<table>
<thead>
<tr>
<th>Title</th>
<th>On splitting theorems for CAT(0) spaces (General and Geometric Topology and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Hosaka, Tetsuya</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2004, 1370: 126-129</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/25468">http://hdl.handle.net/2433/25468</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
On splitting theorems for CAT(0) spaces

字都大学教育学部
保坂 哲也 (Tetsuya Hosaka)

The purpose of this note is to introduce main results of my recent paper [7] about splitting theorems for CAT(0) spaces.

We say that a metric space $X$ is a geodesic space if for each $x, y \in X$, there exists an isometry $\xi : [0, d(x, y)] \to X$ such that $\xi(0) = x$ and $\xi(d(x, y)) = y$ (such $\xi$ is called a geodesic). Also a metric space $X$ is said to be proper if every closed metric ball is compact.

Let $X$ be a geodesic space and let $T$ be a geodesic triangle in $X$. A comparison triangle for $T$ is a geodesic triangle $\overline{T}$ in the Euclidean plane $\mathbb{R}^2$ with same edge lengths as $T$. Choose two points $x$ and $y$ in $T$. Let $\bar{x}$ and $\bar{y}$ denote the corresponding points in $\overline{T}$. Then the inequality

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})$$

is called the CAT(0)-inequality, where $d_{\mathbb{R}^2}$ is the natural metric on $\mathbb{R}^2$. A geodesic space $X$ is called a CAT(0) space if the CAT(0)-inequality holds for all geodesic triangles $T$ and for all choices of two points $x$ and $y$ in $T$.

A proper CAT(0) space $X$ can be compactified by adding its ideal boundary $\partial X$, and $X \cup \partial X$ is a metrizable compactification of $X$ ([2], [4]).

A geometric action on a CAT(0) space is an action by isometries which is proper ([2, p.131]) and cocompact. We note that every CAT(0) space on which some group acts geometrically is a proper space ([2,
Details of CAT(0) spaces and their boundaries are found in [2] and [4].

In [7], we first proved the following splitting theorem which is an extension of Proposition II.6.3 in [2].

**Theorem 1.** Suppose that a group $\Gamma = \Gamma_1 \times \Gamma_2$ acts geometrically on a CAT(0) space $X$. If $\Gamma_1$ acts cocompactly on the convex hull $C(\Gamma_1 x_0)$ of some $\Gamma_1$-orbit, then there exists a closed, convex, $\Gamma$-invariant, quasi-dense subspace $X' \subset X$ such that $X'$ splits as a product $X_1 \times X_2$ and there exist geometric actions of $\Gamma_1$ and $\Gamma_2$ on $X_1$ and $X_2$, respectively. Here each subspace of the form $X_1 \times \{x_2\}$ is the closed convex hull of some $\Gamma_1$-orbit.

Using this theorem, we also proved the following splitting theorem which is an extension of Theorem II.6.21 in [2].

**Theorem 2.** Suppose that a group $\Gamma = \Gamma_1 \times \Gamma_2$ acts geometrically on a CAT(0) space $X$. If the center of $\Gamma$ is finite, then there exists a closed, convex, $\Gamma$-invariant, quasi-dense subspace $X' \subset X$ such that $X'$ splits as a product $X_1 \times X_2$ and the action of $\Gamma = \Gamma_1 \times \Gamma_2$ on $X' = X_1 \times X_2$ is the product action.

We also showed the following splitting theorem in more general case.

**Theorem 3.** Suppose that a group $\Gamma = \Gamma_1 \times \Gamma_2$ acts geometrically on a CAT(0) space $X$. Then there exist closed convex subspaces $X_1, X_2, X'_1, X'_2$ in $X$ such that

1. $X_1 \times X'_2$ and $X'_1 \times X_2$ are quasi-dense subspaces of $X$,
2. $X'_1$ and $X'_2$ are quasi-dense subspaces of $X_1$ and $X_2$ respectively,
3. $\Gamma_1$ and $\Gamma_2$ act geometrically on $X_1$ and $X_2$ respectively, and
4. some subgroups of finite index in $\Gamma_1$ and $\Gamma_2$ act geometrically on $X'_1$ and $X'_2$ respectively.

