Conjecture Related to König-Egerváry Theorem

Daisuke Hosaka and Akihiro Yamamura
Akita University
1-1, Tegata Gakuen-machi, Akita 010-8502 Japan

Abstract

In this paper, we consider an extension of König-Egerváry theorem into three-dimensional arrays and make a conjecture related to an extension of the König-Egerváry theorem. Performing computer experiment, we give counterexamples of the conjecture

1 Introduction

A graph is called *bipartite* if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y. A matching in a graph is a set of pairwise non-adjacent edges. A covering of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set.

Theorem (König-Egerváry Theorem)**.** *In any bipartite graph, the cardinality of a maximum matching is equal to the cardinality of a minimum covering.*

It can be restated in terms of matrices as follows. Let A be a $(0, 1)$-matrix. A set M of 1's of A is a matching of A if no two 1's in M lie in the same line. A set C of lines is a covering of A if lines in C together cover all the 1's of A. Here the term "line" means either a row or a column of A. We denote the cardinality of a maximum matching of A by $m_2(A)$, and the cardinality of a minimum covering of A by $c_2(A)$. König-Egerváry theorem can be translated as $m_2(A) = c_2(A)$ for any $(0, 1)$-matrix A. We consider a certain extension of König-Egerváry theorem to three-dimensional array.

2 Extension of König-Egerváry Theorem

We define a line in a three-dimensional array A similarly to that in a matrix and define a matching in A, a covering of A similar to the 2-dimensional case. We denote the cardinality of a maximum matching in A by $m_3(A)$, and the cardinality of a minimum covering of A by $c_3(A)$.

2.1 Properties of three-dimensional array

Theorem. *Let A be a $(0,1)$-three-dimensional array. For any matching M of A and any covering C of A,*

$$|M| \leq |C|.$$

Proof. We denote the elements in M by $1_1, \ldots, 1_n$, where $n = |M|$. By the definition of a covering, for any 1_i, there exists $l_i \in C$ such that $1_i \in l_i$. By the definition of a matching, $i \neq j \Rightarrow l_i \neq l_j$. Therefore, C has at least $|M|$ elements. □

Corollary. *For any $(0,1)$-three-dimensional array A,*

$$m_3(A) \leq c_3(A).$$

□

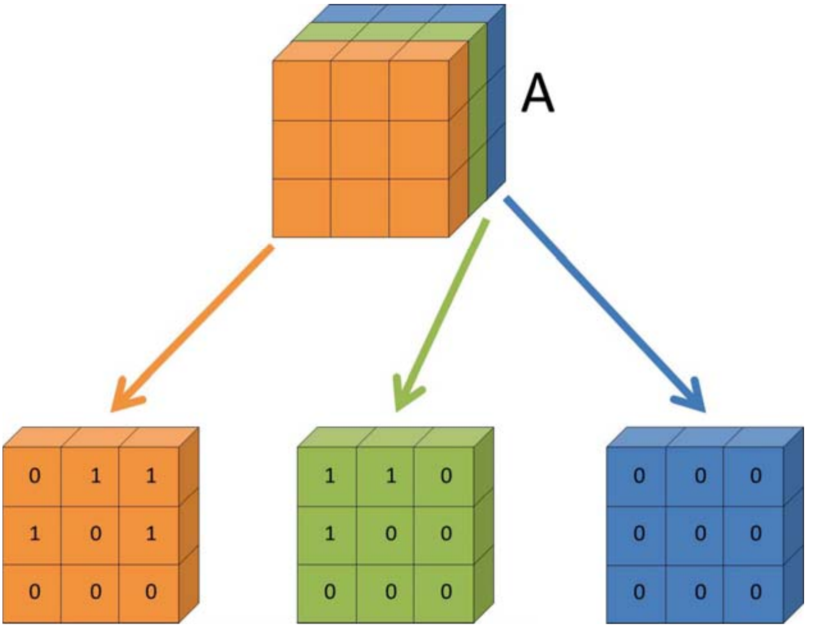
Now we conjecture that

$$m_3(A) = c_3(A)$$

for any $(0,1)$-three-dimensional array A.

2.2 Experiment

We verified $m_3(A) = c_3(A)$ for any $2 \times 2 \times 2$ $(0,1)$-array A by computer experiment. Next, we tried to check whether $m_3(A) = c_3(A)$ for any $3 \times 3 \times 3$ $(0,1)$-three-array A and then we found a counterexample in which $m_3(A) = 3 < 4 = c_3(A)$ as follows.



We performed a computer experiment on Windows 7 whose memories are 4.00 GB. Its CPU is Inter(R) Core(TM) i5-3470 CPU @ 3.20GHz. The program is made by Python and spent about half a month calculating $m_3(A)$ and $c_3(A)$ of 141,750 $3 \times 3 \times 3$ $(0, 1)$-array A. We found 3,434 counterexamples. We also checked the values of $m_3(A)$ and $c_3(A)$ for each counterexample A and found that $c_3(A) - m_3(A) = 1$ for all counterexamples.

References

[1] Egerváry, E., On combinatorial properties of matrices. Mat. Lapok. 1931, Vol. 38, p. 16-28.

[2] Aharoni, R., Ryser's Conjecture for Tripartite 3-Graphs. Combinatorica. 2001, Vol. 21, Issue 1, p. 1-4.