<table>
<thead>
<tr>
<th>Title</th>
<th>Extension of Bing Maps (General and Geometric Topology and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Kato, Hisao; Matsuhashi, Eiichi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2004), 1370: 102-116</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/25466">http://hdl.handle.net/2433/25466</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Extension of Bing Maps

筑波大学数学系 加藤久男 (Hisao Kato)
筑波大学数理物質科学研究科 松橋英士 (Eiichi Matsuhashi)

Abstract

In [7], M. Levin proved that the set of all Bing maps of a compact metric space to the unit interval constitutes a $G_\delta$-dense subset of the space of maps. In [6], J. Krasinkiewicz independently proved that the set of all Bing maps of a compact metric space to an $n$-dimensional manifold ($n \geq 1$) constitutes a $G_\delta$-dense subset of the space of maps. In [9], J. Song and E. D. Tymchatyn solved some problems of J. Krasinkiewicz [6]: They proved that the set of all Bing maps of a compact metric space to a nondegenerate connected polyhedron (or a 1-dimensional locally connected continuum) constitutes a $G_\delta$-dense subset of the space of maps. In this note, by using methods of Levin [7] and Krasinkiewicz [6], we prove the extension theorem of Bing maps which is slightly precise than the theorem of J. Song and E. D. Tymchatyn.

1 Introduction

In this note, all spaces are separable metrizable spaces and maps are continuous functions. We denote the unit interval $[0, 1]$ by $I$. An arc is a space which is homeomorphic to $I$. If $X$ is a compact metrizable space and $Y$ is a space, $C(X, Y)$ denotes the space of all continuous maps from $X$ to $Y$ endowed with sup metric. A compact metrizable space is called a compactum, and a continuum means a connected compactum. A map $f$ is called a $\epsilon$-map if all diameters of fibers of $f$ are smaller than $\epsilon$. A continuum is said to be indecomposable if it is not sum of two proper subcontinua. A compactum is called a Bing compactum (or said to be hereditarily indecomposable) if each of its subcontinua is indecomposable. A map is called a Bing map if each of its fibers is a Bing compactum. In [7], M. Levin proved the following theorem.
Theorem 1 (M. Levin [7]) For each compactum $X$, the set of all Bing maps in $C(X, I)$ is a $G_δ$-dense subset in $C(X, I)$.

On the other hand, J. Krasinkiewicz proved the next theorem independently.

Theorem 2 (J. Krasinkiewicz [6]) Let $X$ be a compactum and let $Y$ be an $n$-dimensional manifold ($n \geq 1$). Then the set of all Bing maps in $C(X, Y)$ is a $G_δ$-dense subset in $C(X, Y)$.

Note that Theorem 2 is a generalization of Theorem 1. In [6], J. Krasinkiewicz poses the following problem: If $Y$ in Theorem 2 is an other space (for example, dendrite, dendroid, polyhedron, locally connected continuum, the Menger universal curve, AR, ANR), does Theorem 2 hold? In [9], J. Song and E. D. Tymchatyn solved the problems of J. Krasinkiewicz: In particular, they proved the following:

Theorem 3 (J. Song and E. D. Tymchatyn [9]) The set of all Bing maps of a compact metric space to a nondegenerate connected polyhedron (or a 1-dimensional locally connected continuum) constitutes a $G_δ$-dense subset of the space of maps.

In this note, we prove the following theorem by using methods of Levin [7] and Krasinkiewicz [6], which is more precise than the above theorem of J. Song and E. D. Tymchatyn. The proofs are somewhat different from one of J. Song and E. D. Tymchatyn [9]. For case of graphs, we use an idea of M. Levin [7], and for general case of polyhedra, we will use an idea of J. Krasinkiewicz [6].

Theorem 4 (Extension Theorem of Bing Maps) Let $X$ be a compactum and let $A$ be a closed subset in $X$. Let $\mathcal{K}$ be a finite simplicial complex such that $|\mathcal{K}|$ is a nondegenerate connected polyhedron and let $\mathcal{L}$ be a subcomplex of $\mathcal{K}$. If $f : A \rightarrow |\mathcal{L}|$ is a Bing map and $\tilde{f} : X \rightarrow |\mathcal{K}|$ is a map with $\tilde{f}|A = f$ and $\tilde{f}^{-1}(|\mathcal{L}|) = A$, then for any $\epsilon > 0$ there exists a Bing map $g : X \rightarrow |\mathcal{K}|$ such that $g|A = f$ and $d(\tilde{f}, g) < \epsilon$.

As a corollary, we obtain the theorem of J. Song and E. D. Tymchatyn. Also, we investigate surjective Bing maps from continua to polyhedra.
2 Preliminaries

In this section, first we give some definitions which are used in this paper.

Notation 5 Let $X$ be a space and let $d$ be a metric on $X$. We denote the identity map of $X$ by $id_X$. For a subset $A \subset X$ and $\delta > 0$, denote $B(A, \delta) = \{x \in X \mid$ there exists $a \in A$ such that $d(x, a) < \delta\}$, $\text{diam}A = \sup\{d(x, y) \mid x, y \in A\}$, $\text{cl}A = \{x \in X \mid$ if $U$ is a neighborhood of $x$ then $U \cap A \neq \emptyset\}$, $\text{int}A = \{x \in X \mid$ there exists a neighborhood $V$ of $x$ such that $V \subset A\}$. If $\mathcal{A}$ is a family of subsets of $X$, denote $\text{mesh}\mathcal{A} = \sup\{\text{diam}A \mid A \in \mathcal{A}\}$. If $\sigma$ is a simplex, we denote the boundary of $\sigma$ by $\partial\sigma$. If $\mathcal{K}$ is a simplicial complex and $n \in \mathbb{N}$, we denote $\mathcal{K}^{(n)} = \{\sigma \in \mathcal{K} \mid \dim\sigma \leq n\}$ and $|\mathcal{K}| = \bigcup_{\sigma \in \mathcal{K}} \sigma$. For each arc $I$ and $x \neq y \in I$, $[x, y]_I$ means an arc in $I$ from $x$ to $y$ and $[x, y)_I$, $(x, y]_I$, $(x, y)_I$ mean $[x, y]_I \setminus \{y\}$, $[x, y]_I \setminus \{x\}$, $[x, y]_I \setminus \{x, y\}$ respectively.