A CAT(0) space $X$ is said to have the geodesic extension property if every geodesic can be extended to a geodesic line $\mathbb{R} \rightarrow X$. Concerning
CAT(0) spaces with the geodesic extension property, we obtained the following theorem as an application of the above splitting theorems.

**Theorem 4.** Suppose that a group $\Gamma = \Gamma_1 \times \Gamma_2$ acts geometrically on a CAT(0) space $X$ with the geodesic extension property. Then $X$ splits as a product $X_1 \times X_2$ and there exist geometric actions of $\Gamma_1$ and $\Gamma_2$ on $X_1$ and $X_2$, respectively. Moreover if $\Gamma$ has finite center, then $\Gamma$ preserves the splitting, i.e., the action of $\Gamma = \Gamma_1 \times \Gamma_2$ on $X = X_1 \times X_2$ is the product action.

Let $Y$ be a compact geodesic space of non-positive curvature. Then the universal covering $X$ of $Y$ is a CAT(0) space by the Cartan-Hadamard theorem (cf. [2, p.193, p.237]), and we can think of $Y$ as the quotient $\Gamma \backslash X$ of $X$, where $\Gamma$ is the fundamental group of $Y$ acting freely and properly by isometries on $X$. As an application of Theorem 2, we showed the following splitting theorem which is an extension of Corollary II.6.22 in [2].

**Theorem 5.** Let $Y$ be a compact geodesic space of non-positive curvature. Suppose that the fundamental group of $Y$ splits as a product $\Gamma = \Gamma_1 \times \Gamma_2$ and that $\Gamma$ has trivial center. Then there exists a deformation retract $Y'$ of $Y$ which splits as a product $Y_1 \times Y_2$ such that the fundamental group of $Y_i$ is $\Gamma_i$ for each $i = 1, 2$.

A group $\Gamma$ is called a CAT(0) group, if $\Gamma$ acts geometrically on some CAT(0) space. Theorem 3 implies the following.

**Theorem 6.** $\Gamma_1$ and $\Gamma_2$ are CAT(0) groups if and only if $\Gamma_1 \times \Gamma_2$ is a CAT(0) group.

In [3], Croke and Kleiner proved that there exists a CAT(0) group $\Gamma$ and CAT(0) spaces $X$ and $Y$ such that $\Gamma$ acts geometrically on $X$ and $Y$ and the boundaries of $X$ and $Y$ are not homeomorphic. A CAT(0) group $\Gamma$ is said to be rigid, if $\Gamma$ determines the boundary up to homeomorphism of a CAT(0) space on which $\Gamma$ acts geometrically. Then we denote $\partial \Gamma$ as the boundary of the rigid CAT(0) group $\Gamma$. 
A conclusion in [1] implies that if \( \Gamma \) is a rigid CAT(0) group, then \( \Gamma \times \mathbb{Z}^n \) is also a rigid CAT(0) group for each \( n \in \mathbb{N} \). In [9], Ruane proved that if \( \Gamma_1 \times \Gamma_2 \) is a CAT(0) group and if \( \Gamma_1 \) and \( \Gamma_2 \) are hyperbolic groups (in the sense of Gromov) then \( \Gamma_1 \times \Gamma_2 \) is rigid. Concerning rigidity of products of rigid CAT(0) groups, we can obtain the following theorem from Theorem 3 which is an extension of these results.

**Theorem 7.** If \( \Gamma_1 \) and \( \Gamma_2 \) are rigid CAT(0) groups, then so is \( \Gamma_1 \times \Gamma_2 \), and the boundary \( \partial(\Gamma_1 \times \Gamma_2) \) is homeomorphic to the join \( \partial \Gamma_1 \ast \partial \Gamma_2 \) of the boundaries of \( \Gamma_1 \) and \( \Gamma_2 \).

**REFERENCES**


**DEPARTMENT OF MATHEMATICS, UTSUNOMIYA UNIVERSITY, UTSUNOMIYA, 321-8505, JAPAN**

*E-mail address:* hosaka@cc.utsunomiya-u.ac.jp