Modal Logic and Spatial Reasoning

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Abstract. In accessibility semantics, a formula is interpreted into a set in a graph and a modality is interpreted into an operation over sets. When it is applied to a graph which represents the structure of a figure, a formula describes a structure of a figure. By using this, modal logic describes spatial reasoning.

1 Introduction

1.1 Preliminary of Logic

The meaning of a sentence is a proposition. For example, the meaning of a sentence '20 Feb. 2019 is Wednesday.' is a proposition. On the other hand, the meaning of a sentence with a hole is a property. For example, the meaning of a sentence with a hole '( ) is Wednesday.' is a property, so to say, Wednesdayness.

The denotation of a proposition is its truth value, and the denotation of a property is a set which is its extension. For example, as for the sentence '20 Feb. 2019 is Wednesday.', its denotation is true. On the other hand, as for the sentence with a hole '( ) is Wednesday.', its denotation is the set $\{x|x \text{ is Wednesday.}\}$, which is its extension.

In the standard semantics of formal logic, a closed formula is interpreted into a proposition, and its denotation is a truth value. On the other hand, a formula with holes is interpreted into a property, and its denotation is a set which is its extension. In formal logic, a formula with holes is sometimes called a predicate. Otherwise, a predicate only refers to an atomic formula with holes.

In contrast, there are non-standard semantics. the most well-known non-standard semantics is accessibility semantics, which interprets a formula into a set of nodes in a graph. Accessibility semantics is only for propositional logic, thus it does not have interpretations of formulae with holes.

Accessibility semantics is a semantics for first order propositional modal logic, which is a first order propositional logic with modality. Modality is a non-classical unary operator over formulae. Modalities in standard semantics refer to hearsay, logical consequence, and so forth, which are out of the domain of mathematics. On the other hand, in accessibility semantics, modalities refer to operations over sets.

Formal Syntax	Standard Semantics	Accessibility Semantics
Formula	Proposition/Truth value	Property/Set
Formula with holes	Property/Set	—
Modality	Believe, Logical consequence, etc. Non-mathematics	Operation over sets

When the accessibility semantics is applied to a graph which represents the structures of figures in the plane, modalities are regarded as operations over figures, and modal logic describes spatial structures.

1.2 Aim of This Study

The aim of this study is to use modal logic for describing spatial reasoning, according to accessibility semantics which interprets modalities into operations over figures.

This study is inspired by a study by Prof. Sano Katsuhiko et-al. [1]. Both their study and this study of ours discuss properties of figures with modal logic. They discussed properties of figures drawn by pixels in a grid. We discuss properties of figures represented by graphs.

1.3 Outline

The construction of this article is as below. Section 2 explains modal logic with accessibility semantics. Section 3 explains 3-level model of spatial structure. Section 4 explains how modal logic describes spatial reasoning.

2 Modal Logic with Accessibility Semantics

2.1 Syntax

This subsection defines the syntax of our modal logic.

Definition 2.1.1 (Formula) The set of *formulae* of our modal logic is defined by the following syntax.

$$\text{Formula } F ::= \mathit{Var} | \top | \neg F | F \wedge F | \Box F | \blacksquare F | \mathsf{A} F,$$

where Var is the set of propositional variables.

We use the following abbreviations:

$$\bot \equiv \neg\top,$$

$F \to G \equiv \neg(F \wedge \neg G)$,
$\Diamond F \equiv \neg\Box\neg F$,
$\blacklozenge F \equiv \neg\blacksquare\neg F$,
$\mathsf{E}F \equiv \neg\mathsf{A}\neg F$.
The order of priority is $\neg$, $\blacksquare$, $\Diamond$, $\blacklozenge$, A, E, $\wedge$ and $\to$, where $\neg$ is high and $\to$ is low.

2.2 Accessibility Semantics

Accessibility semantics interprets a formula by a binary relation $\models$ under a graph G and an evaluation V.