Now we will give the definition of $D$-crooked. The definition of $D$-crooked was originally introduced in [2], and the definition below was given in [7].

Definition 6 Let $\mathcal{D} = \{(F_0, F_1, V_0, V_1) \mid F_0, F_1$ are disjoint closed subsets in $\mathbb{R}^n$ and $V_0, V_1$ are disjoint open neighborhoods of $F_0, F_1$ in $\mathbb{R}^n\}$ and $D = (F_0, F_1, V_0, V_1) \in \mathcal{D}$. A subspace $X \subset \mathbb{R}^n$ is $D$-crooked if there exists an open neighborhood $U$ of $X$ in $\mathbb{R}^n$ such that for any map $f : I \to U$ with the property $f(0) \in F_0$ and $f(1) \in F_1$, there exist $t_0, t_1$ with $0 < t_0 < t_1 < 1$ such that $f(t_0) \in V_1$ and $f(t_1) \in V_0$. A map is said to be $D$-crooked if each of its fibers is $D$-crooked.

Clearly, subspaces of $D$-crooked spaces are also $D$-crooked. M. Levin obtained the following propositions in [7].

Proposition 7 (M. Levin [7]) If $A \subset \mathbb{R}^n$ is $D$-crooked, then there exists a neighborhood $U \subset \mathbb{R}^n$ of $A$ such that $U$ is $D$-crooked.

Proposition 8 (M. Levin [7]) A compactum $A \subset \mathbb{R}^n$ is a Bing compactum if and only if $A$ is $D$-crooked for each $D \in \mathcal{D}$.

Proposition 9 (M. Levin [7]) There exist $D_1, D_2, \ldots \in \mathcal{D}$ such that for any compactum $A \subset \mathbb{R}^n$, $A$ is a Bing compactum if and only if $A$ is $D_i$-crooked for each $i = 1, 2, \ldots$.
The next theorem was proved by R. H. Bing. Many authors used the theorem to reach important conclusions (for example, the theorem is used in the proof of Theorem 1).

**Theorem 10** (R. H. Bing [2]) *Let* $X$ *be a compactum and let* $A$, $B$ *be disjoint closed subsets in* $X$. *Then there exists a Bing compactum* $L$ *such that* $L$ *is a partition between* $A$ *and* $B$.

Now, we recall the definition of inverse limits. Let $\{X_i, f_i\}_{i=1}^{\infty}$ be a double sequence of spaces $X_i$, called coordinate spaces, and maps $f_i : X_{i+1} \to X_i$, called bonding maps. Then inverse limit of $\{X_i, f_i\}_{i=1}^{\infty}$, denoted by $\lim_{\leftarrow} \{X_i, f_i\}$, is the subspace of $\prod_{i=1}^{\infty} X_i$ defined by $\lim_{\leftarrow} \{X_i, f_i\} = \{(x_i) \in \prod_{i=1}^{\infty} X_i | f_i(x_{i+1}) = x_i \text{ for each } i = 1, 2, \ldots\}$. For $Y = \lim_{\leftarrow} \{X_i, f_i\}$ and $i = 1, 2, \ldots$, a map $p_i : Y \to X_i$ is called a $i$-th projection if $p_i$ satisfies $p_i((x_j)_{j=1}^{\infty}) = x_i$ for each $(x_j)_{j=1}^{\infty} \in Y$. It is well known that every $n$-dimensional continuum is an inverse limit of $n$-dimensional compact connected polyhedra with onto bonding maps.

### 3 Bing maps to Peano curves

A space is called a *Peano space* if the space is locally connected. A space $X$ is called a *Peano curve* if $X$ is a 1-dimensional Peano continuum. In this section, we prove the theorem of J. Song and E. D. Tymchatyn for graphs by using Levin's idea [7].

**Theorem 11** (J. Song and E. D. Tymchatyn [9]) *Let* $X$ *be a compactum and let* $Y$ *be a Peano curve. Then the set of all Bing maps in* $C(X, Y)$ *is a* $G_\delta$-dense subset in $C(X, Y)$.

Before we prove Theorem 11, we prove some lemmas. The next lemma follows from Theorem 10 which plays very important role in the proof of Lemma 13.

**Lemma 12** *Let* $X$ *be a compactum and let* $F_1, F_2, \ldots, F_k$ $(k \geq 2)$ *be pairwise disjoint closed subsets in* $X$. *Then there exist pairwise disjoint open subsets $U_1, U_2, \ldots, U_k$ such that* $F_i \subset U_i$ *for* $i = 1, 2, \ldots, k$ *and* $X \setminus \bigcup_{i=1}^{k} U_i$ *is a Bing compactum.*
Proof. We will prove Lemma 12 by the induction on \( k \). For \( k = 2 \), Lemma 12 holds by Theorem 10. Suppose that Lemma 12 holds for \( k = 2, 3, \ldots, n - 1 \) \( (n \geq 3) \). Let \( F_1, F_2, \ldots, F_n \) be pairwise disjoint closed subsets in \( X \). By the inductive assumption there exist pairwise disjoint open subsets \( U_1, U_2, \ldots, U_{n-2}, V_{n-1} \) such that \( F_1 \subset U_1, \ldots, F_{n-2} \subset U_{n-2}, F_{n-1} \subset V_{n-1} \) and \( L_1 = X \setminus \left( \bigcup_{i=1}^{n-2} U_i \cup V_{n-1} \right) \) is a Bing compactum. Since \( F_{n-1} \) and \( (X \setminus V_{n-1}) \cup F_n \) are disjoint, there exist disjoint open subsets \( U_{n-1}, W_{n-1} \) such that \( F_{n-1} \subset U_{n-1}, (X \setminus V_{n-1}) \cup F_n \subset W_{n-1} \) and \( X \setminus (U_{n-1} \cup W_{n-1}) \) is a Bing compactum. Let \( U_n = W_{n-1} \setminus (X \setminus V_{n-1}) \). We see that \( F_i \subset U_i \) for \( i = 1, 2, \ldots, n \) and \( U_1, U_2, \ldots, U_n \) are pairwise disjoint. And since \( X \setminus (U_1 \cup U_2 \cup \cdots \cup U_n) = L_1 \cup (X \setminus (U_{n-1} \cup W_{n-1})) \) and \( L_1, X \setminus (U_{n-1} \cup W_{n-1}) \) are pairwise disjoint Bing compacta, \( X \setminus (U_1 \cup U_2 \cup \cdots \cup U_n) \) is a Bing compactum. So \( U_1, U_2, \ldots, U_n \) have the required property. This completes the proof.

