Infinite sequences of non-Weierstrass numerical semigroup with odd conductor [1]

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Abstract

We construct infinite sequences of non-Weierstrass almost symmetric numerical semigroups with a fixed odd conductor through dividing by three.

1 Introduction

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0 \backslash H$ is finite. The cardinality of $\mathbb{N}_0 \backslash H$ is called the *genus* of H, which is denoted by $g(H)$. In this article H always stands for a numerical semigroup. We set

$$c(H) = \min\{c \in \mathbb{N}_0 \mid c + \mathbb{N}_0 \subseteqq H\},$$

which is called the *conductor* of H. It is known that $c(H) \leqq 2g(H)$. A numerical semigroup H is said to be *symmetric* if $c(H) = 2g(H)$. This semigroup has the following symmetric property: For $\gamma \in \mathbb{N}_0$ we have $\gamma \notin H$ if and only if $2g(H) - 1 - \gamma \in H$. A numerical semigroup H is said to be *quasi-symmetric* if $c(H) = 2g(H) - 1$. We set

$$PF(H) = \{\gamma \in \mathbb{N}_0 \backslash H \mid \gamma + h \in H \text{ , all } h \in H > 0\},$$

whose elements are called *pseudo-Frobenius numbers* of H. We have $c(H) - 1 \in PF(H)$. We set $t(H) = \sharp PF(H)$, which is called the *type* of H.

Remark 1.1 *We have* $c(H) + t(H) \leqq 2g(H) + 1$. *(For example, see [6].)*

A numerical semigroup H is said to be *almost symmetric* if the equality $c(H) + t(H) = 2g(H) + 1$ holds.

Remark 1.2 *i)* H *is symmetric if and only if* $t(H) = 1$. *In this case* H *is almost symmetric.*
ii) If H *is quasi-symmetric, then* $t(H) = 2$. *The converse does not hold. In this case* H *is also almost symmetric.*
iii) If $c(H) = 2g(H) - 2$, *then* $t(H) = 2$ *or* 3.

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[1] This paper is an extended abstract and the details will appear elsewhere.
This work was supported by JSPS KAKENHI Grant Number18K03228.

We set $PF^*(H) = PF(H)\backslash\{c(H) - 1\}$.

Remark 1.3 ([6]) *If H is almost symmetric, then it has the following symmetric property: The map sending γ to $c(H) - 1 - \gamma$ induces a bijection on $PF^*(H)$. The converse is true.*

A *curve* means a projective non-singular irreducible algebraic curve over an algebraically closed field k of characteritic 0. For a pointed curve (C, P) we denote by $H(P)$

$$\{\alpha \in \mathbb{N}_0 \mid \exists \text{ a rational function } f \text{ on } C \text{ such that } (f)_\infty = \alpha P\}.$$

Then $H(P)$ is a numerical semigroup of genus $g(C)$ where $g(C)$ is the genus of C. A numerical semigroup H is said to be *Weierstrass* if there exists a pointed curve (C, P) with $H(P) = H$.

For any integer $t \geqq 2$ we set

$$d_t(H) = \{h' \in \mathbb{N}_0 \mid th' \in H\},$$

which is a numerical semigroup. In this article we are interested in the case $t = 3$. Our main result is the following:

Theorem 1.4 *For any $u \geqq 1$ there exist infinite sequences*

$$H_0 \overset{d_3}{\longleftarrow} H_1 \overset{d_3}{\longleftarrow} H_2 \overset{d_3}{\longleftarrow} \cdots \overset{d_3}{\longleftarrow} H_{i-1} \overset{d_3}{\longleftarrow} H_i \overset{d_3}{\longleftarrow} \cdots$$

of non-Weierstrass numerical semigroups H_i with $c(H_i) = 2g(H_i) - (2u - 1)$ and $t(H_i) = 2u$, hence, H_i is almost symmetric for any i.

2 Non-Weierstrass almost symmetric numerical semigroups

In this section we find numerical semigroups in the starting points of the infinite sequences in Theorem 1.4. For a numerical semigroup H the least positive integer in H is denoted by $m(H)$, which is called the *multiplicity* of H. First we state the key lemmas for constructing the numerical semigroups.

Lemma 2.1 *Let u be an integer with $u \geqq 1$ and H be a numerical semigroup. Let g be an integer with $g > 4u - 3$, $g \not\equiv u$ mod 3 and*

$$g > \max\{m(H) - 2u, u + \frac{3}{2}(c(H) - 1) + \frac{m(H)}{2}, 2u + 2c(H) - 3\}.$$

We set

$$\tilde{H} = 3H \cup \{g + 2u + 3\mathbb{N}_0\} \cup \{2g - 2u - 3r \mid r \in \mathbb{Z}\backslash H\}.$$

Then we have
i) $\tilde{H}$ is a numerical semigroup and $g(\tilde{H}) = g$.
ii) $d_3(\tilde{H}) = H$ and $c(\tilde{H}) = 2g(\tilde{H}) - (2u - 1)$.

See [2] for the proof. We give an example which we get by applying the above lemma.

Example 2.1 In Lemma 2.1 let $u = 3$, $H = \langle 2, 3\rangle$ and $g = 10$. Then $\tilde{H}$ is equal to

$$3\langle 2, 3\rangle \cup \{10 + 6 + 3\mathbb{N}_0\} \cup \{20 - 6 - 3r \mid r \in \mathbb{Z}\backslash\langle 2, 3\rangle\}$$

$$= \langle 6, 9\rangle \cup \{16, 19, 22, \ldots\} \cup \{11, 17, 20, 23, \ldots\} = \langle 6, 9, 11, 16, 19\rangle.$$

In this case, $c(\tilde{H}) = 15 = 2g(\tilde{H}) - 5$.