Definition 2.2.1 (Semantic Graph) A graph $G = (N, E)$ is a directed graph which is a pair of the set of nodes N and the set of edges E where $E \subset N \times N$. This G is called a *semantic graph.* For $n, n' \in N$, $n \to n'$ stands for $(n, n') \in E$ A subset $V \subset N \times \mathit{Var}$ is an evaluation of propositional variables.

Definition 2.2.2 (Interpretation) The binary relation $\models$ between a node $n \in N$ and a formula F is defined by induction of F:
$n \models A$ iff $(n, A) \in V$ for $A \in \mathit{Var}$;
$n \models \top$ always holds;
$n \models \neg F$ iff $n \not\models F$;
$n \models F \wedge G$ iff $n \models F$ & $n \models G$;
$n \models \Box F$ iff, for any $n' \in N$, if $n \to n'$ then $n' \models F$;
$n \models \blacksquare F$ iff, for any $n' \in N$, if $n' \to n$ then $n' \models F$;
$n \models \mathsf{A}F$ iff, for any $n' \in N$, $n' \models F$.

2.3 Modality as a Spatial Operator

The modalities are interpreted into operators over sets, because modalities are operators over formulae, and formulae are interpreted into sets. When the sets are regarded as figures, the modalities are regarded as spatial operators.

We use the notation $[\![F]\!]$ for a formula F to refer to the denotation of F with is $[\![F]\!] = \{n | n \models F\}$. According to the definition of $\models$, the operations of modalities are described as the followings by using $[\![\ ]\!]$.
$[\![\Box F]\!] = \{n \in N | \text{for any } n', \text{ if } n \to n' \text{ then } n' \in [\![F]\!]\}$,
$[\![\blacksquare F]\!] = \{n \in N | \text{for any } n', \text{ if } n' \to n \text{ then } n' \in [\![F]\!]\}$,
$[\![\Diamond F]\!] = \{n \in N | n \to n' \in [\![F]\!]\}$,
$[\![\blacklozenge F]\!] = \{n \in N | n' \to n,\ n' \in [\![F]\!]\}$,
$[\![\mathsf{A}F]\!] = N$ if $[\![F]\!] = N$, and $[\![\mathsf{A}F]\!] = \emptyset$ otherwise,
$[\![\mathsf{E}F]\!] = \emptyset$ if $[\![F]\!] = \emptyset$, and $[\![\mathsf{E}F]\!] = N$ otherwise.

Figures 1–3 describes the operations of modalities over an example.

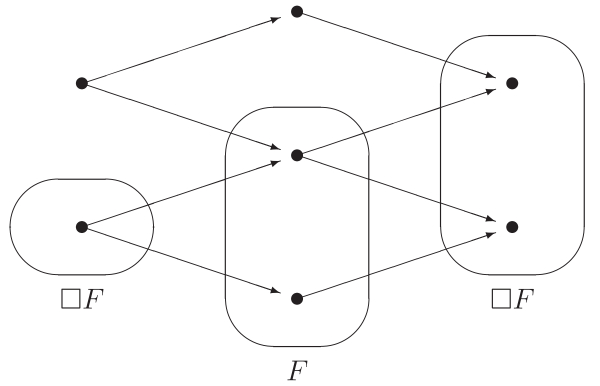


Figure 1. $[\![\Box F]\!] = \{n \in N | \text{for any } n', \text{ if } n \to n' \text{ then } n' \in [\![F]\!]\}$

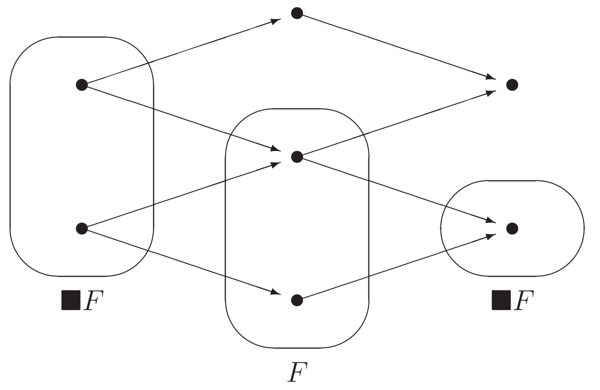


Figure 2. $[\![\blacksquare F]\!] = \{n \in N | \text{for any } n', \text{ if } n' \to n \text{ then } n' \in [\![F]\!]\}$