The proof of the next lemma is inspired by the proof of Theorem 1. Let us recall that a compactum \( X \) is called a graph if \( X \) is a 1-dimensional polyhedron.

**Lemma 13** Let \( X \) be a compactum and let \( G \) be a connected graph. Then the set of all Bing maps in \( C(X, G) \) is a \( G_\delta \)-dense subset in \( C(X, G) \).

Proof. Let \( X \subset \mathbb{R}^n \) be a compactum, \( f \in C(X, G) \) and \( \varepsilon > 0 \). Set \( D \) as in Definition 6 and \( D_1, D_2, \ldots \in D \) as in Proposition 9. Put \( D_i(X, G) = \{ g \in C(X, G) \mid g \text{ is a } D_i \text{-crooked map} \} \) for each \( i = 1, 2, \ldots \).

By Proposition 8 and 9, \( \{ g \in C(X, G) \mid g \text{ is a Bing map} \} = \bigcap_{i=1}^{\infty} D_i(X, G) \).

By Baire theorem it is sufficient to show that \( D_i(X, G) \) is an open dense subset in \( C(X, G) \).

Claim 1. \( D_i(X, G) \) is an open subset in \( C(X, G) \). This result has been proved in [7]. For completeness, we give the proof.

Proof of Claim 1. Let \( g \in D_i(X, G) \). By Proposition 7 and Since \( g \) is a closed map, we can take an open cover \( V \) of \( G \) such that \( f^{-1}(V) \) is \( D_i \)-crooked for each \( V \in V \). Let \( \delta \) be a Lesbegue number of the restriction of this cover to \( g(X) \). Then \( h \in D_i(X, G) \) for each \( h \in C(X, G) \) with \( d(h, g) < \delta/2 \).

Claim 2. \( D_i(X, G) \) is a dense subset of \( C(X, G) \).

Proof of Claim 2. Let \( f \in C(X, G) \) and \( \varepsilon > 0 \). Take a simplicial complex \( \mathcal{K} \) of \( G \) such that mesh \( \mathcal{K} < \varepsilon \). At first we will show that \( f \) can be
approximated by a map $f' \in C(X, G)$ with the property that $f^{-1}(p)$ is a Bing compactum for each $p \in \mathcal{K}^{(0)}$. Let $\{p_j\}_{j=1}^{m} = \mathcal{K}^{(0)} \setminus \{p \in \mathcal{K}^{(0)} \mid p \text{ is an endpoint of } G\}$. For $j = 1$, let $I_{1\ell}, I_{2\ell}, \ldots, I_{1k} \in \mathcal{K}^{(1)}$ be all edges which contain $p_1$ as their endpoint, and let $p_{1\ell}$ be the other endpoint of $I_{1\ell}$ for $\ell = 1, 2, \ldots, k$. Take $r_{\ell} \in I_{1\ell} \setminus \{p_{1\ell}, p_{1\ell+1}\}$ for $\ell = 1, 2, \ldots, k$. Let $A_1 = \bigcup_{\ell=1}^{k} I_{1\ell}$.

Since $f^{-1}([r_1, p_{11}], I_{11}), f^{-1}([r_2, p_{12}], I_{12}), \ldots, f^{-1}([r_k, p_{1k}], I_{1k})$ are pairwise disjoint closed subsets in $f^{-1}(A_1)$, by Lemma 12, there exist open subsets $U_1, U_2, \ldots, U_k$ in $f^{-1}(A_1)$ such that $f^{-1}([r_{\ell}, p_{1\ell}], I_{1\ell}) \subset U_\ell$ for $\ell = 1, 2, \ldots, k$, and $L_1 = f^{-1}(A_1) \setminus \bigcup_{\ell=1}^{k} U_\ell$ is a Bing compactum. Now, we construct $f_\ell : L_1 \cup U_\ell \to I_{1\ell}$ such that $f_\ell|^{-1}([r_{\ell}, p_{1\ell}], I_{1\ell}) = f|^{-1}([r_{\ell}, p_{1\ell}], I_{1\ell})$ and $f_\ell^{-1}(p_{1\ell}) =$ $L_1$ for $\ell = 1, 2, \ldots, k$. We can take a map $f_\ell : L_1 \cup U_\ell \to I_{1\ell}$ defined by $f_\ell(x) = f_\ell(x)$ if $x \in L_1 \cup U_\ell \setminus f^{-1}((r_{\ell}', p_{1\ell}')]_{I_{1\ell}})$ and $f_\ell(x) = f(x)$ if $x \in f^{-1}((r_{\ell}', p_{1\ell}')]_{I_{1\ell}})$ has the required property. Define $f_\ell : I_{1\ell} \to A_1$ by $f_\ell'(x) = f_\ell(x)$ if $x \in L_1 \cup U_\ell$ for $\ell = 1, 2, \ldots, k$. Define $f_1 : X \to G$ by $f_1(x) = f(x)$ if $x \in \text{cl}(X \setminus f^{-1}(A_1))$ and $f_1(x) = f_1'(x)$ if $x \in f^{-1}(A_1)$. Then $d(f, f_1) < \epsilon$ and $f_1^{-1}(p_{1\ell})$ is a Bing compactum.