Lemma 2.2 *Let u, H, g and $\tilde{H}$ be as in Lemma 2.1. We set $m = m(H)$. Then we have*

$$PF(\tilde{H}) \subseteqq \{g + 2u + 3l - 3m \mid 0 \leqq l \leqq m - 1\} \cup \{2g - 2u + 3m - 3m\}.$$

Hence, $t(\tilde{H}) \leqq m + 1$. Moreover, $s_{max}(\tilde{H}) = 2g - 2u + 3m$.

See [2] for the proof.

Theorem 2.3 *Let u, H, g and $\tilde{H}$ be as in Lemma 2.2. Moreover, assume that $2u - 1 \geqq m(H)$ and $g \geqq 2u + 2c(H) - 1$. Then we have the following:*
i) $t(\tilde{H}) = m(H) + 1$.
ii) If $2u - 1 = m(H)$, then $c(\tilde{H}) + t(\tilde{H}) = 2g(\tilde{H}) + 1$, hence $\tilde{H}$ is almost symmetric

Using Lemmas 2.1 and 2.2 we can prove the above statement. See [2] for the details of the proof.

To construct the desired non-Weierstrass numerical semigroups we need the known facts.

Remark 2.4 (Oliveira-Stöhr [7]) *A numerical semigroup $\tilde{H}$ satisfies that $H = d_3(\tilde{H})$ is non-Weierstrass. If $g(\tilde{H}) \geqq 15g(H) + 11$ and $g(\tilde{H}) \not\equiv 1$ mod 3, then $\tilde{H}$ is non-Weierstrass.*

Remark 2.5 (Buchweitz [1] and Komeda [4]) *Let m be an integer with $m \geqq 13$. Then there exists a non-Weierstrass numerical semigroup H with $m(H) = m$.*

Remark 2.6 (Komeda [3]) *Let $m = 8$ or 12. Then there exists a non-Weierstrass numerical semigroup H with $m(H) = m$.*

Corollary 2.7 *Let u be an integer with $u \geqq 7$. Then there exists a non-Weierstrass almost symmetric numerical semigroup $\tilde{H}$ with $c(\tilde{H}) = 2g(\tilde{H}) - (2u - 1)$.*

Proof. Let m be an odd integer with $m \geqq 13$. We set $u = \dfrac{m+1}{2}$. Then $u \geqq 7$. Using Remark 2.5, Lemma 2.1, Remark 2.4 and Theorem 2.3 in this order we get the desired non-Weierstrass numerical semigroups. □

Corollary 2.8 *Let u be an integer with $1 \leqq u \leqq 6$. Then there exists a non-Weierstrass almost symmetric numerical semigroup $\tilde{H}$ with $c(\tilde{H}) = 2g(\tilde{H}) - (2u - 1)$.*

Proof. If $u = 2, 3, 4$ (resp. 5, resp. 6), then we take a non-Weierstrass 8-semigroup (resp. 12-semigroup, resp. 14-semigroup H). We construct a non-Weierstrass numerical semigroup $\tilde{H}$ with $t(\tilde{H}) = 2u$ (see Komeda [2] for the details of the proof). In the case $u = 1$, the result is due to Oliveira-Stöhr [7]. □

We give an example in the case $u = 3$, namely, a non-Weierstrass almost symmetric numerical semigroup $\tilde{H}$ with $c(\tilde{H}) = 2g(\tilde{H}) - 5$.

Example 2.2 In Corollary 2.8 let $u = 3$. Let $H = \langle 8, 12, 18, 22, 45, 49 \rangle$, which is a non-Weierstrass numerical semigroup of genus 31 from [3]. Let $g = 15g(H)+11 = 15\times31+11 = 476$. We set $\tilde{H} = 3H \cup \{476 + 2 \times 3 + 3\mathbb{N}_0\} \cup \{2 \times 476 - 2 \times 3 - 3r \mid r \in \mathbb{Z}\backslash H\}$. Then $PF(\tilde{H}) = \{482+3l-24 \mid l = 3, \ldots, 7\} \cup \{952-6+24-24\}$, i.e., $t(\tilde{H}) = 6$. Hence $\tilde{H}$ is a non-Weierstrass almost symmetric numerical semigroup of genus 476 with $c(\tilde{H}) = 2g(\tilde{H}) - 5$.

3 Proof of Theorem 1.4

To prove Theorem 1.4 we need the following Lemma:

Lemma 3.1 (Komeda [2]) *For any $t \geqq 1$ we can construct an infinite sequence of numerical semigroups H_i with $c(H_i) = 2g(H_i) - t$ as follows:*

$$H_0 \overset{d_3}{\longleftarrow} H_1 \overset{d_3}{\longleftarrow} H_2 \overset{d_3}{\longleftarrow} \cdots \overset{d_3}{\longleftarrow} H_{i-1} \overset{d_3}{\longleftarrow} H_i \overset{d_3}{\longleftarrow} \cdots$$

Here, H_0 is any almost symmetric numerical semigroup with $c(H_0) = 2g(H_0) - t$ and H_i is also almost symmetric for any $i \geqq 1$.

We take H_0 in Lemma 3.1 as the non-Weierstrass almost numerical semigroup $\tilde{H}$ in Corollaries 2.7 and 2.8. Using Remark 2.4 and the method of construction of Lemma 3.1 (see Komeda [2]) we get Theorem 1.4.

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