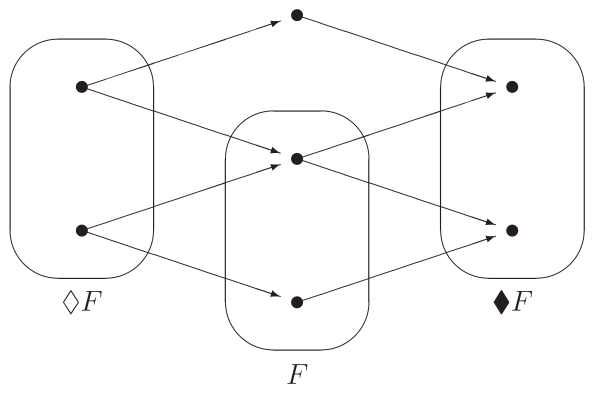


Figure 3. $[\![\Diamond F]\!] = \{n \in N | n \to n' \in [\![F]\!]\}, \ [\![\blacklozenge F]\!] = \{n \in N | n' \to n, \ n' \in [\![F]\!]\}$

2.4 Axiomatisation

The logic of accessibility semantics is axiomatised as below.

Definition 2.4.1 (Axiomatic System) The axiomatic system is defined by the follwoing rules:

From $F \to G$ and F, infer G; From F, infer $\Box F$, $\blacksquare F$ and $\mathsf{A}F$;
All tautologies;
$\Box(F \to G) \to \Box F \to \Box G$; $\blacksquare(F \to G) \to \blacksquare F \to \blacksquare G$;
$\mathsf{A}(F \to G) \to \mathsf{A}F \to \mathsf{A}G$; $\mathsf{A}F \to F$; $\mathsf{A}F \to \mathsf{AA}F$; $\mathsf{EA}F \to \mathsf{A}F$;
$F \to \Box\blacklozenge F$; $F \to \blacksquare\Diamond F$; $\mathsf{A}F \to \Box F$; $\mathsf{A}F \to \blacksquare F$.

Theorem 2.4.2 *This axiomatic system is sound and complete for accessibility semantics.*

3 3-level Model of Spatial Structure

3.1 3-level Model

A 3-level model is a model of spatial structure, which is originated from PLCA expression by Prof. Takahashi Kazuko et-al. [2].

A 3-level model is a triple (M_2, M_1, M_1), where
– M_2 is a finite set of *arias*,
– M_1 is a finite set of *lines*, and
– M_0 is a finite set of *points*,
with some constraints. The detailed definition is given below.

Definition 3.1.1 (Point) A *point* is an element of $\mathbf{R}^2$.

Definition 3.1.2 (Line) A subset $l \subset \mathbf{R}^2$ is a *line* if there are functions $f_1, f_2 \in C^1[0,1]$ such that:
– $l = \{(f_1(t), f_2(t)) \in \mathbf{R}^2 | t \in [0,1]\}$,
– $f_1'(t)^2 + f_2'(t)^2 > 0$, and
– $t \mapsto (f_1(t), f_2(t))$ is injective.
The *end points* of l is defined as:

$$\mathrm{End}(l) = \{(f_1(0), f_2(0)), (f_1(1), f_2(1))\}.$$

Figure 4 shows a point and a line.

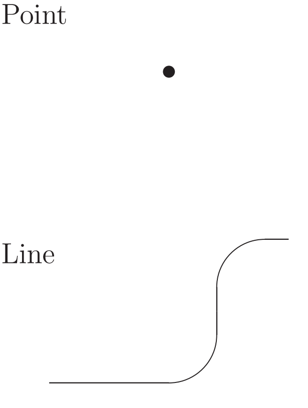
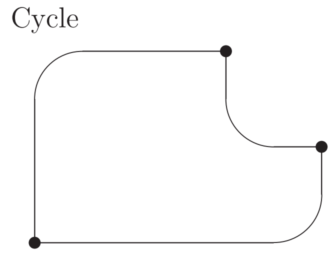


Figure 4.