If the step above has been done for $j \leq n - 1$ $(2 \leq n \leq m)$, then do the same step for $j = n$. Then $f$ can be approximated by a map $f_n : X \to G$ such that $f^{-1}(p_1)$ is a Bing compactum for $j = 1, 2, \ldots, n$. So we can take a map $f_1 : X \to G$ such that $d(f, f_1) < \epsilon$ and $f_1^{-1}(p_1)$ is a Bing compactum for each $j = 1, 2, \ldots, m$. And we may assume that $f_1(x)$ is not an endpoint of $G$ for each $x \in X$. So $f$ can be approximated by a map $f_1 : X \to G$ such that $f_1^{-1}(p)$ is a Bing compactum for each $p \in \mathcal{K}^{(0)}$. So we may assume that $A = \bigcup_{p \in \mathcal{K}^{(0)}} f^{-1}(p)$ is a Bing compactum.

Now, we will use an idea of the proof of [7, Theorem 1.8]. Let $D_1 = (F_0, F_1, V_0, V_1) \in \mathcal{D}$. Take closed neighborhoods $E_0, E_1$ of $F_0, F_1$ such that $F_0 \subset E_0 \subset V_0$ and $F_1 \subset E_1 \subset V_1$. Since $D_1 = (E_0, E_1, V_0, V_1) \in \mathcal{D}$ and $A$ is a Bing compactum, by Proposition 8 $A$ is $D_1'$-crooked. By Proposition 7 there exists a neighborhood $B$ of $A$ such that $B$ is $D_1'$-crooked. We claim that $H = B \cup \text{int} E_0 \cup \text{int} E_1$ is $D_1$-crooked.

Let $\varphi : \mathbb{N} \to H$ be a map with $\varphi(0) \in F_0$ and $\varphi(1) \in F_1$. Let $b_0 = \max\{b \in \mathbb{N} : \varphi(b) \in E_0\}$ and $b_1 = \min\{b \in \mathbb{N} : b > b_0 \text{ and } \varphi(b) \in E_1\}$. Since $B$ is $D_1'$-crooked, there exist $t_0, t_1 \in \mathbb{N}$ with $b_0 < t_0 < t_1 < b_1$ such that $\varphi(t_0) \in V_1$ and $\varphi(t_1) \in V_0$. So $H$ is $D_1$-crooked.

Since $(X \setminus H) \cap F_0 = \phi = (X \setminus H) \cap F_1$, $X \setminus H$ is $D_1$-crooked and since $(X \setminus H) \cap A = \phi$, $A \cup (X \setminus H)$ is $D_1$-crooked. Let $\mathcal{K}^{(1)} = \{I_1, I_2, \ldots, I_s\}$. For
each \( I_j \in \mathcal{K}^{(1)} \), let \( p_{j_1}, p_{j_2} \) be the endpoints of \( I_j \), and \( X_j = f^{-1}(I_j) \) and \( S_j = (X \setminus H) \cap X_j \). Define \( g_j : X_j \to I_j \) such that \( g_j^{-1}(p_{j_1}) = f^{-1}(p_{j_1}) \cup S_j \) and \( g_j^{-1}(p_{j_2}) = f^{-1}(p_{j_2}) \) for \( j = 1, 2, \ldots, s \). Define \( g : X \to G \) by \( g(x) = g_j(x) \) if \( x \in X_j \). Then, \( d(f, g) < \varepsilon \) and for each \( y \in G \), \( g^{-1}(y) \subset H \) or \( g^{-1}(y) \subset (X \setminus H) \cup A \). In both cases, \( g^{-1}(y) \) is \( D_\varepsilon \)-crooked. This completes the proof.

**Remark 14** In the proof of Claim 1 we only use the fact that \( X \) is compact. So for each compactum \( X \) and space \( Y \), the set of all Bing maps in \( C(X, Y) \) is a \( G_\delta \)-subset in \( C(X, Y) \).

The following definition was given in [6].

**Definition 15** Let \( Y \) be a space. We say that \( Y \) is *free* if for every compactum \( X \) the set of all Bing maps in \( C(X, Y) \) is a dense subset in \( C(X, Y) \).

A map \( f : X \to Y \) is called an *n-dimensional map* if \( \dim f^{-1}(y) \leq n \) for each \( y \in Y \). Note that 0-dimensional maps are Bing maps. By the theorem of Hurewicz for mappings and dimension, we see that if \( X \) is a compactum and \( P \) is a polyhedron such that \( \dim X > \dim P \), then there is no 0-dimensional map \( f \) from \( X \) to \( P \).

We need the next lemma.

**Lemma 16** Let \( Y \) be a space. If for each \( \varepsilon > 0 \) there exist a free compactum \( Z \) and maps \( p : Y \to Z \) and \( q : Z \to Y \) such that \( d(q \circ p, \text{id}_Y) < \varepsilon \) and \( q \) is a 0-dimensional map, then \( Y \) is a free space.

Proof. Let \( X \) be a compactum and let \( h : X \to Y \) be a map. By the assumption there exists a free compactum \( Z \) and maps \( p : Y \to Z \) and \( q : Z \to Y \) such that \( d(q \circ p, \text{id}_Y) < \varepsilon \) and \( q \) is a 0-dimensional map. Since \( q \) is uniformly continuous, there exists \( \delta > 0 \) such that if \( a, b \in Z \) satisfy \( d(a, b) < \delta \), then \( d(q(a), p(b)) < \varepsilon \). Since \( Z \) is free, there exists a Bing map \( \varphi : X \to Z \) such that \( d(p \circ h, \varphi) < \delta \). Let \( \psi = q \circ \varphi \), then \( \psi \) is a Bing map because \( q \) is 0-dimensional and \( \varphi \) is a Bing map. And \( d(h, \psi) = d(h, q \circ \varphi) \leq d(h, q \circ p \circ h) + d(q \circ p \circ h, q \circ \varphi) < \varepsilon + \varepsilon = 2\varepsilon \). So \( Y \) is a free space.

Now, we will give the proof of Theorem 11.

