Superprime Rings

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The talk presented was a preliminary report and an introduction to the subject.

A ring in which every (two sided) ideal is an idempotent is called a fully idempotent ring. An example of such a ring includes the class of Von Numan regular rings. In fact, it is a wellkown and easy to show that every one sided ideal of Von Numan regular rings is an idempotent.
If a ring R is fully idempotent, then for any ideals J, K of R, $J \cap K = JK$. Hence an ideal P in a fully idempotent ring R is prime if and only if $J \cap K \subseteq P$ implies $J \subseteq P$, or $K \subseteq P$. On the other hand, in the ring $\mathbb{Z}_8$, $<2> \cap <4> = <4>$ but $<4>$ is not a prime ideal of $\mathbb{Z}_8$.
We define a prime ideal P in an arbitry ring R to be superprime if
$\bigcap_{i \in I} J_i \subseteq P \Rightarrow J_i \subseteq P$ for some i, where J_i is an ideal of R. A ring in witch 0 is superprime will be called a superprime ring.

The speaker has long been investgated the structure of rings in which every ideal is prime.
An example of such rings includes the ring R of all liner transformations $f: V \to V$ of a vector space V over a field F. We are mainly interested in the structure of fully prime rings with a superprime ideal.

Theorem 1 [1, Theorem 1.2]: A ring R is fully prime if and only if R is fully idempotent and ideals in R is linearly ordered.

Proposition 2: A superprime ring is primitive if and only if it is semiprimitive.
Proof: By definition, the intersection of all nonzero ideals of a superprime ring is nonzero, and

hence it is the minimal nonzero two sided ideal. If 0 is not a primitive ideal, the ring cannot be semiprimitive since the Jacobson radical must then contains the minimal nonzero two sided ideal.

A commutative fully prime ring is a field**.** Since a superprime ring is in particular prime, the minimal nonzero ideal is an idempotent. Hence, a commutative suprprime ring is also a field. The center of a fully prime ring is either a field or zero ([1, Theorem 1.3]). We ask: what can we say about the center of a superprime ring?

It is wellknown that a prime ring with a minimal right ideal is primitive.

Theorem 1: A right Noetherian fully prime superprime ring R is primitive. Further, if R is not simple, then R contains no minimal right ideals.

Sketch of a proof: By Nakayama's lemma and Theorem 1, R is semiprimitive. Hence by Proposition 2, R is primitive. It can be shown that Soc (R) is either 0 or R. Suppose that Soc (R) $\neq 0$. Then since R is prime, Soc (R) is the intersection of nonzero ideals of R. Since R is not simple (but fully prime right Noetherian), we have a contradiction.

A prime semiprimitive but not a primitive ring is not superprime. The ring of integers is an obious example. A Von Numan regular ring is semiprimitive but there is a wellknown example of prime Von Numan regular ring that is not primitive. We ask: is a semiprimitive fully prime ring superprimitive?

We conclude this preriminaly report with the following conjucture: Let R be a fully prime ring. The following statements are equivalent:

(a) R is primitive.

(b) R is semiprimitive

(c) R is superprime.

Reference

[1] W.D. Blair and H. Tsutsui, Fully Prime Rings, Comm. Albebra 22 (1994), no. 13, 5389-5400.