Definition 3.1.3 (Cycle) A list of lines $(l_1, l_2, .., l_n)$ is a *cycle* if
– $l_i \cap l_{i+1} = \mathrm{End}(l_i) \cap \mathrm{End}(l_{i+1}) \neq \emptyset$,
– $l_n \cap l_1 = \mathrm{End}(l_n) \cap \mathrm{End}(l_1) \neq \emptyset$, and
– $\bigcup_i l_i$ is homeomorphic to a circle.

$|(l_1, l_2, .., l_n)| = \bigcup_i l_i$, which is called the *image* of a cycle $(l_1, l_2, .., l_n)$.



Definition 3.1.4 (Area) A subset $a \subset \mathbf{R}^2$ is a *area* if a is homeomorphic to the disc $\{(x, y)|x^2 + y^2 \leq 1\}$, and the boundary ∂a is the image of a circle.

Figures 4 and 5 show a cycle and a area.

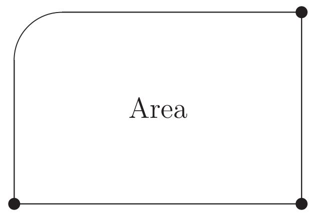
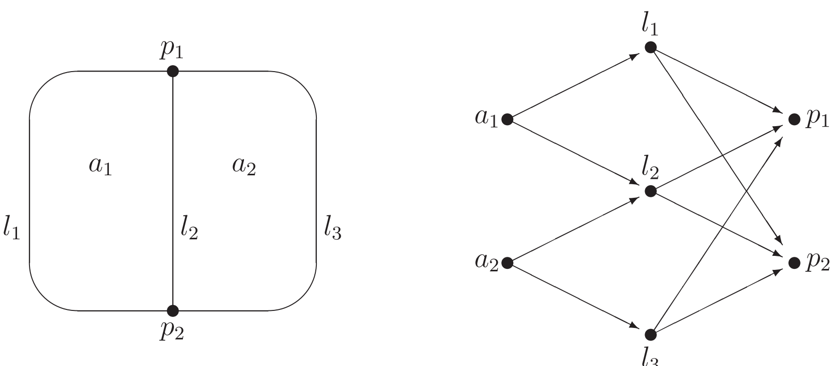


Figure 5.

Definition 3.1.5 (3-level Model) A triple $M = (M_2, M_1, M_0)$ is a 3-level model if
– M_2 is a finite set of areas;
– M_1 is a finite set of lines;
– M_0 is a finite set of points;
– For each $a, a' \in M_2$, if $a \neq a'$ then $a \cap a' \subset \partial a$;
– $\bigcup_{a \in M_2} \partial a = \bigcup_{l \in M_1} l$;
– For each $l, l' \in M_1$, if $l \neq l'$ then $l \cap l' \subset \mathrm{End}(l)$; and
– $M_0 = \bigcup_{l \in M_1} \mathrm{End}(l)$.

The left part of Figure 6 shows a 3-level model.



$M = (\{a_1, a_2\}, \{l_1, 1_2, l_3\}, \{p_1, p_2\})$

Figure 6.

Definition 3.1.6 (Concact Graph) For a 3-level model $M = (M_2, M_1, M_0)$, the *contact graph* $G(M) = (N(M), E(M))$ is defined as:

$N(M) = M_2 \cup M_1 \cup M_0$,
$E(M) = \{(a, l) \in M_2 \times M_1 | l \subset a\} \cup \{(l, p) \in M_1 \times M_0 | p \in l\}$.

The right part of Figure 6 is the contact graph of the 3-level model in the left part of Figure 6.