Proof of Theorem 11. By Remark 14, it is sufficient to show that \( Y \) is free. So we will show that \( Y \) satisfies the condition of Lemma 16. Let \( h \in C(X, Y) \)
and $\varepsilon > 0$. Since $Y$ is a 1-dimensional continuum, $Y$ can be written as $Y = \lim \downarrow \{G_i, f_i\}_{i=1}^{\infty}$, where $G_i$ is a graph and $f_i : G_{i+1} \to G_i$ is surjective for $i = 1, 2, \ldots$ Since $Y$ is Peano continuum, there exists $\varepsilon_1 > 0$ such that if $x, y \in Y$ satisfy $d(x, y) < \varepsilon_1$, then there exists an arc $A$ in $Y$ such that $A$ contains $x$ and $y$ as its endpoints and $\text{diam} A < \varepsilon$. Let $\varepsilon_2 = \min\{\varepsilon, \varepsilon_1\}$. Take $i$ sufficient large so that the projection $p_i : Y \to G_i$ is an $\varepsilon_2$-mapping. Since $p_i$ is a closed map to a compactum, there exists $\varepsilon_3 > 0$ such that if $B \subset G_i$ satisfies $\text{diam} B < \varepsilon_3$, then $\text{diam} p_i^{-1}(B) < \varepsilon_2$. Let $K$ be a subdivision of $G_i$ with $\text{mesh} K < \varepsilon_3$. Let $\mathcal{K}^{(0)} = \{v_j\}_{j=1}^{m}$ and $\mathcal{K}^{(1)} = \{I_{\ell}\}_{\ell=1}^{n}$. Take $a_{j} \in p^{-1}(v_j)$ for $j = 1, 2, \ldots$ Let $I_{\ell} \in \mathcal{K}^{(1)}$ and let $v_{\ell_{1}}, v_{\ell_{2}}$ be endpoints of $I_{\ell}$. Since $\text{diam} I_{\ell} < \varepsilon_3$, it follows that $\text{diam} (p_{\ell_{1}}^{-1}(I_{\ell})) < \varepsilon_2$. Take $a_{\ell_{1}} \in p^{-1}(v_{\ell_{1}}) \cap \{a_{j}\}_{j=1}^{m}$ and $a_{\ell_{2}} \in p^{-1}(v_{\ell_{2}}) \cap \{a_{j}\}_{j=1}^{m}$. Since $d(a_{\ell_{1}}, a_{\ell_{2}}) < \varepsilon_2$, there exists an embedding $q_{\ell} : I_{\ell} \to Y$ such that $\text{diam} (q_{\ell}(I_{\ell})) < \varepsilon$, $q_{\ell}(p_{\ell_{1}}) = a_{\ell_{1}}$ and $q_{\ell}(p_{\ell_{2}}) = a_{\ell_{2}}$. Define $q_i : G_i \to Y$ by $q_i(x) = q_{\ell}(x)$ if $x \in I_{\ell}$ for $\ell = 1, 2, \ldots, n$. Then $d(id_{Y}, q_{i} \circ p_{i}) < 2\varepsilon$, and $|q_{i}^{-1}(y)| < \infty$ for each $y \in Y$. So $Y$ satisfies the condition of Lemma 16. This completes the proof.

Remark 17 In the proof of Lemma 13, we used an idea of M. Levin (see the proof of [7, Theorem 1.8]). Also, we can prove Lemma 13 by using an idea of J. Krasinkiewicz [6, Lemma (5.2)] (compare the proof of Lemma 13 with the proofs of Lemma 22, 23 and Theorem 24 in the next section).

4 Bing maps to polyhedra

In this section, by the method of J. Krasinkiewicz [6] we prove Theorem 24 and as an application of this theorem, we show the theorem of J. Song and E. D. Tymchatyn: the set of all Bing maps in $C(X, P)$ is a $G_{\delta}$-dense subset in $C(X, \mathbb{P})$, where $X$ is any compactum and $P$ is any nondegenerate connected polyhedron. The next definition was given in [6].

Definition 18 Let $X$ be a compactum and let $p \in C(X, I)$. We say that $X$ is folded relatively $p$ (folded rel $p$) if there exist closed subsets $F_0$, $F_{1/2}$, $F_1$ such that

1) $F_0 \cup F_{1/2} \cup F_1 = X$.
2) $F_0 \cap F_1 = \emptyset$.
3) $p^{-1}(0) \subset F_0$, $p^{-1}(1) \subset F_1$.
4) $F_0 \cap F_{1/2} \subset p^{-1}([1/2, 1])$, $F_{1/2} \cap F_1 \subset p^{-1}([0, 1/2])$. 


A subset \( X' \subset X \) is said to be **folded rel \( p \)** if \( X' \) is folded rel \( p|X' \). A map \( f \) from \( X \) to a compactum \( Y \) is said to be **folded rel \( p \)** if \( f^{-1}(y) \) is folded rel \( p \) for each \( y \in Y \).

**Lemma 19** (J. Krasinkiewicz [6]) Let \( X \) be a compactum and let \( Y \) be a space. Then for each \( p \in C(X, \mathbb{I}) \) we have:

1. If \( X \) is folded rel \( p \), then for each \( q \in C(Y, X) \) \( Y \) is folded rel \( p \circ q \). In particular every subset of \( X \) is folded \( p \).

2. If \( F \) is a subset of \( X \) folded rel \( p \), then some neighborhood of \( F \) in \( X \) is folded rel \( p \).

**Lemma 20** (J. Krasinkiewicz [6]) For each compactum \( X \), there exists \( \mathcal{P} = \{p_i\}_{i=1}^{\infty} \subset C(X, \mathbb{I}) \) such that a closed subset \( B \subset X \) is a Bing compactum if and only if \( B \) is folded rel \( p_i \) for each \( i = 1, 2, \ldots \).

**Lemma 21** (J. Krasinkiewicz [6]) Let \( X \) be a compactum and let \( Y \) be a space. Then for each \( p \in C(X, \mathbb{I}) \), the set \( \{f \in C(X, Y) \mid f \text{ is folded rel } p \} \) is an open subset in \( C(X, Y) \).

