. Let $S = \{s, t, u\}$ and let
$$W = \langle S \mid s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (us)^3 = 1 \rangle.$$ Then $(W, S)$ is a Coxeter system and $W^{\{s\}}$ is quasi-dense in $W$. On the other hand, for any $\alpha \in \partial \Sigma(W, S)$, $W\alpha$ is a finite-points set and not dense in $\partial \Sigma(W, S)$ which is a circle. Thus we can not omit the assumption "$m(s_0, t_0) = \infty$" in Theorem 1.

We showed the following lemma in [10].

Lemma 3. Let $(W, S)$ be a Coxeter system. Suppose that there exist a maximal spherical subset $T$ of $S$ and $s_0 \in S$ such that $m(s_0, t) \geq 3$ for each $t \in T$ and $m(s_0, t_0) = \infty$ for some $t_0 \in T$. Then $W^{\{s_0\}}$ is quasi-dense in $W$.

As an application of Theorems 1 and 2, we can obtain the following corollary from Lemma 3.

Corollary 4. Let $(W, S)$ be a Coxeter system. Suppose that there exist a maximal spherical subset $T$ of $S$ and an element $s_0 \in S$ such that $m(s_0, t) \geq 3$ for each $t \in T$ and $m(s_0, t_0) = \infty$ for some $t_0 \in T$. Then
(1) $W\alpha$ is dense in $\partial \Sigma(W, S)$ for some $\alpha \in \partial \Sigma(W, S)$, and
(2) $\{w^\infty \mid w \in W \text{ such that } o(w) = \infty\}$ is dense in $\partial \Sigma(W, S)$.

Example. The Coxeter system defined by the diagram in Figure 1 is not hyperbolic in Gromov sense, since it contains a copy of $\mathbb{Z}^2$, and it satisfies the condition of Corollary 4.
REFERENCES


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