4 Modal Logic and Spatial Reasoning

For 3-level model M, its contact graph $G(M)$ is regarded as a semantic graph for modal logic. Then, modal logic describes some properties of M.

We write $M, V \models F$ when $n \models F$ for any $n \in N(M)$ under $G(M)$ and V.

The axioms are sound but not complete for the contact graphs. For example, $\Diamond\Diamond\Diamond\top$ is not refutable by the axioms. Then it is satisfiable by a semantics graph, namely the graph in Figure 7, but not by any contact graphs.

$$\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$$

Figure 7.

The followings are the formulae which mean the structures of 3-level models.

– $M, V \models \mathsf{E}X$:
This means that $[\![X]\!]$ is not empty.
– $M, V \models X \rightarrow \blacksquare\bot$:
This means that$[\![X]\!]$ consists of only areas.
– $M, V \models Y \rightarrow (\Diamond\top \wedge \blacklozenge\top)$:
This means that$[\![Y]\!]$ consists of only lines.
– $M, V \models Z \rightarrow \Box\bot$:
This means that $[\![Z]\!]$ consists of only points.

Figure 8 shows these situations.

M, V:

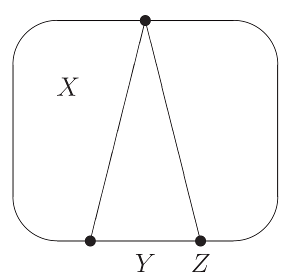


Figure 8.

This formula means that $[\![X]\!]$ and $[\![Y]\!]$ consist of areas:

$M, V \models \mathsf{A}(\Box\neg X \wedge \Box\neg Y)$,

and this formula means that $[\![X]\!]$ contacts $[\![Y]\!]$ at a point:

$M, V \models \mathsf{A}\neg(\blacklozenge X \wedge \blacklozenge Y) \wedge \mathsf{E}(\blacklozenge\blacklozenge X \wedge \blacklozenge\blacklozenge Y)$,

as is shown in Figure 9.

The follwoging situations are shown in Figure 10. This formula means that $[\![X]\!]$ and $[\![Y]\!]$ consist of areas:

$M, V \models \mathsf{A}(\Box\neg X \wedge \Box\neg Y)$.

This formula means that $[\![X]\!]$ contacts $[\![Y]\!]$ on a line:

$M, V \models \mathsf{E}(\blacklozenge X \wedge \blacklozenge Y \wedge \Diamond\top)$.

This formula means that $[\![X]\!]$ and $[\![Y]\!]$ share a point:

$M, V \models \mathsf{E}(\blacklozenge\blacklozenge X \wedge \blacklozenge\blacklozenge Y)$.

The axiomatic system derives

$\vdash \mathsf{E}(\blacklozenge X \wedge \blacklozenge Y \wedge \Diamond\top)) \to \mathsf{E}(\blacklozenge\blacklozenge X \wedge \blacklozenge\blacklozenge Y)$.

This means that, if two areas conatct on a line, then the two share a point.

Thus, modal logic describes spatial reasoning.

M, V:

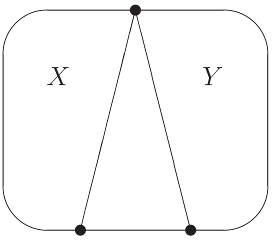


Figure 9.

M, V:

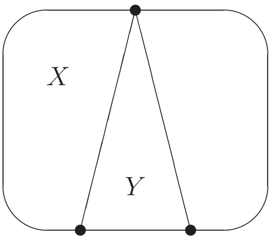


Figure 10.

References

[1] G. Sindoni, K. Sano, J.G. Stell: 'Axiomatizing Discrete Spatial Relations', Relational and Algebraic Methods in Computer Science, 2018, LNCS 11194, pp. 113–130.

[2] K. Takahashi, T. Sumitomo, I. Takeuti: 'On Embedding a Qualitative Representation in a Two-Dimensional Plane', Spatial Cognition & Computation 8, pp. 4–26, 2008.