The next lemma is a key lemma in this paper. The proof is based on an idea of J. Krasinkiewicz [6, Lemma (5.2)].

**Lemma 22** Let \( X \) be a compactum and let \( A \) be a closed subset in \( X \). Let \( \varepsilon > 0 \), \( \sigma^n (n \geq 1) \) an \( n \)-dimensional simplex, and \( p : X \to \mathbb{I} \) a map. If \( f : A \to \partial \sigma^n \) is a Bing map and \( \tilde{f} : X \to \sigma^n \) is a map with \( \tilde{f}|A = f \), then there exists a map \( g : X \to \sigma^n \) such that \( g|A = f \), \( d(\tilde{f}, g) < \varepsilon \) and \( g \) is folded rel \( p \).

Proof. Let \( \varepsilon > 0 \). Let \( f : A \to \partial \sigma^n \) be a Bing map and let \( \tilde{f} : X \to \sigma^n \) be a map with \( \tilde{f}|A = f \). Let \( \varphi : \sigma^n \times \mathbb{I} \to \sigma^n \) and \( \psi : \sigma^n \times \mathbb{I} \to \mathbb{I} \) be projections. We may assume that \( \tilde{f} \) satisfies \( \tilde{f}^{-1}(\partial \sigma^n) = A \). Since \( \tilde{f} \) is a closed map, by (2) of Lemma 19 there exists \( \mathcal{V} \) which is a family of open subsets in \( \sigma^n \) such that \( \partial \sigma^n \subset \bigcup \mathcal{V} \) and \( \tilde{f}^{-1}(\mathcal{V}) \) is folded rel \( p \) for each \( V \in \mathcal{V} \). Let \( r = d(\partial \sigma^n, \sigma^n \setminus \bigcup \mathcal{V}) \) and let \( Z = \{y \in \sigma^n \mid d(y, \partial \sigma^n) \geq r/2\} \). There exists \( \delta \geq 0 \) such that if \( B \subset \sigma^n \) satisfies \( \text{diam} B < \delta \) and \( B \cap \{y \in \sigma^n \mid d(y, \partial \sigma^n) = r/2\} \neq \emptyset \) then \( B \) is contained in some member of \( \mathcal{V} \). Let \( \mathcal{U} = \{U_i\}_{i=1}^{k} \) be a finite family of open \( n \)-discs in \( \sigma^n \) such that \( Z \subset \bigcup \mathcal{U} \) and \( \text{mesh} \mathcal{U} < \min\{\delta/2, r/2, \varepsilon\} \). We can assume that no proper subfamily of \( \mathcal{U} \) covers \( Z \). Take \( a_i \in U_i \setminus \bigcup_{j \neq i} U_j \) for \( i = 1, 2, \ldots \). There exist compact sets
$Z_1, Z_2, \ldots, Z_k$ such that $\bigcup_{i=1}^k Z_i = Z$ and $Z_i \subset U_i$ for $i = 1, 2, \ldots, k$. For each $i = 1, 2, \ldots, k$ there exists a PL $n$-disc $D_i$ in $U_i$ such that $Z_i \subset \text{int} D_i$. For each $i = 1, 2, \ldots, k$, there exists an open $n$-disc $G_i$ such that $D_i \cup \{a_i\} \subset G_i \subset \text{cl} G_i \subset U_i$. For each $i = 1, 2, \ldots, k$, there exists a neighborhood $W_i$ of $a_i$ such that $W_i \subset G_i \setminus \bigcup_{j \neq i} G_j$. Let $O_1, O_2, \ldots, O_k$ be pairwise disjoint open intervals in $(0,1/2)$. For each $i = 1, 2, \ldots, k$, take $r_i, s_i, t_i \in O_i$ such that $r_i < s_i < t_i$. For each $i = 1, 2, \ldots, k$, there exists PL $(n+1)$-disc $E_i$ such that $(a_i, 3/4) \in \text{int} E_i \subset E_i \subset G_i \times O_i \cup W_i \times (t_i, 1)$, $D_i \times [r_i, t_i] \cap E_i \subset \partial D_i \times [r_i, t_i]$ and $Q_i = D_i \times [r_i, t_i] \cup E_i$ is closed PL $(n+1)$-disc. Since $\{G_i \times O_i \cup W_i \times (t_i, 1)\}_{i=1}^k$ is pairwise disjoint and $Q_i \subset G_i \times O_i \cup W_i \times (t_i, 1)$ for $i = 1, 2, \ldots, k$, $Q_1, Q_2, \ldots, Q_k$ are pairwise disjoint. So there exists an isotopy $H_t : \sigma^n \times I \to \sigma^n \times I$ such that $H_t|(\sigma^n \times I \setminus \bigcup_{i=1}^k \text{int} Q_i) = \text{id}_{\sigma^n \times I}$, $H_t|Q_i$ is a homeomorphism of $Q_i$ to itself and $H_t(Z_i \times \{s_i\}) \subset \psi^{-1}((1/2,1))$ for $i = 1, 2, \ldots, k$. Let $g = \varphi \circ H_1^{-1} \circ (\tilde{f} \times p) : X \to \sigma^n$. Since $H_1|\partial \sigma^n \times I = \text{id}_{\sigma^n \times I}$, $g|A = f$. Since mesh $U < \varepsilon$, $d(\tilde{f}, g) < \varepsilon$. Let $y \in \sigma^n$. Now we consider next three cases.

Case 1. If $y \in \sigma^n \setminus U$, then there exists $V \in \mathcal{V}$ such that $y \in V$. Since $g^{-1}(y) = f^{-1}(y) \subset \tilde{f}^{-1}(V)$, $g^{-1}(y)$ is folded rel $p$.

Case 2. Suppose that $y \in U \setminus (\sigma^n \setminus Z)$. Let $U_1, U_2, \ldots, U_t$ be the all members of $\mathcal{U}$ which contain $y$. Let $U' = \bigcup_{i=1}^t U_i$. Since $U' \cap \{y \in \sigma^n | d(y, \partial \sigma^n) = r/2\} \neq \emptyset$ and $\text{diam} U' < \delta$, there exists $V' \in \mathcal{V}$ such that $U' \subset V'$. Then $g^{-1}(y) = (\tilde{f} \times p)^{-1} \circ H_1 \circ \varphi^{-1}(y) \subset (\tilde{f} \times p)^{-1} \circ \varphi^{-1}(U') = \tilde{f}^{-1}(V') \subset \tilde{f}^{-1}(V')$. So $g^{-1}(y)$ is folded rel $p$.

Case 3. Case 3. Suppose that $y \in Z$ and there exists $i = 1, 2, \ldots, k$ such that $y \in Z_i$. Since $H_1(\{y\} \times I) = H_1(\{y\} \times [0, s_i]) \cup H_1(\{y\} \times [s_i, t_i]) \cup H_1(\{y\} \times [t_i, 1])$, $H_1(\{y\} \times I)$ is folded rel $\psi$. Since $g^{-1}(y) = (\tilde{f} \times p)^{-1} \circ H_1 \circ \varphi^{-1}(y) = (\tilde{f} \times p)^{-1} \circ H_1(\{y\} \times I)$ and by (1) of Lemma 19 $g^{-1}(y)$ is folded rel $\psi \circ (\tilde{f} \times p) = p$.

So $g$ is folded rel $p$. This completes the proof.

**Lemma 23** Let $X$ be a compactum and let $A$ be a closed subset in $X$. Let $\varepsilon > 0$, $\sigma^n$ ($n \geq 1$) an $n$-dimensional simplex. If $f : A \to \partial \sigma^n$ is a Bing map and $\tilde{f} : X \to \sigma^n$ is a map with $\tilde{f}|A = f$, then there exists a Bing map $g : X \to \sigma^n$ such that $d(\tilde{f}, g) < \varepsilon$ and $g|A = f$. 
Proof. Set \( P \) as in Lemma 20. Let \( C(X, f|A) = \{ g \in C(X, \sigma^n) | g|A = f \} \).
For each \( p_i \in P \), let \( C(X, f|A, p_i) = \{ g \in C(X, f|A) | g \) is folded rel \( p_i \}. \)
Let \( B(X, f|A) = \{ g \in C(X, f|A) | g \) is a Bing map \}. Since \( B(X, f|A) = \bigcap_{i=1}^{\infty} C(X, f|A, p_i) \), by Lemma 21, 22 and Baire theorem \( B(X, f|A) \) is dense in \( C(X, f|A) \). This completes the proof.

The following theorem is a more precise result than the theorem of J. Song and E. D. Tymchatyn.

**Theorem 24** (Extension Theorem of Bing Maps) Let \( X \) be a compactum and let \( A \) be a closed subset in \( X \). Let \( K \) be a finite simplicial complex such that \( |K| \) is a nondegenerate connected polyhedron and let \( L \) be a subcomplex of \( K \). If \( f : A \to |L| \) is a Bing map and \( f : X \to |K| \) is a map with \( f|A = f \) and \( f^{-1}(|L|) = A \), then for any \( \epsilon > 0 \) there exists a Bing map \( g : X \to |K| \) such that \( g|A = f \) and \( d(f, g) < \epsilon \).

Proof. First, we prove the following claim:

The set \( C_{v_0}(X, |K|) = \{ f \in C(X, |K|) | f^{-1}(v) \) is a Bing compactum for each vertex \( v \in K^0 \} \) is a \( G_{\delta} \)-dense subset of \( C(X, |K|) \).

Let \( v = v_0 \in K^0 \) and let \( p : X \to I \) be a map. We shall prove that \( C_{v}(X, |K|, p) = \{ f \in C(X, |K|) | f^{-1}(v) \) is folded rel \( p \} \) is an open and dense subset of \( C(X, |K|) \). We can easily see that \( C_{v}(X, |K|, p) \) is an open set of \( C(X, |K|) \). We prove that \( C_{v}(X, |K|, p) \) is dense in \( C(X, |K|) \).

Let \( \epsilon > 0 \) and \( 0 < \alpha < \beta < 1 \). Consider the star \( St(v, K) = \cup \{ \sigma \in K | v \in \sigma \} \) of \( K \) with \( v \). For each simplex \( \sigma = [v_0, v_1, .., v_m] \in K (v_0 = v, m \geq 1) \), put

\[
\sigma_{\alpha} = \{ \sum_{i=0}^{m} t_i v_i | t_i \geq 0 (i = 0, 1, 2, .., m), \sum_{i=0}^{m} t_i = 1, t_0 \geq \alpha \}
\]

\[
\sigma_{\beta} = \{ \sum_{i=0}^{m} t_i v_i | t_i \geq 0 (i = 0, 1, 2, .., m), \sum_{i=0}^{m} t_i = 1, t_0 \geq \beta \}.
\]

Let

\[
M = \bigcup_{v \in \sigma, \sigma_{\alpha}} \sigma_{\alpha}, \quad N = \bigcup_{v \in \sigma, \sigma_{\beta}} \sigma_{\beta}.
\]

Choose positive numbers \( s_0, s_1, \) and \( s_2 \) with \( 0 < s_0 < s_1 < 1/2 < s_2 < 1 \). Consider the following set

\[
Z = (M \times [s_0, s_1]) \cup (cl(M - N) \times [s_1, s_2]) \subset St(v, K) \times [0, 1].
\]

For each \((m\text{-dimensional})\) simplex \( \sigma \) containing \( v \) \((m \geq 1)\), put \( \sigma_Z = Z \cap (\sigma \times I) \). Also, consider the following map \( \phi : Z \to T = ([0, 1 - \alpha] \times [s_0, s_1]) \cup \)
Consider that each simplex can be homeomorphic to an $m$-cell for $m > 0$. Let $h: |\mathcal{K}| \times I \to |\mathcal{K}| \times I$ be the natural projection. Hence for each simplex $\sigma$, $h_{\sigma}(q(\sigma \times I) - q(\sigma Z)) = \sigma \times I$ and $h_{\sigma}(q(\sigma Z)) = D$ and $h_{\sigma}(q(H_{\sigma})) = \{v\} \times I$, where

$$H_{\sigma} = \{v\} \times ([0, s_0] \cup [s_1, s_2]) \cup (\sigma \alpha \times \{s_0\}) \cup (H_{\alpha} \times [s_0, s_2]) \cup$$

$$(H_{\beta} \times [s_1, s_2]) \cup (\sigma \alpha \times \{s_2\}) \cup (\sigma \beta \times \{s_1\}),$$

and

$$H_{\alpha} = \{\Sigma_{i=0}^{m} t_i v_i \mid t_i \geq 0 \ (i = 0, 1, 2, \ldots, m), \ \Sigma_{i=0}^{m} t_i = 1, \ t_0 = \alpha\},$$

$$H_{\beta} = \{\Sigma_{i=0}^{m} t_i v_i \mid t_i \geq 0 \ (i = 0, 1, 2, \ldots, m), \ \Sigma_{i=0}^{m} t_i = 1, \ t_0 = \beta\}.$$

Also, choose a map $u': E = (St(v, \mathcal{K}) \times I) \cup_u D \to St(v, \mathcal{K}) \times I$ such that $u'(St(v, \mathcal{K}) \times I) = id$ and $u'^{-1}(\{v\} \times I) = \{v\} \times I$. By using these maps, we can obtain a map $h: |\mathcal{K}| \times I \to |\mathcal{K}| \times I$ such that for each simplex $[v, v_1, \ldots, v_m] \in \mathcal{K}$ containing $v$, $h([v_1, \ldots, v_m] = id$ and $h^{-1}(\{v\} \times I) = \bigcup_{v \in \mathcal{K}} H_{\sigma}$. Consider the map $g = \varphi \circ h \circ (f \times p): X \to |\mathcal{K}|$, where $\varphi: |\mathcal{K}| \times I \to |\mathcal{K}|$ is the natural projection. Since $H_{\sigma}$ is crooked with respect to $p$, we see that $g^{-1}(v)$ is folded rel $p$ (see the following figures below). Since we can choose a positive number $\alpha$ with $1 - \alpha < \epsilon$, we see that $d(f, g) < \epsilon$. Hence we see that $C_{\epsilon}(X, |\mathcal{K}|) = \{f \in C(X, |\mathcal{K}|) \mid f^{-1}(v) \text{ is a Bing compactum}\}$ is a $G_{\delta}$-dense subset of $C(X, |\mathcal{K}|)$. Then

$$C_{\mathcal{K}^0}(X, |\mathcal{K}|) = \cap_{v \in \mathcal{K}^0} C_{\epsilon}(X, |\mathcal{K}|)$$

is a $G_{\delta}$-dense subset of $C(X, |\mathcal{K}|)$. Hence the claim is true.
Let \( \dim|\mathcal{K}| = n \). For each \( j = 0, 1, \ldots, n \), let \( A_j = |L| \cup |\mathcal{K}(j)| \). By the claim, we may assume that \( \tilde{f}|\tilde{f}^{-1}(A_0) : \tilde{f}^{-1}(A_0) \to A_0 \) is a Bing map. Put \( \tilde{g}_0 = \tilde{f} \). Note that for each simplex \( \sigma \in \mathcal{K} \), the boundary \( \partial \sigma \) is a Z-set of \( \sigma \). By Lemma 23, we have a Bing map \( g_1 : \tilde{g}_0^{-1}(A_1) \to A_1 \) such that \( g_1|\tilde{g}_0^{-1}(A_0) = \tilde{g}_0|\tilde{g}_0^{-1}(A_0) \). By the homotopy extension theorem, we may assume that there is a map \( \tilde{g}_1 : X \to |K| \) such that \( \tilde{g}_1 \) is an extension of \( g_1 \), and \( \tilde{g}_1^{-1}(A_1) = \tilde{g}_0^{-1}(A_1) \). If we continue this process, we have a Bing map \( g = g_n : X \to |\mathcal{K}| \) such that \( g|A = f \) and \( d(\tilde{f}, g) < \epsilon \). This completes the proof.

The next result is the theorem of J. Song and E. D. Tymchatyn.

**Corollary 25** (J. Song and E. D. Tymchatyn [9]) Let \( X \) be a compactum and let \( P \) be an \( n \)-dimensional connected polyhedron \((n \geq 1)\). Then the set of all Bing maps in \( C(X, P) \) is a \( G_\delta \)-dense subset in \( C(X, P) \).

**Proof.** If we put \( A = |\mathcal{L}| = \phi \) in Theorem 24, we obtain this theorem.

**Corollary 26** (J. Song and E. D. Tymchatyn [9]) Let \( M \) be a Menger manifold with \( \dim M \geq 1 \). Then the set of all Bing maps in \( C(X, M) \) is a \( G_\delta \)-dense subset in \( C(X, M) \) (see [1] for properties of Menger manifolds).

**Proof.** We only prove that \( M \) is free. Let \( \varepsilon > 0 \). There exists a non-degenerate connected polyhedron \( P \subset M \) and map \( p : M \to P \) such that
$d(x, p(x)) < \varepsilon$ for each $x \in \mathbf{M}$ (see [1]). Let $q : \mathbf{P} \to \mathbf{M}$ be a natural embedding. Then $q$ is 0-dimensional and $d(q \circ p, id_{\mathbf{M}}) < \varepsilon$. By Lemma 16 and Corollary 25, $\mathbf{M}$ is free. This completes the proof.

References


Hisao Kato
Institute of Mathematics
University of Tsukuba
Ibaraki, 305-8571 Japan
e-mail: hisakato@sakura.cc.tsukuba.ac.jp

Eiichi Matsumashashi
Institute of Mathematics
University of Tsukuba
Ibaraki, 305-8571 Japan
e-mail: matsuhashieichi@mail.goo.ne.jp